Math. Model. Nat. Phenom.
Vol. 10, No. 3, 2015, pp. 6–15
DOI: 10.1051/mmnp/201510302

The Problem with Hilbert's 6th Problem

M. Slemrod [*]

Department of Mathematics, University of Wisconsin, Madison Madison, WI 53706 USA

Abstract. This paper reviews earlier results of the author regarding the hydrodynamic limit problem for the Boltzmann equation. In particular the key points are that the work of Gorban&Karlin suggests that Korteweg hydrodynamics is implied by the Boltzmann equation and this correspondence is validated by comparison with experimental, analytical, and numerical results. But if the correspondence is indeed valid then passage to limiting compressible Euler equations will not be generally possible after any time for which the the Euler system fails to have smooth solutions.

Dedicated to my friend Alexander Gorban on the occasion of his sixtieth birthday

Keywords and phrases: Boltzmann equation, hydrodynamic limit, Korteweg hydrodynamics, capillarity, viscosity

Mathematics Subject Classification: 35Q20, 82B40, 76A02

1. Introduction

In their recent review articles published in the Bulletin of the American Mathematical Society, Gorban&Karlin [15] and Saint-Raymond [38] address in elegant detail Hilbert's sixth problem [22] and related topics. Hence I will not repeat many of the subjects raised in their papers. Here I want to reinforce some comments I have made in the past with respect the very specific issue Hilbert raised, i.e. develop "mathematically the limiting processes ... which lead from the atomistic view to the laws of motion of continua." In contemporary research this has been taken as passage from the kinetic theory of Boltzmann to the continuum theory of Euler as the Knudsen number $\epsilon \to 0$. So far the results in this direction have been limited to the case when we are in a fluid domain where the Euler equations have have smooth solutions [31],[50]. These results are reviewed in the monograph [38] of Saint-Raymond. So of course the difficulty must enter when the situations where the Euler equations have lost smoothness due to formation of singularities. But here the issue becomes foggy: how can we see precisely where things go bad? Analysis is often much better at predicting when things go right by virtue of possibly overly strong sufficient conditions. So to help resolve this issue it is helpful to break the problem up into separate pieces where each one may both be simply explained and validated. I view this as a three step process.

---

[*] Corresponding author. E-mail: `slemrod@math.wisc.edu`

1. Replace the Boltzmann equation by an "equivalent" systems of macroscropic balance laws valid on the Boltzmann equation's invariant manifold [15] for conservation of the macroscopic states: density, momentum, and energy.
2. Validate the replacement given in Step 1 is physically correct.
3. Show that this "equivalent" system of balance laws will *not* allow for passage to the Euler equations as the small parameter Knudsen number ϵ tends to zero.

It is this sequence of steps that provide the basis for this paper and they will now be addressed.

2. From the Boltzmann equation to Korteweg's theory of capillarity: $CE \Longrightarrow Korteweg$

Our starting point is the Boltzmann equation itself which describes the evolution of a perfect gas in terms of a particle distribution function $f(t,x,\xi)$ which gives the probability of finding a particle of gas at time t, position x, moving with velocity ξ. The equation itself is $\partial_t f + \xi\cdot\nabla_x f = Q(f,f)/\epsilon$ where ϵ is the Knudsen number measuring the mean free distance between collisions. Q is the collision operator and given by

$$Q(f,f) = \iint (f(\xi')f(\xi_1') - f(\xi)f(\xi_1))b(\xi-\xi_{1,}\omega)d\xi_1 d\omega,$$
$$\xi' = \xi + (\xi-\xi_1)\cdot\omega\omega,\ \xi_1' = \xi_1 - (\xi-\xi_1)\cdot\omega\omega.$$

One develops macroscopic theory from the Boltzmann equation by multiplication by the vector $(1,\xi,\xi\cdot\xi)$ and intgegration over all $\xi\in\mathbb{R}^3$.

The integrals on the right hand side vanish as $(1,\xi,\xi\cdot\xi)$ are collison invariants and hence one can reassociate terms on the left hand side to make them appear as the usual balance laws of mass, linear momentum and energy (see for example [11]). This is a system of moments of f , the first five appearing under the time differentiation ∂_t. But the system is not closed and hence one may attempt to continue the moment process which of course will never close at at any finite set of moments. Closure rules must then be supplied either by taking a finite system of moments and expressing any additional moments not appearing in the time evolution term ∂_t in terms of the evolving ones. The most well known of this truncation is known as Grad's 13 moment system [20,21] and a thorough exposition is found in the book of Struchtrup [46].

The second appealing approach is to try to find an approximation to the solution f of the Boltzmann equation via a formal asymptotic expansion $f(t,x,\xi,\epsilon) = \sum \epsilon^n f_n(t,x,\xi)$. This expansion is called the Hilbert expansion but a careful reading of Maxwell's 1879 paper [30] shows that Maxwell himself had the germ of the idea three decades before Hilbert's 1912 paper [23]. We substitute the expansion into the Boltzmann equation, balance terms of equal powers of ϵ and obtain an infinite set of coupled nonlinear partial differential equations for the coefficients $f_n(t,x,\xi)$. It is important to note that the moment procedure and the Hilbert expansion are not mutually exclusive: one may take a list of moments, make a closure such as the one suggested by Grad, and then do a Hilbert type expansion to express the higher order flux terms in terms of the traditional macroscopic states of mass, momentum, and energy. This is in fact the route taken by Gorban&Karlin in [16–18,24,25].

Explicit computation of all the coefficients $f_n(t,x,\xi)$ in the Hilbert expansion would be a formidable task. On the other hand computation of $f_0(t,x,\xi)$ is easy since $Q(f_0,f_0) = 0$ and $f_0(t,x,\xi) = \frac{\rho(t,x)}{(2\pi\theta(x,t))^{3/2}}\exp\left(-\frac{|\xi-u(t,x)|^2}{2\theta(x,t)}\right)$ for some $\rho>0$ density, $\theta>0$ temperature, $u\in\mathbb{R}^3$ velocity. Substitution of f_0 into the five balance laws arising from the first five moments yields the classical compressible Euler equations

$$\begin{aligned}
&\partial_t\rho + \operatorname{div}(\rho u) = 0,\\
&\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u + (\rho\theta)\mathbf{I}) = 0,\\
&\partial_t(\rho\,|u|^2 + 3\rho\theta) + \operatorname{div}(u(\rho|u|^2 + 5(\rho\theta))) = 0.
\end{aligned}$$

Of course the formal limit as $\epsilon \to 0$ suggests the motivation behind Hilbert's program as illucidated in his sixth problem: the formal limit of the solution of the Boltzmann equation as $\epsilon \to 0$ does indeed yield the compressible Euler equations of macroscopic gas dynamics. Even more is true. Computation of $f_1(t, x, \xi)$ is possible and insertion into the five balance laws gives the order ϵ correction to the compressible Euler equations, i.e. the compressible Navier-Stokes-Fourier system (NSF) given by

$$\begin{aligned}
&\partial_t \rho + \partial_i(\rho u_i) = 0,\\
&\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) = \partial_j(T_{ij}),\\
&\partial_t(\frac{1}{2}\rho u_i u_i + \rho e) + \partial_j((\frac{1}{2}\rho u_i u_i + \rho e)\rho u_j) = \partial_j(u_i T_{ij}) - \partial_i q_i,\\
&T = T^E + T^V, e = e^E, q = q^F, \text{ where } e^E = \frac{3}{2}\theta,\ q^F = -\kappa \nabla\theta
\end{aligned}$$

Here T^E, T^V denote the elastic, viscous contributions to the Cauchy stress tensor T and are given by

$$\begin{aligned}
&T^E_{ij} = -p\delta_{ij}, p = \rho\theta,\\
&T^V_{ij} = \lambda(trD)\delta_{ij} + 2\mu D_{ij}, D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \lambda = -\frac{2}{3}\mu, \mu > 0,
\end{aligned}$$

and $\mu(\rho, \theta), \kappa(\rho, \theta)$ are the coefficients of viscosity and heat conduction. Again the formal limit as $\epsilon \to 0$ yields the compressible Euler equations.

A more efficient way to compute the Hilbert expansion is write the expansion in terms of the macroscopic state variables $M = (\rho, \rho u, \rho E), E = \frac{1}{2}|u|^2 + e$ of density, momentum, and energy so that dependence of $f_n(t, x, \xi)$ on x, t only occurs through M and its higher space derivatives. This rewriting is called the Chapman-Enskog expansion [11, 12] and hence we have $f(t, x, \xi, \epsilon) = \sum \epsilon^n f_n(\nabla^n M(t, x), \xi)$ where the symbol $\nabla^n M(t, x)$ is used to denote terms depending on space derivatives up to order n. The results of truncation of the Chapman-Enskog expansion at order zero and one yield the compressible Euler and Navier-Stokes-Fourier systems, respectively, and motivated Burnett [7, 8] to go the next order in ϵ thus obtaining the system that bears his name: the Burnett equations. To write them down in their entirety is both a chore and irrelevant.

The main point is that the Burnett equations yield the rest state of a gas unstable to high wave number perturbations. This instability was first recognized by Bobylev [2–4] and not surprisingly is called the Bobylev instability. The occurence of the Bobylev instability at Burnett order in itself does not invalidate the value of the Chapman-Enskog expansion, it only casts aspersion on *truncations* of the Chapman-Enskog expansion. In fact what mathemtician when teaching the concept of infinite series would rely on truncation when indeed it is the tail of the series that contains the information as to convergence or divergence of the series? It was the need to approximate the entire sum of the Chapman-Enskog expansion that motivated Rosenau [35, 36] and Bobylev [3, 4] to pursue their approximate summation techniques. But if approximate summation is nice it goes without saying that *an exact summation* is better. Hence it was to my great excitement when I discovered the work of Alexander Gorban and Ilya Karlin [16–18, 24, 25] who were able to produce such an exact sum for the linearized Grad moment equations. It this summation I now discuss since it is the key to my story.

We start with the 13-moment Grad equations obtained from taking moments of the Boltzmann equations, then we close the system according to the rule suggested by Grad and linearize about the rest state of zero velocity and unit density. This gives us the linearized 13 -moment Grad equations written as

$$\begin{aligned}
&\partial_t\rho + \operatorname{div} u = 0,\\
&\partial_t u = -\nabla p - \nabla\theta - \operatorname{div}\sigma,\\
&\partial_t\theta = -\frac{2}{3}(\operatorname{div} u + \operatorname{div} q),\\
&\partial_t\sigma = -(\nabla u + (\nabla u)^\top) + \frac{4}{3}(\operatorname{div} u)\mathbf{I} - \frac{2}{5}(\nabla q + (\nabla q)^\top) + \frac{8}{15}(\operatorname{div} q)\mathbf{I} - \frac{\sigma}{\epsilon},\\
&\partial_t q = -\frac{5}{2}\nabla\theta - \operatorname{div}\sigma - \frac{2}{3\epsilon}q.
\end{aligned}$$

This is a system of 13 equations for the 13 unknowns $\rho, u \in \mathbb{R}^3, \theta, q \in \mathbb{R}^3$, σ is a 3×3 symmetric and traceless matrix. Gorban&Karlin then proceed to further simplifications taking $p = \rho + \theta$ to obtain Grad's 10 moment system

$$\begin{aligned}
&\partial_t p + \frac{5}{3}\operatorname{div} u = 0,\\
&\partial_t u = -\nabla p - \operatorname{div}\sigma,\\
&\partial_t\sigma = -(\nabla u + (\nabla u)^\top) + \frac{4}{3}(\operatorname{div} u)\mathbf{I} - \frac{\sigma}{\epsilon},
\end{aligned}$$

and then consider the special one dimensional version of the 10 moment system

$$\begin{aligned}
&\partial_t p + \frac{5}{3}\partial_x u = 0,\\
&\partial_t u = -\partial_x p - \partial_x\sigma,\\
&\partial_t\sigma = -\frac{4}{3}\partial_x u - \frac{\sigma}{\epsilon}.
\end{aligned}$$

Here we have a very tractible system of 3 equations in the 3 scalar unknowns p, u, σ where σ is now the xx component of the stress. Admittedly this system is definitely not the Boltzmann equation but what is lost in generality is more than regained in understanding. So Gorban&Karlin proceed to perform the Chapman-Enskog expansion on this simple system by writing σ in the form $\sigma_{CE} = \sum\limits_{n=1} \epsilon^n\sigma^{(n)}$. The first few terms of the expansion are easy to find and we see $\sigma_{CE} = -\frac{4}{3}(\epsilon\partial_x u + \epsilon^2\partial_{xx}p + \epsilon^3\partial_{xxx}u + ..)$.Thus there is alternation between space derivatives of u and space derivatives of p. In order to sum the series it is natural to resort to Fourier transform and we denote the Fourier transforms with a super-imposed "ˆ". Again it is quite obvious that if we rescale $x/\epsilon = x'$ the ϵ disappears from σ_{CE}. Take the Fourier transform with respect to x' we have $\hat\sigma_{CE} = -ikA(k^2)\hat u - k^2 B(k^2)\hat p$, $A(k^2) = \sum\limits_{n=0} -ika_n(-k^2)^{n.}, B(k^2) = \sum\limits_{n=0} b_n(-k^2)^{n.}$.

Now we get to hard part. What are A and B? That is to say what are the exact sums of the power series defining A, B? When I first encountered this issue it seemed to me a Herculean task even for the simple example considered here. But indeed Gorban&Karlin did find beautiful and valuable formulas for A, B. The formulas are $A = \frac{B}{1-k^2B}$,and with C defined as $C = k^2B, C$ satisfies the fundamental cubic equation -$\frac{5}{3}(1-C)^2(C+\frac{4}{5}) - \frac{C}{k^2} = 0$. This gives us a good excuse to recall Cardano's formula for the roots of a cubic and since we are only interested in the unique real root we find $C < 0, k \neq 0, C(0) = 0, C$ is monotone decreasing for increasing $k^2, C(k^2) \to -\frac{4}{5}$ as $k^2 \to \infty$. From this result we immediately see the obvious yet profound result: A, B *are eternally paired.* You cannot have one without the other. In terms of the original Chapman- Enskog expansion for σ_{CE} it means that dependence on u is forever linked to dependence on p. Anyone who thinks they can justify using purely dependence on u , say as in NSF , based on the Chapman-Enskog expansion is guilty of wishful thinking. It is just not true.

The next important result follows from insertion of $\hat{\sigma}_{CE}$ into the Fourier transformation of the two balance laws $\partial_t p + \frac{5}{3}\partial_x u = 0$ and $\partial_t u = -\partial_x p - \partial_x \sigma$, i.e.

$$\begin{aligned}&\partial_t \hat{p} - \frac{5}{3}\hat{u} = 0,\\ &\partial_t \hat{u} = ik\hat{p} + ik(-ikA(k^2)\hat{u} - k^2 B(k^2)\hat{p}).\end{aligned}$$

Here for convenience we have written x instead of x'. Multiplication by the complex conjugates (c.c) of $\hat{p}, \hat{u}$ respectively yields the energy equality

$$\frac{1}{2}\partial_t \int_{-\infty}^{\infty} \frac{3}{5}|\hat{p}|^2 + |\hat{u}|^2 dk + \frac{1}{2}\partial_t \int_{-\infty}^{\infty} -\frac{3}{5}|\hat{p}|^2 k^2 B(k^2) dk + \int_{-\infty}^{\infty} [c.c.(\widehat{\partial_x p})]\hat{u} + [c.c.(\widehat{\partial_x u})]\hat{p} dk = \int_{-\infty}^{\infty} k^2 A(k^2)|\hat{u}|^2 dk$$

and with an application of Parseval's identity we have

$$\frac{1}{2}\partial_t \int_{-\infty}^{\infty} \frac{3}{5}\left|p\right|^2 + \left|u\right|^2 dx + \frac{1}{2}\partial_t \int_{-\infty}^{\infty} -\frac{3}{5}|\hat{p}|^2 k^2 B(k^2) dk = \int_{-\infty}^{\infty} k^2 A(k^2)\left|\hat{u}\right|^2 dk.$$

Now associate terms as follows:

- mechanical energy $=\frac{1}{2}\int_{-\infty}^{\infty} \frac{3}{5}\left|\hat{p}\right|^2 + |\hat{u}|^2 dx$,
- capillarity energy$=\frac{1}{2}\int_{-\infty}^{\infty} -\frac{3}{5}\left|\hat{p}\right|^2 k^2 B(k^2) dk$,
- viscous dissipation$=\int_{-\infty}^{\infty} k^2 A(k^2)\left|\hat{u}\right|^2 dk$,

and we have the energy formula

$$\partial_t(\text{mechanical energy}) + \partial_t(\text{capillarity energy}) = \text{viscous dissipation}.$$

Notice that since A, B are negative for $k \neq 0$ the capillarity energy is positive and the viscous dissipation is negative as they should be to make physical sense. The importance of the energy equality was succinctly summarized by Gorban&Karlin in their BAMS survey [15] and I now paraphrase their remarks. The bulk capillarity terms in fluid mechanics were introduced into the Navier-Stokes equations by Korteweg [26] (for a review of of some further results see [41–45]). Such terms appear naturally in theories of phase transitions, such as van der Waals liquids [51], the Ginzburg-Landau equations [27], and Cahn-Hilliard equations [9,10]. Surprisingly, as we have just seen such terms are also found in the ideal gas dynamics as a consequence of the Chapman-Enskog expansion. Finally, in the energy formula for the exact sum of the Chapman-Enskog expansion, the viscous dissipation decreases the total energy where the total energy is the sum of the mechanical and capillarity energies.

Let us now review where we are in our story. We started with the Boltzmann equation and desired to replace it by macroscopic hydrodynamic equations. Historically this was attempted either by taking moments of the Boltzmann equation or the Chapman-Enskog expansion. Here following Gorban&Karlin we took a hybrid approach and computed the Chapman-Enskog expansion for a linearized set of moment equations and this expansion could be exactly summed to show that our closed hydrodynamic system has the features associated with a non-local linear version of Korteweg's theory of capillarity. Hence let us ask the question: what should an exact summation of the Chapman-Enskog expansion for the full non-linear Boltzmann equation look like when we require this exact summation possess stresses that are second order in spacial gradients, material frame indifferent, and upon linearization yield macroscopic linear balance laws that resemble linear Korteweg theory? But with this proviso how could we make any guess other than that the non-linear hydrodynamics we desire should itself be Korteweg's theory (or something very close) in its full non-linear form. Hence after this introduction of Korteweg's theory into the hydrodynamic story it is time to record the full non-linear Korteweg theory and we take the form given by Dunn&Serrin in [14].

Denote by ρ density, u_i velocity, T_{ij} Cauchy stress tensor, e specific internal energy, θ temperature, q heat flux, p pressure, and c capillarity coefficient. Then the balance laws of mass, momentum, and energy are given by

$$\begin{aligned}
&\partial_t\rho + \partial_i(\rho u_i) = 0,\\
&\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) = \partial_j(T_{ij}),\\
&\partial_t(\frac{1}{2}\rho u_i u_i + \rho e) + \partial_j((\frac{1}{2}\rho u_i u_i + \rho e)\rho u_j) = \partial_j(u_i T_{ij}) - \partial_i q_i,\\
&T = T^E + T^V + T^K, e = e^E + e^K, q = q^F + q^K.
\end{aligned}$$

Here T^E, T^V, T^K denote the elastic,viscous, and Korteweg contributions to the Cauchy stress tensor T and are given by

$$\begin{aligned}
T^E_{ij} &= -p\delta_{ij}, p = \rho\theta,\\
T^V_{ij} &= \lambda(trD)\delta_{ij} + 2\mu D_{ij}, D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \lambda = -\frac{2}{3}\mu, \mu > 0,\\
T^K_{ij} &= \{\rho c\triangle\rho + \rho(\partial_\rho c)M + 2\rho(\partial_M c)d\otimes d\cdot\nabla^2\rho + \rho(\partial_\theta c)g\cdot d\}\delta_{ij} - c d_i d_j.
\end{aligned}$$

The quantities λ, μ denote the viscosity coefficients, $c = c(\rho, \theta, M)$ is the capillarity coefficient, $d = \nabla\rho, g = \nabla\theta$, $M = d\cdot d$.

The specific internal energy has the usual contribution $e^E = \frac{3}{2}\theta$ as well as the Korteweg contribution $e^K = \frac{c-\theta\partial_\theta c}{2\rho}d\cdot d$.The heat flux q is given by the sum the Fourier contribution $q^F = -\kappa g$,and the Korteweg term $q^K = c\rho tr(D)\nabla\rho$.

Of particular note is that this set of equations is material frame indifferent and hence compatible with this basic postulate of continuum mechanics. In addition it contains the classical viscous component of the stress tensor as well as the Korteweg component that was found in our Chapman-Enskog expansion. The extra addition to the internal energy which we found from the exact sum of the Chapman-Enskog expansion makes its appearance in general Korteweg theory as $e^K = \frac{c-\theta\partial_\theta c}{2\rho}d\cdot d$.

3. Validation of Korteweg's theory as a representation of the sum of the Chapman-Enskog expansion

Of course the title of this section is provocative. How could one possibly prove such a result with its myriad of difficulties? At best so far we have only a formal computation based on the exact sum of the Chapman-Enskog expansion for a linear finite moment Grad system. But I suggest that at this point it is adviseable to drop the complexity of a direct proof and rely on the simplicity of comparisons with solutions to boundary value problems for rarefied gas dynamics where the NSF theory is known to fail and kinetic theory is historically the main tool. (I note Struchtrup and his co-authors have taken a similar tack when establishing validity of their improved Grad like systems [46–48].) Of course this may require perhaps the unusual step of the analyst to consider laboratory experiment as a tool.

The boundary value problem we consider is one of thermal transpiration and often called in modern terms terms thermal creep. The theoretical treatment of the problem goes back to the classic paper of Maxwell [29]. Maxwell was looking for an explanation to the surface flow phenomena discovered by Kundt and Warburg for dilute gases and Helmhotz and Piotrowski for liquids. Maxwell notes that it was Reynolds who realized that the classical NSF theory fails to predict the observed experiments and Reynolds suggested that it is a problem calling for examination via the kinetic theory of gases. In fact Maxwell wrote "this phenomenon, to which professor Osborne Reynolds has given the name *thermal transpiration* was discovered entirely by him. He was the first to point out that a phenomenon of this kind was a necessary consequence of the kinetic theory of gases.." As Maxwell's own paper on kinetic theory [30] had just recently appeared and Maxwell was familiar with Boltzmann's theory as well, he had all the relevant tools to address the problem. Maxwell like us was searching for the next correction to NSF theory that he hoped would give desired predictions. But here Maxwell got into trouble. He, like Burnett some fifty years later, had the belief that if he could find the next order ϵ correction using his own version of the Chapman-Enskog expansion he would find the correct theory. Maxwell derived a part of what we now call the Burnett contribution to the Cauchy stress tensor and then remarked that having obtained his system "the final equations of motion" that "I have not, attempted to enter into the calculation of the steady motion".

So what are the problems of thermal creep that we and Maxwell consider? Here we take two problems of two dimensional steady flow of a dilute gas, namely take $(u_{1,}u_2) = (u, v), (x_1, x_2) = (x, y)$, (a) channel flow $\theta = \theta_1 \left|x\right|^s, v(x, y) = 0, |y| = H/2$, where $-\infty < x < \infty, -H/2 < y < H/2$,(b) half plane flow $\theta = \theta_1 \left|x\right|^s, v(x, y) = 0, y = 0$, where $-\infty < x < \infty, 0 < y$. Maxwell and Reynolds were interested in the case when there was no pressure gradient to drive the flow and like them we take isobaric motion $p = 1$.Then we observe that a solution of the NSF equations is nothing more than $(u, v) = (0, 0)$ with $\mathrm{div}(\kappa\nabla\theta) = 0$ where θ satisfies the relevant boundary conditions. It was precisely the fact that NSF predicts no motion of the gas that Reynolds and Maxwell knew that kinetic theory was called for. With today's computer power a numerical simulation of the solution to the Boltzmann equation may be performed and the results have been given in the papers of Ohwada et al [32, 33]. Furthermore Chen, Liu,Takata [13], Takata,Funugane [49] have given a careful rigorous analysis of the linearized Boltzmann equation that leads to a velocity profile for the channel flow problem consistent with the numerical results of Ohwada et al [32,33]. With these results in mind it was quite reasonable for Kim, Lee, Slemrod [25] to compare these very precise results with an elementary analysis of the flow problem using non-linear Korteweg theory. The results are given in detail in [25] where the Korteweg theory leads to extremely simple self similar systems of ordinary differential equations that can be essentially solved in closed form. Moreover the correspondence with the above mentioned numerical and analytical results for the Boltzmann equation appear quite good. For example for the channel flow problem with $s = 1$ the Korteweg system has a solution of the form $u = U(y), v = 0, \theta = x\Theta(y), \rho = x^{-1}R(y), R\Theta = 1$ with $\mu_0 U - KR^{-1}/L = const.$ when the capillarity coefficient is given by the power law $c = K\rho^a\theta^b, b - a = 4, K > 0$, for Maxwell molecules and L is a typical length along the walls..

So the velocity profile U is determined by the scaled density profile R. Conveniently R satisfies the ordinary differential equation $RR'' - 5(R')^2 = C_1 R^4$ where C_1 is a constant. This equation has a phase portrait in the R', R phase plane shown in Figure 2 of [25] where free constants are used to satisty the boundary conditions $R = 1/\theta_1$ at $y = \pm H/2$. We then solve for the velocity profile U from the profile of R. The qualitative picture obtained from Korteweg theory is consistent with Figures 1 and 2 of Chen, Liu, Takata [13] and Figure 8 of Ohwada et al [32,33]. Moreoever one should keep in mind that the results of Korteweg theory are exact solutions to a non-linear system and there was no resort to linearization as was done in Chen, Liu, Takata [13] and Ohwada et al [32, 33]. Of course most important of all is that if there is no capillarity contibution to the Cauchy stess, i.e $K = 0$, then we have $\mu_0 U = const.$ and no slip boundary conditions will give $U = 0$, i.e. no flow. So the linkage of viscosity to capillarity as was seen from Gorban&Karlin's exact summation of the Chapman-Enskog expansion is validated by the observed velocity profile. Futhermore since $K > 0$ the flow goes from left to right in the channel, again consistent with results from numerical computation and experiment. If by chance we had used the

Burnett system as Maxwell himself was trying to we would have had $K < 0$ and the flow would go in the direction opposite to the one actually observed.

4. Failure of convergence of Korteweg to Euler as the Knudsen number $\epsilon \to 0$

In the first two sections of the this paper I have given arguments based on both Gorban&Karlin's exact summation of the Chapman-Enskog expansion and comparison with analytical, numerical, and experimetal results as why Korteweg theory gives a simple, material frame indifferent, representation of the hydrodynamics of an ideal gas which is valid even when the gas is dilute. But now we come to the consequence of this derivation: one cannot in general pass to the limit of Korteweg's system (and hence from the Boltzmann equation) to the compressible Euler equations of gas dynamics. The reason is quite self evident and more or less staring us in the face. Recall the Korteweg system motivated by the exact summation results (displayed in Section 1 above) of Gorban&Karlin should have been written in terms of space variable $x' = x/\epsilon$.Hence if we now revert to the original x dependence of $T^V, T^K, e^K, q^F.q^K$ the Knudsen number ϵ now appears in our Korteweg system and we write the system as before but where λ, μ are proportional to ϵ and c is proportional to ϵ^2.In particular $\rho e^K = \frac{c - \theta \partial_\theta c}{2} d \cdot d$ for a power law choice of $c = \epsilon^2 c_0 \rho^a \theta^b$ has the form $\rho e^K = \epsilon^2 c_0 (1-b) \rho^a \theta^b \left|\nabla\rho\right|^2 /2$. So unless we are in the very lucky case of $b = 1$ the coefficient of $\left|\nabla\rho\right|^2$ will be non-zero. Furthermore if we desire that the limit as $\epsilon \to 0$ of the Korteweg system approach the Euler equations we must have $\epsilon^2 c_0 (1-b) \rho^a \theta^b \left|\nabla\rho\right|^2 /2 \to 0$ as $\epsilon \to 0$ in the sense of distributions. But without any further assumptions this is a hopeless task. It is as if we have a sequence of functions f_ϵ in $L^2(\Omega)$ satisfying a bound $\|\epsilon f_\epsilon\| < const.$ and ask that $\epsilon\ f_\epsilon \to 0$ *weakly* in $L^2(\Omega)$ as $\epsilon \to 0$. Without *a priori* information on f_ϵ this cannot be done. So for example if we assumed $\nabla\rho$ is bounded in $L^2(\Omega)$ for Ω our relevant space-time domain and where the bound is independent of ϵ then the limit could be taken to derive Hilbert's desired result.

The Korteweg system in itself cannot provide such an *a priori* bound. Hence it is no surprise that if one assumes the limit Euler system has smooth enough solutions so that the desired *a priori* bound is satisfied then we can indeed recover the Euler system from Boltzmann equation. This gives (admittedly a possibly over simplified) explanation for the results of Nishida [31], Ukai&Asano [50], on the ability to pass from the Boltzmann equation to the compressible Euler equations. (See also the paper of Saint-Raymond [39] for a related result for the incompressible Euler equations.). The reader will notice that our conclusion is similar to the well known result of Lax and Levermore [28] for the singular limit of the Korteweg - deVries equation $\partial_t u + u \partial_x u = \epsilon^2 K \partial_{xxx} u$ as $\epsilon \to 0$, i.e. once smooth solutions to the limit system $\partial_t u + u \partial_x u = 0$ breakdown all bets are off and there will be no convergence to the desired limit.

Now it could be argued that our Korteweg system has a viscous term and a better analogy would be with the KdV-Burgers system $\partial_t u + u \partial_x u = \epsilon \partial_{xx} u + \epsilon^2 K \partial_{xxx} u$. If the capillarity coefficient K is sufficiently small, viscosity would dominate and passage to the limit might be accomplished as in the paper of Schonbek [40]. But alas I have shown in [43] based on the results of Gorban&Karlin that while it is appealing mathematically to make such an assumption it does not follow from Gorban&Karlin's results. In fact just the reverse is true: capillarity dominates viscosity and our Korteweg system can be expected to generate oscillations just as in the Lax-Levermore limiting process.

Finally I note that all that has been said above was motivated by Boltzmann's kinetic theory for a rarefied, monatomic gas. This leaves open the issue of finding the correct extended fluid dynamics for a moderate dense gas or polyatomic gas. Here relevant references are [1, 5, 6, 34].

Acknowledgements. The author was supported by a Simons Foundation Collaborative Research Grant 232531. The author also thanks the referees for their valuable remarks.

References

[1] T. Arima, A. Mentrelli, T. Ruggeri. *Molecular extended thermodynamics of rarefied polyatomic gases and wave velocities for increasing number of moments.* Annals of Physics, 345 (2014), 111–140.

[2] A.V. Boby'lev. *Instabilities in the Chapman-Enskog expansion and hyprbolic Burnett equations.* J.Statistical Physics 124 (2-4) (2006), 371–399.

[3] A.V. Boby'lev. *The Chapman-Enskog and Grad methods for solving the Boltzmann equation.* Soviet Phys. Dokl., 27 (1982), 29–31.

[4] A.V. Boby'lev. *Generalized Burnett hydrodynamics.* J. Statistical Physics, 132 (3) (2008), 569–580.

[5] C. Borgnakke, P.S. Larsen. *Statistical collision model for Monte-Carlo simulation of polyatomic gas mixtures.* J. Computational Physics, 18 (1975), 405–420.

[6] J-F. Bourgat, L. Desvillettes, P. Le Tallec, B. Perthalme. *Microreversible collisions for polyatomic gases and the Boltzmann theorem.* European J. Mech. Fluids, 13 (1994), 237–254.

[7] D. Burnett. *The distibution of velocities in a slightly non-uniform gas.* Proc. London Math. Soc., S2-39 (1) (1935), 385–430.

[8] D. Burnett. *The distibution of molecular velocities and the mean motion in a non-uniform gas.* Proc. London Math. Soc., S2-40 (1) (1936), 382–435.

[9] J.W. Cahn. *Free energy of a non-uniform system. II. Thermodynamic basis.* J. Chemical Physics, 30 (1959), 1121–1124.

[10] J.W. Cahn, J.E. Hilliard. *Free energy of a non-uniform system II. Thermodynamic basis.* J. Chemical Physics, 28 (1958), 258–266.

[11] C. Cercignani. The Boltzmann equation and its applications. Series Applied Mathematical Sciences, V. 67. Springer, New York, 1988.

[12] S. Chapman, T.G. Cowling. The Mathematical Theory of Nonuniform Gases, third edition. Series Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990.

[13] I-K. Chen, T-P. Liu, S. Takata. *Boundary singularity for thermal transpiration problem of the linearized Boltzmann equation.* Archive for Rational Mechanics and Analysis, 212 (2014), 575–595.

[14] J.E. Dunn, J. Serrin. *On thermomechanics of interstitial working.* Archive for Rational Mechanics and Analysis, 88 (2) (1985), 95–133.

[15] A.N. Gorban, I.V. Karlin. *Hilbert's 6th problem: exact and approximate hydrodynamic manifolds for kinetic equations.* Bulletin of the American Mathematical Society (N.S.), 51 (2) (2014), 187–246.

[16] A.N. Gorban, I.V. Karlin. *Structure and approximations of the Chapman-Enskog expansion.* Soviet Physics JETP, 73 (1991), 637–641.

[17] A.N. Gorban, I.V. Karlin. *Structure and approximations of the Chapman-Enskog for the linearized Grad equations.* Transport Theory and Statistical Physics, 21 (1-2) (1992), 101–117.

[18] A.N. Gorban, I.V. Karlin. *Short wave limit of hydrodynamics: a soluble model.* Phys. Rev. Letters, 77 (1996), 282–285.

[19] H. Grad. *On the kinetic theory of rarefied gases.* Communications on Pure and Applied Mathematics, 2 (1949), 331-407.

[20] H. Grad. *Principles of the kinetic theory of gases.* In Handbuch der Physik (heraus-gegeben von S.Flugge), Bd. 12, Thermodynamik der Gase. Springer, Berlin, 205–294, 1958.

[21] D. Hilbert. *Mathematical problems.* Bulletin of the American Mathematical Society, 8 (10) (1902) 437–479.

[22] D. Hilbert. *Begrundung der kinetische Gastheorie.* Math. Ann., 72 (1912), 562–577.

[23] I.V. Karlin, A.N. Gorban. *Hydrodynamics from Grad's equations:what can we learn from exact solutions?* Ann. Phys., 11 (10-11) (2002), 783–833.

[24] I.V. Karlin, A.N. Gorban, G. Dukek, T.F. Nonnenmacher. *Dynamic correction to oment approximations.* Phys. Rev. E, 57 (2) (1998), 1668–1672.

[25] Y-J. Kim, M-G. Lee, M. Slemrod. *Thermal creep of a rarefied gas on the basis of non-linear Korteweg-theory.* Archive for Rational Mechanics and Analysis, 215 (2) (2015), 353–379.

[26] D.J. Korteweg. *Sur la forme que prennent les equations du mouvements des fluides si l'on tient compte des forces capillaires causes par des variations de densite.* Arch. Neerl. Sci. Exactes. Ser II 6 (1901), 1–24.

[27] L.D. Landau, E.M. Lifschitz. Statistical Physics, Addison-Wesley, Reading, MA, 1969.

[28] P.D. Lax, C.D. Levermore. *The small dispersion limit of the Korteweg-deVries equation* I. Communications in Pure and Applied Mathematics, 36 (1983), 253–290.

[29] J.C. Maxwell. *On the stresses in rarefied gases arising from inequalities in temperature.* Phil. Trans. Roy. Soc. (London), 170 (1879), 231–256.

[30] J.C. Maxwell. *On the dynamic theory of gases.* Phil. Trans. Roy. Soc. (London), 157 (1879) 49–88.

[31] T. Nishida. *Fluid dynamic limit of the nonlinear Boltzmann equation at the level of the compressible Euler equations.* Comm. Math. Physics, 61 (1978), 119–148.

[32] T. Ohwada, Y. Sone, K. Aoki. *Numerical analysis of the shear and thermal creep flows of a rarefied gas over a plane wall on the basis of the linearized Boltzmann equation for hard sphere molecules.* Physics of Fluids A-Fluid Dynamics, 1(9) (1989), 1588–1599.

[33] T. Ohwada, Y. Sone, K. Aoki. *Numerical analysis of the Poiseuille and thermal transpiration flows between 2 parallel plates on the basis of the Boltzmann equation for hard sphere molecules.* Physics of Fluids A-Fluid Dynamics, 1 (12) (1989), 2042–2049.

[34] M. Pavic, T. Ruggeri, S. Simic. *Maximum entropy principle for rarefied polyatomic gases.* Physica A, 392 (2013), 1302–1317.

[35] P. Rosenau. *Extension of the Landau-Ginzburg free energy functions to high gradient domains.* Physical Review A, 3 (1989), 6614–6617.

[36] P. Rosenau. *Extending hydrodynamics via regularization of the Chapman-Enskog expansion.* Physical Review A, 90 (1989), 7193–7196.

[37] L. Saint-Raymond. *A mathematical PDE perspective of the Chapman-Enskog expansion.* Bulletin of the American Mathematical Society (N.S.), 51 (2) (2014), 247–276.

[38] L. Saint-Raymond. Hydrodynamic limits of the Boltzmann equation. Lecture Notes in Mathematics, V. 1971, Springer, Berlin 2009.

[39] L. Saint-Raymond. *From Boltzmann's kinetic theory to Euler's equations.* Physica D, 237 (2008), 2028–2036.;

[40] M.E. Schonbek. *Convergence of solutions to nonlinear dispersive equations.* Communications in Partial Differential Equations, 7(1982), 959–1000.

[41] M. Slemrod. *Admissibility criteria for propagating phase boundaries in a van der Waals fluid.* Archive for Rational Mechanics and Analysis, 81 (4) (1983), 301–315.

[42] M. Slemrod. *Dynamic phase transitions in a van der Waals fluid.* J. Differential Equations, 52 (1) (1984), 1–23.

[43] M. Slemrod. *Chapman-Enskog*$\Longrightarrow$*viscosity-capillarity.* Quarterly of Applied Mathematics, 70 (3)(2012), 613–624.

[44] M. Slemrod. *Admissibility of weak solutions for the compressible Euler equations,* $n \geq 2$. Philosophical Transactions Roy. Soc. A, 371 (3) (2013), 1–11.

[45] M. Slemrod. *From Boltzmann to Euler: Hilbert's 6th problem revisited.* Computers and Mathematics with Applications, 65 (10) (2013), 1497–1501.

[46] H. Struchtrup. *Macroscopic transport equations for rarefied gas flows.* Series Interaction of Mathematics and Mechanics. Springer, Berlin, 2005.

[47] H. Struchtrup. *Macroscopic transport models for rarefied gas flows: a brief review.* IMA J. Appl.Math., 76 (5) (2011), 672–697.

[48] P. Taheri, A.S. Rana, M. Torrilhon, H. Struchtrup. *Macroscopic description of steady and unsteady rarefaction effects in boundary value problems of gas dynamics.* Continuum Mechanics and Thermodynamics, 21 (6) (2009), 423–443.

[49] S. Takata, H. Funagane. *Singular behavior of a rarefied gas on a plane boundary.* J. Fluid. Mechanics, 717 (2013), 30–47.

[50] S. Ukai, K. Asano. *The Euler limit and initial layers of the nonlinear Boltzmann equation.* Hokkaido Math. J., 12 (1983), 311–332.

[51] J.D. van der Waals. *Theorie thermodynamique de la capillarite, dans l'hypothese d'une variation continue de densite.* Archives Neerlandaises des Sciences exactes et naturelles, 28 (1895), 121–209.