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Exactly Solvable Models and Bifurcations: the Case of the Cubic NLS with a δ or a δ' Interaction in Dimension One

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Abstract. We explicitly give all stationary solutions to the focusing cubic NLS on the line, in the presence of a defect of the type Dirac's delta or delta prime. The models prove interesting for two features: first, they are exactly solvable and all quantities can be expressed in terms of elementary functions. Second, the associated dynamics is far from being trivial. In particular, the NLS with a delta prime potential shows two symmetry breaking bifurcations: the first concerns the ground state and was already known. The second emerges on the first excited state, and up to now had not been revealed. We highlight such bifurcations by computing the nonlinear and the no-defect limits of the stationary solutions.

Keywords and phrases: cubic NLS, stationary solutions, symmetry-breaking bifurcation

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1. Introduction

In recent years, the spectacular development of experimental techniques in condensed matter and ultracold gases, has greatly increased the interest in the mathematical modeling of the dynamics of Bose-Einstein condensates, i.e. in the Gross-Pitaevskii equation

$$i\partial_t u(t,x) \;=\; -\Delta u(t,x) \pm \lambda |u(t,x)|^2 u(t,x) + V(x)u(t,x), \qquad t \in \mathbb{R},\, x \in \mathbb{R}^d, \lambda > 0$$

where the unknown u is the wave function of the condensate, the potential V models the action of an external field, and the cubic term summarizes the effects of the two-body interactions among the particles in the condensate. It is well-known that the coupling constant $\pm\lambda$ is proportional to the scattering length of such a two-body interaction.

The function V can be characterized by the same length scale of the condensate, for instance when it models the trap confining the system, or by a much shorter length scale, for instance when it describes the effect of an inhomogeneity or of an impurity.

In this paper we focus on the latter case, restrict our study to the so-called cigar-shaped condensates, i.e. effectively one-dimensional systems, and specialize to the choices

$$1.\; V(x) = -\alpha\delta_0,\; \alpha > 0. \qquad 2.\; V(x) = -\gamma\delta_0',\; \gamma > 0. \tag{1.1}$$

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While choice 1. can be described as the action of a delta-shaped potential to be understood in the sense of distribution and it turns out to be a form perturbation of the free laplacian, the formalization of choice 2. requires to make resort to the self-adjoint extension theory due to von Neumann and Krein ([5,6]). Using such theory, one ends up by defining the singular interactions in (1.1) by means of suitable boundary conditions (see [6, 7]).

In the present paper we consider the equations

$$i\partial_t u(t,x) \;=\; H_j u(t,x) - \lambda |u(t,x)|^2 u(t,x), \qquad t \in \mathbb{R},\, x \in \mathbb{R},\, \lambda > 0,\, j = 1,2\,, \tag{1.2}$$

where the Hamiltonian operator H_1 and H_2 are defined as follows:

- H_1 is the quantum Hamiltonian operator associated to an attractive Dirac's delta potential. Its precise definition is the following:

$$\begin{aligned} D(H_1) \;&:=\; \{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cup H^1(\mathbb{R}),\ \psi'(0+) - \psi'(0-) = -\alpha\psi(0+) = -\alpha\psi(0-)\} \\ (H_1\psi)(x) \;&=\; -\psi''(x). \end{aligned} \tag{1.3}$$

- H_2 is the quantum Hamiltonian operator associated to an attractive delta prime potential.

$$\begin{aligned} D(H_2) \;&:=\; \{\psi \in H^2(\mathbb{R}\backslash\{0\}), \quad \psi'(0+) = \psi'(0-), \quad \psi(0+) - \psi(0-) = -\gamma\psi'(0+)\} \\ (H_2\psi)(x) \;&=\; -\psi''(x). \end{aligned} \tag{1.4}$$

Thinking of applications to BEC, the natural mathematical environment is the *energy space*. It is known (see e.g. [2]) that the energy space for the Dirac's delta case is

$$Q_\delta := H^1(\mathbb{R})$$

and, given $\psi \in Q_\delta$, the related energy functional reads

$$E_{\alpha\delta}(\psi) \;:=\; \frac{1}{2}\int_{\mathbb{R}} |\psi'(x)|^2\,dx - \frac{\alpha}{2}|\psi(0)|^2 - \frac{\lambda}{4}\int_{\mathbb{R}} |\psi(x)|^4\,dx.$$

On the other hand, for the delta prime interaction the energy space reads

$$Q_{\delta'} := H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-).$$

and, given $\psi \in Q_{\delta'}$, the related energy functional reads

$$\begin{aligned} E_{\gamma\delta'}(\psi) \;:=\; & \frac{1}{2}\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} |\psi'(x)|^2\,dx + \int_{\varepsilon}^{+\infty} |\psi'(x)|^2\,dx\right] - \frac{1}{2\gamma}|\psi(0+) - \psi(0-)|^2 \\ & - \frac{\lambda}{4}\int_{\mathbb{R}} |\psi(x)|^4\,dx. \end{aligned} \tag{1.5}$$

In [2] it was proven that the problem given by (1.2) is globally well-posed for any initial data in the related energy space. Moreover, L^2-norm and energy are conserved quantities.

As already stressed, the main focus of the present paper is given by the stationary states of (1.2). We call *stationary state* any square-integrable (in the variable x) solution to equation (1.2) of the type

$$u_\omega(t,x) \;=\; e^{i\omega t}\psi_\omega(x).$$

As a consequence, the function ψ_ω must solve the *nonlinear stationary Schrödinger equation*

$$H_j\psi_\omega - \lambda|\psi_\omega|^2\psi_\omega + \omega\psi_\omega \;=\; 0, \qquad j = 1,2$$

that can be rephrased as the problem of finding a function ψ_ω in the appropriate energy space that satisfies the equation

$$-\psi_\omega'' - \lambda|\psi_\omega|^2\psi_\omega + \omega\psi_\omega \;=\; 0, \qquad x \neq 0 \tag{1.6}$$

and fulfils the boundary conditions already given in (1.3) or (1.4):

– in the case of the Dirac's delta potential

$$\psi'_\omega(0+) - \psi'_\omega(0-) \;=\; -\alpha\psi_\omega(0+) \;=\; -\alpha\psi_\omega(0-); \tag{1.7}$$

– in the case of the delta prime interaction

$$\psi_\omega(0+) - \psi_\omega(0-) \;=\; -\gamma\psi'_\omega(0+) \;=\; -\gamma\psi'_\omega(0-). \tag{1.8}$$

Some remarks are in order concerning the meaning of the two singular interactions introduced.
It is well known that the δ interaction is the norm resolvent limit of a family of one dimensional Schrödinger operators with scaled potentials of the form $-\frac{d}{dx^2} + \frac{1}{\epsilon}V(\frac{x}{\epsilon})$, where V is an integrable function and in the limit it turns out that $\alpha = \int_{\mathbb{R}} V(x)\,dx$. This justifies the name of δ potential often used.
On the contrary, the δ prime interaction is not the limit, in any sense, of a sequence of Schrödinger operators with potentials related to the derivative of a Dirac's delta. It is however the norm-resolvent limit of a more complicated family of Schrödinger operators. For example, an approximation through a combination of three scaled δ potentials can be constructed, but the scaling is nonlinear as the δ centres get closer in the limit process. A more complete analysis of these and other related phenomena is given in [9, 11] and references therein. Like the δ potential, the δ prime interaction can also be considered as an effective operator replacing, after the scaling limit, more regular but not exactly solvable Schrödinger operators. Nevertheless, while the δ potential and its regularizations are in some sense elementary, all the regularizations of the δ prime interaction are characterized by some internal structure. There is a seemingly more natural singular Schrödinger operator called δ prime *potential* too, or dipole interaction, which is obtained as a suitable limit of scaled Schrödinger operators with a potential which is a derivative of a δ sequence. This self-adjoint operator is again a member of the family of point interactions on the line and has received some interest in the recent literature (see [15] for the construction and [4] for applications to NLS). It is however not treated in this paper. Finally, it is worth remembering that several different generalizations to graphs of the δ prime potential on the line exist in literature (see e.g. [1, 10]).
Finally, concerning the δ prime interaction H_2, we notice that the extremal cases of $\gamma = \infty$ and $\gamma = 0$ correspond respectively to decoupled half lines with Neumann boundary conditions, and to the standard second derivative on the line. In both cases they describe, with different interpretations, the disappearance of the defect.

Now we turn back to the problem of finding the stationary states. It is well-known that any real L^2 solution to (1.6) must be of the following type:

$$\psi_\omega(x) \;:=\; \begin{cases} \pm\sqrt{\frac{2\omega}{\lambda}}\operatorname{sech}(\sqrt{\omega}(x-x_1)), & x<0 \\[2ex] \pm\sqrt{\frac{2\omega}{\lambda}}\operatorname{sech}(\sqrt{\omega}(x-x_2)), & x>0\;. \end{cases} \tag{1.9}$$

So the problem of finding stationary states reduces to the issue of determining signs and the parameters x_1 and x_2 in order to fulfil the matching conditions (1.7) or (1.8).

In this paper we find all stationary solutions to equation (1.2) for both linear Hamiltonian operators H_1 and H_2 and perform two kinds of limit on them: the so-called *linear limit*, i.e. $\lambda \to 0+$ and the *no-defect limit*, which amounts to $\alpha \to 0+$ for the Dirac's delta potential and, according to the previous considerations, $\gamma \to 0+$ or $\gamma \to +\infty$ for the delta prime potential.

It is well-known that the standard nonlinear Schrödinger equation in dimension one is a completely integrable system ([21]); adding a perturbation, the complete integrability is broken but some features of exact solvability are retained. For example, perturbing with a Dirac's delta interaction, the stationary states of the system can still be exactly computed ([8, 12, 13, 16, 19]). In analogy with these cases, the problem of the stationary states for equation (1.2) with a delta prime interaction proves exactly solvable too. In addition, as shown in [3], the structure of the family of ground states can exhibit non-trivial bifurcations (for other one-dimensional models with bifurcations see also [14, 17, 18]).

1.1. Notation

Along with the symbols already introduced for energy spaces and energy functionals, we make use of the following notation:

- The $L^p(\mathbb{R})$-norm of the function ψ is denoted by $\|\psi\|_p$.
- The squared $L^2(\mathbb{R})$-norm of the function ψ will be often referred to as the *mass* of ψ.
- We make use of a family of functionals called *action*, defined as follows: choose $\omega > 0$, then

$$S_\omega(\psi) = E(\psi) + \frac{\omega}{2}\|\psi\|_2^2, \tag{1.10}$$

where E can be either $E_{\alpha\delta}$ or $E_{\gamma\delta'}$ according to the case one is examining.
- For any $A, B > 0, C \in \mathbb{R}$ we define the function

$$\phi(A,B,C;x) := A\,\mathrm{sech}(Bx - C) \tag{1.11}$$

We often omit the argument x making use of the shortened notation $\phi(A,B,C)$.

1.2. A preliminary lemma

The following lemma provides an estimate that will be repeatedly used along the paper.

Lemma 1.1. *Defined the function $\phi(A,B,C)$ as in (1.11), the following estimate holds:*

$$\begin{aligned}&\|\phi(A,B,C) - \phi(A',B',C')\|_{H^1(\mathbb{R}^+)}\\ \le\ & c|A-A'|\|\mathrm{sech}(B\cdot)\|_{H^1(\mathbb{R})} + c|B-B'|\|e^{-\min(B,B')|\cdot|}\|_{H^1(\mathbb{R})} + c|C-C'|\|\mathrm{sech}(B\cdot)\|_{H^2(\mathbb{R})},\end{aligned}$$

where c is a positive constant.

Proof. By triangular inequality

$$\begin{aligned}&\|\phi(A,B,C) - \phi(A',B',C')\|_{H^1(\mathbb{R}^+)}\\ \le\ & |A-A'|\|\phi(1,B,0)\|_{H^1(\mathbb{R})} + |A'|\|\phi(1,B,0) - \phi(1,B',0)\|_{H^1(\mathbb{R})}\\ &+ |A'|\|\phi(1,B',C) - \phi(1,B',C')\|_{H^1(\mathbb{R}^+)}\\ =:&(I) + (II) + (III)\end{aligned}$$

Term (I) is already in its final form, let us work on (II) and (III).
Concerning (II), notice that

$$|\mathrm{sech}(Bx) - \mathrm{sech}(B'x)| \le 4|e^{-Bx} - e^{-B'x}| \le 4|B-B'|e^{-\min(B,B')|x|},$$

thus

$$\|\mathrm{sech}(B\cdot) - \mathrm{sech}(B'\cdot)\|_2^2 \le 16|B-B'|^2\|e^{-\frac{BB'}{B+B'}|\cdot|}\|_2^2.$$

Analogously,

$$\left|\frac{d}{dx}\left(\mathrm{sech}(Bx) - \mathrm{sech}(B'x)\right)\right| \le 6|e^{-Bx} - e^{-B'x}||B-B'|e^{-\min(B,B')|x|},$$

then

$$\|\mathrm{sech}(B\cdot) - \mathrm{sech}(B'\cdot)\|_{H^1(\mathbb{R})} \le c|B-B'|\|e^{-\min(B,B')|\cdot|}\|_{H^1(\mathbb{R})}$$

Finally, concerning term (III), we observe that

$$\widehat{\phi(1,B',C)}(k) - \widehat{\phi(1,B',C)}(k) = \frac{e^{-ikC} - e^{-ikC'}}{B'}\widehat{\mathrm{sech}}\left(\frac{k}{B'}\right),$$

where $\widehat{f}$ denotes the Fourier transform of the function f. Then, a straightforward estimate gives

$$\|\phi(1,B',C) - \phi(1,B',C)\|_{H^1(\mathbb{R})}^2 \le \frac{|C-C'|^2}{|B'|^2}\int_{\mathbb{R}}(1+|k|^2)|k|^2\left|\widehat{\mathrm{sech}}\left(\frac{k}{B'}\right)\right|^2 dk.$$

The proof is complete. □

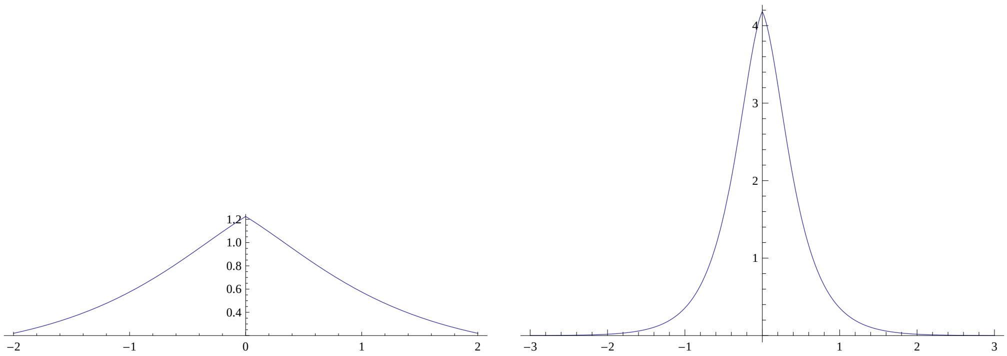

FIGURE 1. $\psi_{\delta\omega}$ for $\alpha = 1$, $\lambda = 1$ and $\omega = 1$ $\omega = 9$.

2. Dirac's delta defect

Here we investigate the stationary states of the system

$$i\partial_t u(t,x) \; = \; H_1 u(t,x) - \lambda |u(t,x)|^2 u(t,x)$$

As shown by several authors ([4, 8, 13, 20]), for every $\omega > \frac{\alpha^2}{4}$ there exists a unique stationary state of the above equation given by

$$\psi_{\alpha\delta,\omega} \; = \; \sqrt{\frac{2\omega}{\lambda}}\mathrm{sech}(\sqrt{\omega}(|x| - \bar{x}_{\alpha,\rho,\lambda})) \tag{2.1}$$

where

$$\bar{x}_{\alpha,\rho,\lambda} = \frac{1}{2\sqrt{\omega}} \log \frac{2\sqrt{\omega} - \alpha}{2\sqrt{\omega} + \alpha}.$$

Notice that $\bar{x}_{\alpha,\rho,\lambda} < 0$.
Stated differently, there is a single C^1 branch of standing waves

$$\omega \ni (\alpha^2/4, +\infty) \mapsto \psi_{\alpha\delta,\omega} \in H^1(\mathbb{R})$$

given explicitly by the above formulas.

The typical shape of $\psi_{\alpha\delta,\omega}$ is represented in Figure 1.

By direct computation, one has the following results for L^2-norm, energy and action:

Proposition 2.1. *The L^2-norm, the energy, and the action of the stationary states given in* (2.1) *read*

$$\|\psi_{\alpha\delta,\omega}\|_2^2 = \frac{4\sqrt{\omega}}{\lambda} - \frac{2\alpha}{\lambda} \; ; \qquad E_{\alpha\delta}(\psi_{\alpha\delta,\omega}) = -\frac{2}{3}\frac{\omega^{\frac{3}{2}}}{\lambda} + \frac{\alpha^3}{12\lambda}$$

$$S_\omega(\psi_{\alpha\delta,\omega}) = \frac{4}{3}\frac{\omega^{\frac{3}{2}}}{\lambda} - \frac{\alpha\omega}{\lambda} + \frac{\alpha^3}{12\lambda} \tag{2.2}$$

Remark 2.2. Notice that one has

$$\lim_{\omega \to \frac{\alpha^2}{4}+} \|\psi_{\alpha\delta,\omega}\|_2^2 \; = \; \lim_{\omega \to \frac{\alpha^2}{4}+} E_{\alpha\delta}(\psi_{\alpha\delta,\omega}) \; = \; \lim_{\omega \to \frac{\alpha^2}{4}+} S_\omega(\psi_{\alpha\delta,\omega}) = 0$$

This can be interpreted as an example of bifurcation from a simple eigenvalue. The nonlinear standing solution $\psi_{\alpha\delta,\omega}$ bifurcates from the vanishing solution in correspondence of the opposite of the linear eigenvalue $\mu = -\omega = -\frac{\alpha^2}{4}$. The opposite is due to the choice of the exponent sign in the definition of a standing wave and it is matter of convention.

Remark 2.3. It was proven in [4, 13] by different methods that such a branch of stationary states is indeed a family of orbitally stable ground states.

Remark 2.4. Since, from (2.2) the squared L^2-norm of the elements of the branch is a monotonically increasing function of the frequency ω, it is possible to parametrize the family of ground states in terms of the squared L^2-norm, that in the following will be denoted by ρ, just by replacing

$$\sqrt{\omega} \;=\; \frac{\lambda\rho}{4} + \frac{\alpha}{2},$$

so one obtains

$$\begin{aligned}
\varphi_{\alpha\delta,\rho}(x) \;&=\; \psi_{\alpha\delta,\omega(\rho)}(x) \;=\; \left(\frac{\sqrt{\lambda}\rho}{2\sqrt{2}} + \frac{\alpha}{\sqrt{2\lambda}}\right) \operatorname{sech}\left(\left(\frac{\lambda\rho}{4} + \frac{\alpha}{2}\right)(x - \bar{x}_{\alpha,\rho,\lambda})\right) \\
\bar{x}_{\alpha,\rho,\lambda}(\rho) \;&=\; \frac{2}{\lambda\rho + 2\alpha} \log \frac{\lambda\rho}{\lambda\rho + 4\alpha}
\end{aligned} \tag{2.3}$$

2.1. Linear limit

Here we study the linear limit *at constant mass* of the ground state (2.3).

Theorem 2.5. *The following limit holds at every fixed mass $\rho \in (0, +\infty)$*

$$\varphi_{\alpha\delta,\rho} \longrightarrow \sqrt{\frac{\rho\alpha}{2}} e^{-\frac{\alpha}{2}|\cdot|} \quad \text{in } H^1(\mathbb{R}), \quad \text{as } \lambda \to 0+ . \tag{2.4}$$

Proof. By triangular inequality,

$$\begin{aligned}
&\left\|\varphi_{\alpha\delta,\rho} - \sqrt{\frac{\rho\alpha}{2}} e^{-\frac{\alpha}{2}|\cdot|}\right\|_{H^1(\mathbb{R})} \\
\leq &\left\|\varphi_{\alpha\delta,\rho} - \phi\left(\frac{\alpha}{\sqrt{2\lambda}}, \frac{\alpha}{2}, \bar{x}_{\alpha,\rho,\lambda}\right)\right\|_{H^1(\mathbb{R})} + \left\|\phi\left(\frac{\alpha}{\sqrt{2\lambda}}, \frac{\alpha}{2}, \bar{x}_{\alpha,\rho,\lambda}\right) - \varphi_1\right\|_{H^1(\mathbb{R})} \\
&+ \left\|\varphi_1 - \sqrt{\frac{\rho\alpha}{2}} e^{-\frac{\alpha}{2}|\cdot|}\right\|_{H^1(\mathbb{R})} \\
= &(I) + (II) + (III)
\end{aligned}$$

where $\varphi_1(x) := \sqrt{\frac{2}{\lambda}} \alpha e^{-\frac{\alpha}{2}(|x| - \bar{x}_{\alpha,\rho,\lambda})}$. Noting that $\varphi_{\alpha\delta,\rho} = \phi\left(\frac{\sqrt{\lambda}\rho}{2\sqrt{2}} + \frac{\alpha}{\sqrt{2\lambda}}, \frac{\lambda\rho}{4} + \frac{\alpha}{2}, \bar{x}_{\alpha,\rho,\lambda}\right)$, term (I) can be estimated by Lemma 1.1 as

$$(I) \;\leq\; c\rho\sqrt{\lambda} \left\|\operatorname{sech}\left(\frac{\alpha}{2}\cdot\right)\right\|_{H^1(\mathbb{R})} + c\rho\lambda \left\|e^{-\frac{\alpha}{2}|\cdot|}\right\|_{H^1(\mathbb{R})}.$$

Concerning term (II), by straightforward computations one has

$$\begin{aligned}
\left\|\phi\left(\frac{\alpha}{\sqrt{2\lambda}}, \frac{\alpha}{2}, \bar{x}_{\alpha,\rho,\lambda}\right) - \varphi_1\right\|_2^2 \;&=\; \frac{4\alpha^2}{\lambda} \int_0^{+\infty} \frac{e^{-\alpha(x - \bar{x}_{\rho,\alpha,\lambda})}}{(e^{\alpha(x - \bar{x}_{\rho,\alpha,\lambda})} + 1)^2} dx \\
&\leq \frac{4\alpha^2}{\lambda} e^{3\alpha(x - \bar{x}_{\rho,\alpha,\lambda})} \int_0^{+\infty} e^{-3\alpha x}\, dx \;\leq\; c\rho^3\alpha^2\lambda^2,
\end{aligned}$$

where, in the last line, we used the definition of $\bar{x}_{\rho,\alpha,\lambda}$. Furthermore,

$$\begin{aligned}
&\left\|\frac{d}{dx}\left(\phi\left(\frac{\alpha}{\sqrt{2\lambda}}, \frac{\alpha}{2}, \bar{x}_{\rho,\alpha,\lambda}\right) - \varphi_1\right)\right\|_2^2 \\
= &\frac{4\alpha^4}{\lambda} \int_0^{+\infty} \left|\frac{e^{-\alpha(x - \bar{x}_{\rho,\alpha,\lambda})}}{e^{\alpha(x - \bar{x}_{\rho,\alpha,\lambda})} + 1} + \frac{e^{\alpha(x - \bar{x}_{\rho,\alpha,\lambda})}}{(e^{\alpha(x - \bar{x}_{\rho,\alpha,\lambda})} + 1)^2}\right|^2 dx \\
\leq &c\alpha^4 \int_0^{+\infty} e^{-\alpha(x - \bar{x}(\rho,\alpha,\lambda))}\, dx \;\leq\; c\alpha^4\rho^3\lambda^2.
\end{aligned}$$

Concerning (III), it is sufficient to notice that

$$\varphi_1(x) = \sqrt{\frac{2}{\lambda}}\alpha\left(\frac{\lambda\rho}{\lambda\rho+4\alpha}\right)^{\frac{\alpha}{\lambda\rho+2\alpha}} e^{-\frac{\alpha}{2}|x|},$$

and since

$$\sqrt{\frac{2}{\lambda}}\alpha\left(\frac{\lambda\rho}{\lambda\rho+4\alpha}\right)^{\frac{\alpha}{\lambda\rho+2\alpha}} \longrightarrow \sqrt{\frac{\rho\alpha}{2}} \quad \text{as } \lambda\to 0+$$

we have that term (III) vanishes too and the proof is complete. □

2.2. No-defect limit

Theorem 2.6. *The following limit holds at every fixed mass $\rho\in(0,+\infty)$:*

$$\varphi_{\alpha\delta,\rho}\longrightarrow \phi\left(\frac{\lambda\rho}{2\sqrt{2}},\frac{\lambda\rho}{4},0\right) \text{ in } H^1(\mathbb{R}), \qquad \alpha\to 0+\,.$$

Proof. The result immediately follows from Lemma 1.1, once observed that

$$\varphi_{\alpha\delta,\rho} = \phi\left(\frac{\sqrt{\lambda}\rho}{2\sqrt{2}}+\frac{\alpha}{\sqrt{2\lambda}},\frac{\lambda\rho}{4}+\frac{\alpha}{2},\bar{x}_{\alpha,\rho,\lambda}\right).$$

□

Remark 2.7. The linear limit coincides with the NLS soliton centred at zero.

3. Delta prime defect

Here we investigate the stationary states of the system

$$i\partial_t u(t,x) \;=\; H_2 u(t,x) - \lambda|u(t,x)|^2 u(t,x).$$

where we recall that H_2 is the Hamiltonian of a δ prime defect on the line described by (1.4). To this purpose, we reduce equation (1.6) with boundary conditions (1.8) to a couple of systems of two equations in two real unknowns. For the convenience of the reader, we preliminarily solve such systems.

3.1. Two algebraic systems

Proposition 3.1. *For any $\omega>0$, the only solution (t_1,t_2) to the system*

$$T_+ := \begin{cases} t_1^2 - t_1^4 &= t_2^2 - t_2^4 \\ t_1^{-1} - t_2^{-1} &= \gamma\sqrt{\omega} \end{cases} \tag{3.1}$$

satisfying the condition $0\le t_1,t_2\le 1$, is given by

$$\begin{aligned} t_1 &= \frac{1-\sqrt{1+\gamma^2\omega}+\sqrt{\gamma^2\omega-2+2\sqrt{1+\gamma^2\omega}}}{2\gamma\sqrt{\omega}} \\ t_2 &= \frac{-1+\sqrt{1+\gamma^2\omega}+\sqrt{\gamma^2\omega-2+2\sqrt{1+\gamma^2\omega}}}{2\gamma\sqrt{\omega}} \end{aligned} \tag{3.2}$$

Proof. Along the proof we consider the case $\gamma = 1$. The generic case can be recovered by the replacement $\omega \to \gamma^2\omega$.
The first equation in (3.1) rewrites as

$$(t_1^2 - t_2^2)(t_1^2 + t_2^2 - 1) \;=\; 0,$$

so the system splits into two subsystems $T_{1,+}$ and $T_{2,+}$, defined by

$$T_{1,+} := \begin{cases} t_1 &= t_2 \\ t_1^{-1} - t_2^{-1} &= \sqrt{\omega} \end{cases}$$

and

$$T_{2,+} := \begin{cases} t_1^2 + t_2^2 &= 1 \\ t_1^{-1} - t_2^{-1} &= \sqrt{\omega} \end{cases}$$

The system $T_{1,+}$ can be solved only for $\omega = 0$, so we are not interested in such a solution.
Let us consider the system $T_{2,+}$. Multiplying the second equation by $t_1 t_2$ one obtains

$$t_2 - t_1 \;=\; \sqrt{\omega} t_1 t_2, \tag{3.3}$$

thus squaring both sides, using the first equation of T_2 and imposing positivity, we obtain

$$t_1 t_2 \;=\; \frac{\sqrt{1+\omega} - 1}{\omega}$$

and so, by (3.3)

$$t_2 - t_1 \;=\; \frac{\sqrt{1+\omega} - 1}{\sqrt{\omega}}. \tag{3.4}$$

By (3.4) and the first equation of $T_{2,+}$, we finally get (3.2). □

Proposition 3.2. *The solutions* (t_1, t_2) *to the system*

$$T_- := \begin{cases} t_1^2 - t_1^4 &= t_2^2 - t_2^4 \\ t_1^{-1} + t_2^{-1} &= \gamma\sqrt{\omega} \end{cases} \tag{3.5}$$

with the condition $0 \le t_1, t_2 \le 1$, *can be classified as follows:*

1. *For* $0 < \omega \le \frac{4}{\gamma^2}$ *there are no solutions.*
2. *For* $\omega > \frac{4}{\gamma^2}$ *there exists the solution*

$$t_1 \;=\; t_2 \;=\; \frac{2}{\gamma\sqrt{\omega}}. \tag{3.6}$$

3. *For* $\omega > \frac{8}{\gamma^2}$ *there exist two further solutions* $(t_{1,+}, t_{2,+})$, $(t_{1,-}, t_{2,-})$ *given by*

$$\begin{aligned} t_{1,\pm} &= \frac{1 + \sqrt{1+\gamma^2\omega} \pm \sqrt{\gamma^2\omega - 2 - 2\sqrt{1+\gamma^2\omega}}}{2\gamma\sqrt{\omega}} \\ t_{2,\pm} &= \frac{1 + \sqrt{1+\gamma^2\omega} \mp \sqrt{\gamma^2\omega - 2 - 2\sqrt{1+\gamma^2\omega}}}{2\gamma\sqrt{\omega}} \end{aligned} \tag{3.7}$$

Proof. As in the previous proof, we put $\gamma = 1$ and to write the final result we perform the change $\omega \to \gamma^2\omega$. We rewrite the first equation in (3.7) as

$$(t_1^2 - t_2^2)(t_1^2 + t_2^2 - 1) \;=\; 0,$$

so the system (3.7) is equivalent to the union of the systems

$$T_{1,-} := \begin{cases} t_1 &= t_2 \\ t_1^{-1} + t_2^{-1} &= \sqrt{\omega} \end{cases}$$

and

$$T_{2,-} := \begin{cases} t_1^2 + t_2^2 &= 1 \\ t_1^{-1} + t_2^{-1} &= \sqrt{\omega} \end{cases}$$

The system $T_{1,-}$ gives the solution (3.6). The condition $\omega > 4$ emerges by the prescription $t_1, t_2 < 1$, so points (1) and (2) are proven.

Let us consider the system $T_{2,-}$. Multiplying the second equation by $t_1 t_2$ one obtains

$$t_1 + t_2 \;=\; \sqrt{\omega} t_1 t_2, \tag{3.8}$$

thus squaring both sides, using the first equation of $T_{2,-}$ and imposing positivity, we have

$$t_1 t_2 \;=\; \frac{1 + \sqrt{1+\omega}}{\omega}$$

and so, by (3.8)

$$t_1 + t_2 \;=\; \frac{1 + \sqrt{1+\omega}}{\sqrt{\omega}}. \tag{3.9}$$

By (3.9) and the first equation of $T_{2,-}$, we finally get (3.7). The positivity of the quantity under the square root of (3.7) gives the condition $\omega > 8$, so the proof is complete. □

3.2. Stationary states for the case with a δ' potential

In the following theorem we explicitly give all solutions to (1.6)-(1.8), modulo multiplication by a phase factor. In particular, we give solutions that are positive on the left real half line. When effective, we explicitly specify the role of the parity symmetry, in order to stress the appearance of double branches of stationary states.

Theorem 3.3. *Fixed $\omega > 0$, the stationary states of equation* (1.2) *can be classified as follows:*

1. *The unperturbed symmetric solitary wave*

$$\psi_\omega^{0,0}(x) \;:=\; \sqrt{\frac{2\omega}{\lambda}} \operatorname{sech}(\sqrt{\omega} x).$$

   Such solutions are present for any $\omega > 0$.

2. *two asymmetric non-changing sign solutions*

$$\psi_{\omega,+}^{x_1,x_2}(x) \;:=\; \begin{cases} \sqrt{\frac{2\omega}{\lambda}} \operatorname{sech}(\sqrt{\omega}(x - x_1)), & x < 0 \\ \sqrt{\frac{2\omega}{\lambda}} \operatorname{sech}(\sqrt{\omega}(x - x_2)), & x > 0, \end{cases}$$

where the first of the two couples of parameters (x_1, x_2) is given by

$$x_1 \;=\; \frac{1}{2\sqrt{\omega}} \log \frac{1+t_1}{1-t_1}, \quad x_2 \;=\; \frac{1}{2\sqrt{\omega}} \log \frac{1+t_2}{1-t_2} \tag{3.10}$$

and the second is obtained by parity symmetry, i.e. performing the change

$$x_1 \longrightarrow -x_2, \quad x_2 \longrightarrow -x_1.$$

The couple (t_1, t_2) appearing in the last lines solves the system (3.1).
Such solutions are present for any $\omega > 0$.

3. *The changing sign solutions, which, in turn, can be classified as*
   (a) *the antisymmetric solution*

$$\psi_{\omega,-}^{\bar{x},-\bar{x}} \;:=\; \begin{cases} \sqrt{\frac{2\omega}{\lambda}}\,\mathrm{sech}(\sqrt{\omega}(x-\bar{x})), & x<0 \\ -\sqrt{\frac{2\omega}{\lambda}}\,\mathrm{sech}(\sqrt{\omega}(x+\bar{x})), & x>0, \end{cases} \tag{3.11}$$

where

$$\bar{x} = \frac{1}{2\sqrt{\omega}} \log \frac{\gamma\sqrt{\omega}+2}{\gamma\sqrt{\omega}-2}. \tag{3.12}$$

Such solutions are present for $\omega > 4/\gamma^2$.

   (b) *two non-symmetric solutions*

$$\psi_{\omega,-}^{x_1,x_2} \;:=\; \begin{cases} \sqrt{\frac{2\omega}{\lambda}}\,\mathrm{sech}(\sqrt{\omega}(x-x_1)), & x<0 \\ -\sqrt{\frac{2\omega}{\lambda}}\,\mathrm{sech}(\sqrt{\omega}(x-x_2)), & x>0, \end{cases} \tag{3.13}$$

where the first of the two couples (x_1, x_2) is given by

$$x_1 \;=\; \frac{1}{2\sqrt{\omega}} \log \frac{1+t_{1,+}}{1-t_{1,+}}, \quad x_2 \;=\; -\frac{1}{2\sqrt{\omega}} \log \frac{1+t_{2,+}}{1-t_{2,+}}, \tag{3.14}$$

while the second is obtained by performing the exchange

$$x_1 \longrightarrow -x_2, \quad x_2 \longrightarrow -x_1.$$

The couple $(t_{1,+}, t_{2,+})$ was defined in (3.7).
Such solutions are present for any $\omega > 8/\gamma^2$.

See Figure 2 and Figure 3 for a representation of the typical behaviours.

Proof. The proof consists in finding the values of the parameters x_1 and x_2 in (1.9) such that the matching conditions (1.8) are realized. It turns out that, assumed the positive sign for the stationary states at $0-$, without loss of generality owing to the phase symmetry of the system, there are two families of solutions: the *changing sign* and the *non changing sign* solutions.
– The fact that the function $\psi_\omega^{0,0}$ is a solution to the problem (1.6), (1.7) can be easily verified by direct inspection.
– In order to find the family of non changing sign solutions $\psi_{\omega,+}^{x_1,x_2}$, one defines $t_i := \tanh(\sqrt{\omega}x_i)$ and obtains that the the couple (t_1, t_2) must solve the system (3.1), studied in Proposition 3.1. So one finally finds the values for x_1 and x_2 given in (3.10).
– In order to find the values of the parameters x_1, x_2 for the family of changing sign solutions $\psi_{\omega,-}^{x_1,x_2}$, one notices that, imposing the matching conditions (1.8), the quantities $t_i = |\tanh(\sqrt{\omega}x_i)|$ must solve the system (3.5), studied in Proposition 3.2. Then, one finds that the possible couples (x_1, x_2) are given by (3.11), (3.12), (3.13), (3.14). □

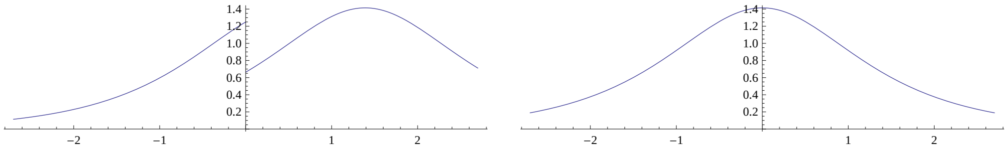

FIGURE 2. $\psi_{\omega,+}^{x_1,x_2}$ for $\gamma = 1$, $\lambda = 1$, $\omega = 9$ and pure soliton with $\omega = 1$.

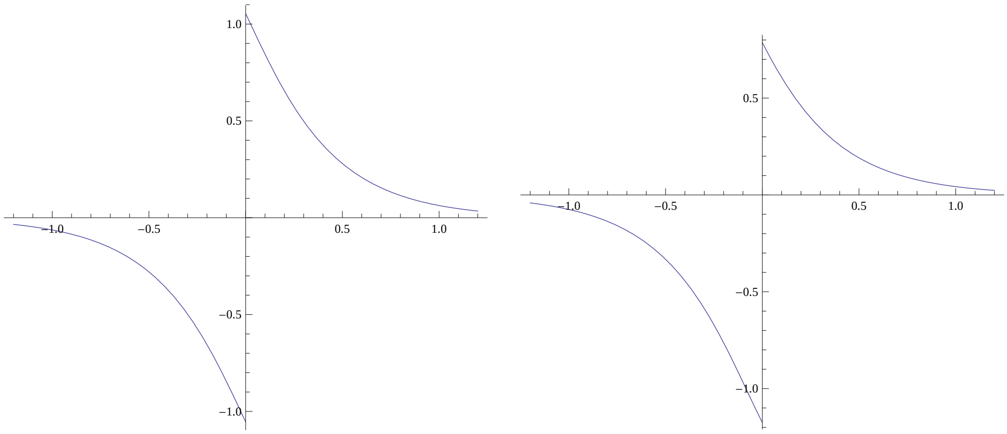

FIGURE 3. $\psi_{\omega,-}^{\bar{x},-\bar{x}}$ and $\psi_{\omega,-}^{x_1,x_2}$ for $\gamma = 1$, $\lambda = 1$, $\omega = 9$.

By direct computations, one can prove the following

Proposition 3.4. *The L^2-norm, the energy, and the action of the stationary states found in Theorem 3.3, are explicitly given by*

1. *For the unperturbed symmetric solitary wave*

$$\|\psi_{\omega,+}^{0,0}\|_2^2 = \frac{4}{\lambda}\sqrt{\omega}, \qquad E_{\gamma\delta'}(\psi_{\omega,+}^{0,0}) = -\frac{2}{3\lambda}\omega^{\frac{3}{2}}, \qquad S_\omega(\psi_{\omega,+}^{0,0}) = \frac{4}{3\lambda}\omega^{\frac{3}{2}}. \tag{3.15}$$

2. *For the asymmetric non-changing sign stationary states*

$$\begin{aligned} \|\psi_{\omega,+}^{x_1,x_2}\|_2^2 &= \frac{4}{\lambda}\sqrt{\omega} + \frac{2}{\lambda\gamma}\sqrt{1+\gamma^2\omega} - \frac{2}{\lambda\gamma} \\ E_{\gamma\delta'}(\psi_{\omega,+}^{x_1,x_2}) &= -\frac{2}{3\lambda}\omega^{\frac{3}{2}} + \frac{2-\gamma^2\omega}{3\lambda\gamma^3}\sqrt{\gamma^2\omega+1} - \frac{2}{3\lambda\gamma^3} \\ S_\omega(\psi_{\omega,+}^{x_1,x_2}) &= \frac{4}{3\lambda}\omega^{\frac{3}{2}} + \frac{2}{3\lambda\gamma^3}(\gamma^2\omega+1)^{\frac{3}{2}} - \frac{\omega}{\lambda\gamma} - \frac{2}{3\lambda\gamma^3} \end{aligned} \tag{3.16}$$

3. *For the antisymmetric solutions*

$$\|\psi_{\omega,-}^{\bar{x},-\bar{x}}\|_2^2 = \frac{4}{\lambda}\sqrt{\omega} - \frac{8}{\gamma\lambda}, \quad E_{\gamma\delta'}(\psi_{\omega,-}^{\bar{x},-\bar{x}}) = \frac{16}{3\gamma^3\lambda} - \frac{2}{3\lambda}\omega^{\frac{3}{2}}, \quad S_\omega(\psi_{\omega,-}^{\bar{x},-\bar{x}}) = \frac{16}{3\gamma^3\lambda} - 4\frac{\omega}{\gamma\lambda} + \frac{4}{3\lambda}\omega^{\frac{3}{2}} \tag{3.17}$$

4. For the asymmetric, changing sign solutions

$$\begin{aligned} \|\psi_{\omega,-}^{x_1,x_2}\|_2^2 &= \frac{4}{\lambda}\sqrt{\omega} - \frac{2}{\lambda\gamma}\sqrt{1+\gamma^2\omega} - \frac{2}{\lambda\gamma} \\ E_{\gamma\delta'}(\psi_{\omega,-}^{x_1,x_2}) &= -\frac{2}{3\lambda}\omega^{\frac{3}{2}} + \frac{\gamma^2\omega-2}{3\lambda\gamma^3}(\sqrt{\gamma^2\omega+1}) - \frac{2}{3\lambda\gamma^3} \\ S_\omega(\psi_{\omega,+}^{x_1,x_2}) &= \frac{4}{3\lambda}\omega^{\frac{3}{2}} + \frac{2}{3\lambda\gamma^3}(\gamma^2\omega+1)^{\frac{3}{2}} - \frac{\omega}{\lambda\gamma} - \frac{2}{3\lambda\gamma^3} \end{aligned} \tag{3.18}$$

Remark 3.5. In any family of stationary states $\psi_{\omega,+}^{0,0}$, $\psi_{\omega,+}^{x_1,x_2}$, $\psi_{\omega,-}^{\bar{x},-\bar{x}}$, and $\psi_{\omega,-}^{x_1,x_2}$, introduced in Theorem 3.3, the L^2-norm is a monotonically increasing function of the frequency ω that diverges at infinity. In other words, it is a bijection of the interval $(0,+\infty)$ into itself.

Remark 3.6. Mass and energy (and consequently action) of $\psi_{\omega,+}^{0,0}$ and $\psi_{\omega,+}^{x_1,x_2}$ vanish for $\omega \to 0+$; therefore, the stationary solutions $\psi_{\omega,+}^{0,0}$ and $\psi_{\omega,+}^{x_1,x_2}$ bifurcate from the vanishing solution in correspondence to the edge of the essential spectrum of the linear operator H_2. Analogously, mass and energy of $\psi_{\omega,-}^{\bar{x},-\bar{x}}$ vanish for $\omega \to \frac{4}{\gamma^2}+$; they bifurcate from the vanishing solution in correspondence with the opposite of the simple eigenvalue $\mu = -\frac{4}{\gamma^2}$ of the linear part H_2.

3.3. Minimizing the action on the Nehari manifold

Here we consider the variational problem given by minimizing the action S_ω, defined in (1.10), at fixed $\omega > 0$ on the natural constraint called the *Nehari manifold*

$$E_{\gamma\delta'}(\psi) - \frac{\lambda}{4}\|\psi\|_4^4 = 0.$$

Proposition 3.7. *1. For any $\omega > 0$,*

$$S_\omega(\psi_{\omega,+}^{0,0}) < S_\omega(\psi_{\omega,+}^{x_1,x_2}). \tag{3.19}$$

2. For any $\omega > \frac{4}{\gamma^2}$,

$$S_\omega(\psi_{\omega,-}^{\bar{x},-\bar{x}}) < S_\omega(\psi_{\omega,+}^{0,0}) < S_\omega(\psi_{\omega,+}^{x_1,x_2}). \tag{3.20}$$

3. For any $\omega > \frac{8}{\gamma^2}$,

$$S_\omega(\psi_{\omega,-}^{x_1,x_2}) < S_\omega(\psi_{\omega,-}^{\bar{x},-\bar{x}}) < S_\omega(\psi_{\omega,+}^{0,0}) < S_\omega(\psi_{\omega,+}^{x_1,x_2}). \tag{3.21}$$

Proof. Inequalities (3.19) and (3.20) follow immediately from (3.15), (3.16), (3.17).
Finally, the first inequality in (3.21) is proven by (3.17) and (3.18). In particular, one gets that the inequality is equivalent to

$$3\gamma^2\omega < 6 + \frac{2}{3}(\gamma^2\omega+1)^{\frac{3}{2}}$$

that is immediately verified for any $\omega > \frac{8}{\gamma^2}$. □

Remark 3.8. 1. Defined the function $\Delta_+(\omega) := S_\omega(\psi_{\omega,+}^{x_1,x_2}) - S_\omega(\psi_{\omega,+}^{0,0})$, one obtains

$$\Delta_+(0) = \frac{d}{d\omega}\Delta_+(0) = 0,$$

while $\frac{d^2}{d\omega^2}\Delta_+(\omega) = \frac{\gamma^2}{2}(\gamma^2\omega+1)^{-1/2} > 0.$
This phenomenon reflects the fact that the stationary state $\psi_{\omega,+}^{x_1,x_2}$ bifurcates from $\psi_{\omega,+}^{0,0}$ at $\omega > 0$.
2. Analogously, defined the function $\Delta_-(\omega) := S_\omega(\psi_{\omega,-}^{\bar{x},-\bar{x}}) - S_\omega(\psi_{\omega,-}^{x_1,x_2})$ one obtains

$$\Delta_-(0) = \frac{d}{d\omega}\Delta_-(0) = 0,$$

while $\frac{d^2}{d\omega^2}\Delta_-(\omega) = \frac{\gamma^2}{2}(\gamma^2\omega+1)^{-1/2} > 0.$
Again, this is related to the fact that the stationary state $\psi_{\omega,-}^{x_1,x_2}$ bifurcates from $\psi_{\omega,-}^{\bar{x},-\bar{x}}$ at $\omega > \frac{8}{\gamma^2}$.

3.4. Minimizing the energy in the manifold of constant mass

Here we consider the variational problem of minimizing the energy $E_{\gamma\delta'}$, defined in (1.5) on the manifold consisting of the function with given L^2-norm $\sqrt{\rho} > 0$ and finite energy. As a consequence of Remark 3.5, one can parametrize the stationary states by the value ρ of their squared L^2-norm $\rho \in (0, +\infty)$, instead of by ω. In the following, given $\rho > 0$, we make use of the notation

$$\varphi^{0,0}_{\rho,+} := \{\psi^{0,0}_{\omega,+},\ \omega \text{ s.t. } \|\psi^{0,0}_{\omega,+}\|_2^2 = \rho\}$$

and analogously for the other families $\varphi^{x_1,x_2}_{\rho,+}$, $\varphi^{\bar{x},-\bar{x}}_{\rho,-}$, and $\varphi^{x_1,x_2}_{\rho,-}$.

Proposition 3.9. *1. For any $\rho > 0$,*

$$E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{\rho,-}) < E_{\gamma\delta'}(\varphi^{0,0}_{\rho,+}) < E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,+}). \tag{3.22}$$

2. For any $\rho > \frac{8}{\gamma\lambda}\left(\sqrt{2}-1\right)$,

$$E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,-}) < E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{\rho,-}) < E_{\gamma\delta'}(\varphi^{0,0}_{\rho,+}) < E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,+}). \tag{3.23}$$

Proof. In order to make the formulas less cumbersome, we show the computations for the particular case $\lambda = 1$, $\gamma = 1$ only. The general formulas can be then recovered by the scaling laws

$$\omega \to \gamma^2\omega, \quad \rho \to \gamma\lambda\rho, \quad E_{\gamma\delta'} \to \gamma^3\lambda E_{\gamma\delta'}, \tag{3.24}$$

that can be directly verified.

Then, we make explicit the dependence of ω, and then of the energy, on ρ. An elementary but lengthy computation shows that

– For the family $\varphi^{0,0}_{\rho,+}$

$$\omega = \frac{\rho^2}{16}, \qquad E_{\gamma\delta'}(\varphi^{0,0}_{\rho,+}) = -\frac{\rho^3}{96}. \tag{3.25}$$

– For the family $\varphi^{x_1,x_2}_{\rho,+}$

$$\begin{aligned}\sqrt{\omega} &= \frac{2\rho + 4 - \sqrt{\rho^2 + 4\rho + 16}}{6}\\ E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,+}) &= -\frac{5}{216}\rho^3 + \frac{1}{54}(\rho^2 + 4\rho + 16)^{\frac{3}{2}} - \frac{5}{36}\rho^2 - \frac{4}{9}\rho - \frac{32}{27}.\end{aligned} \tag{3.26}$$

– For the family $\varphi^{\bar{x},-\bar{x}}_{\rho,-}$

$$\sqrt{\omega} = \frac{\rho}{4} + 2, \qquad E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{\rho,-}) = -\frac{\rho^3}{96} - \frac{\rho^2}{4} - 2\rho. \tag{3.27}$$

– For the family $\varphi^{x_1,x_2}_{\rho,-}$

$$\begin{aligned}\sqrt{\omega} &= \frac{2\rho + 4 + \sqrt{\rho^2 + 4\rho + 16}}{6}\\ E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,-}) &= -\frac{5}{216}\rho^3 - \frac{1}{54}(\rho^2 + 4\rho + 16)^{\frac{3}{2}} - \frac{5}{36}\rho^2 - \frac{4}{9}\rho - \frac{32}{27}.\end{aligned} \tag{3.28}$$

So, the first inequality in (3.22) is immediately proven. For the second inequality, one first extends by continuity the functions $E_{\gamma\delta'}(\varphi^{0,0}_{\rho,+})$ and $E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,+})$ up to the value $\rho = 0$ and obtains

$$\frac{d^j}{d\rho^j}E_{\gamma\delta'}(\varphi^{0,0}_{0+,+}) = \frac{d^j}{d\rho^j}E_{\gamma\delta'}(\varphi^{x_1,x_2}_{0+,+}) = 0, \qquad j = 0, 1, 2.$$

Furthermore,

$$\frac{d^3}{d\rho^3}E_{\gamma\delta'}(\varphi^{0,0}_{0+,+}) \;=\; \frac{d^3}{d\rho^3}E_{\gamma\delta'}(\varphi^{x_1,x_2}_{0+,+}) \;=\; -\frac{1}{16}.$$

The second inequality in (3.22) is then proven by

$$0 \;=\; \frac{d^4}{d\rho^4}E_{\gamma\delta'}(\varphi^{0,0}_{\rho,+}) \;<\; \frac{d^4}{d\rho^4}E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,+}) \;=\; 12(\rho^2+4\rho+16)^{-\frac{5}{2}}.$$

It remains to prove the first inequality in (3.23). First, observe that

$$\begin{aligned} E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{8(\sqrt{2}-1),-}) &= E_{\gamma\delta'}(\varphi^{x_1,x_2}_{8(\sqrt{2}-1),-}) \\ &= -\frac{16}{3}(2\sqrt{2}-1) \end{aligned} \tag{3.29}$$

and

$$\frac{d}{d\rho}E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{8(\sqrt{2}-1),-}) \;=\; \frac{d}{d\rho}E_{\gamma\delta'}(\varphi^{x_1,x_2}_{8(\sqrt{2}-1),-}) \;=\; -4.$$

Now, we consider the second derivative. The statement $\frac{d^2}{d\rho^2}E_{\gamma\delta'}(\varphi^{\bar{x},-\bar{x}}_{\rho,-}) \;>\; \frac{d^2}{d\rho^2}E_{\gamma\delta'}(\varphi^{x_1,x_2}_{\rho,-})$ proves equivalent to $(\rho^2+4\rho^{\frac{1}{2}}+16)^{\frac{1}{2}}(\rho^2+4\rho^{\frac{1}{2}}+10) \;>\; -\frac{11}{16}\rho+\frac{2}{9}$, which is satisfied for any $\rho \geq 8(\sqrt{2}-1)$. The proof is complete.

□

3.5. Linear limit for the stationary states

Theorem 3.10. *For $\lambda \to 0+$, the following limits hold:*

$$\varphi^{0,0}_{\rho,+} \longrightarrow 0, \quad \textit{in the weak topology of} Q_{\delta'} \tag{3.30}$$

$$\varphi^{x_1,x_2}_{\rho,+} \longrightarrow 0, \quad \textit{in the weak topology of} Q_{\delta'} \tag{3.31}$$

$$\varphi^{\bar{x},-\bar{x}}_{\rho,-} \longrightarrow \mathrm{sgn}(x)\sqrt{\frac{2\rho}{\gamma}}e^{-\frac{2}{\gamma}|\cdot|}, \quad \textit{in the strong topology of} Q_{\delta'} \tag{3.32}$$

Proof. Limit (3.30) is immediately proven once observed that, after rescaling (3.24), from (3.25) one has $\sqrt{\omega}=\frac{\lambda\rho}{4}$, so that $\sqrt{\omega}$ vanishes in the limit $\lambda\to 0+$.
To prove (3.31) notice that the first formula in (3.26), after rescaling (3.24), gives

$$\sqrt{\omega} \;=\; \frac{2\gamma\lambda\rho+4-\sqrt{\gamma^2\lambda^2\rho^2+4\gamma\lambda\rho+16}}{6\gamma},$$

so that $\sqrt{\omega}\sim\frac{\lambda\rho}{4}$, as in the previous case.

Limit (3.32) is mapped into limit (2.4) by the replacement $\gamma\to\frac{4}{\alpha}$, so the reader is referred to the proof of Theorem 2.5.

□

Remark 3.11. Notice that the linear limit at fixed mass for the stationary state $\varphi^{x_1,x_2}_{\rho,-}$ is nonsensical: as λ vanishes, the mass ρ of the state is overcome by the threshold $\frac{8}{\lambda\gamma}(\sqrt{2}-1)$ under which there is no such a state. In other words, in order for $\varphi^{x_1,x_2}_{\rho,-}$ to exist, the nonlinearity must be not only present, but also sufficiently strong.

Remark 3.12. A closer look at the limit in (3.31) shows that, in such a case, both x_1 and x_2 positively diverge as $\lambda\to 0+$ (see Theorem 3.3, formula (3.2)). As a consequence, in the linear limit the corresponding stationary state $\varphi^{x_1,x_2}_{\rho,+}$ (or, equivalently, $\psi^{x_1,x_2}_{\omega,+}$) takes the shape of a soliton that runs towards infinity. An analogous statement holds for the stationary state $\varphi^{-x_2,-x_1}_{\rho,+}$. Conversely, this means that, starting from the linear dynamics generated by H_2 and turning on the nonlinearity, two stationary states arise very far away from each other and approach the origin as λ grows.

Remark 3.13. Limit (3.32) gives, as a result, the bound state of the linear Schrödinger equation with a delta prime potential. This means that the stationary state $\varphi_{\rho,-}^{\bar{x},-\bar{x}}$ bifurcates from such a linear bound state, as the nonlinearity is introduced.

3.6. No-defect limits

Theorem 3.14. *For $\gamma \to 0+$, the following limits hold:*

$$\varphi_{\rho,+}^{x_1,x_2} \longrightarrow \phi\left(\sqrt{\frac{\lambda}{2}}\frac{\rho}{2}, \frac{\lambda\rho}{4}, \frac{1}{\sqrt{\lambda\rho}}\log(3+2\sqrt{2})\right), \text{ in the strong topology of } Q_{\delta'} \tag{3.33}$$

$$\varphi_{\rho,-}^{\bar{x},-\bar{x}} \longrightarrow 0, \text{ in the sense of distributions} \tag{3.34}$$

Proof. Limit (3.33) is obtained by observing that, by the first identity in (3.26) with the rescaling (3.24), $\omega \to \frac{\lambda\rho}{4}$ as $\gamma \to 0+$. As a consequence, by (3.2) one has that both t_1 and t_2 tend to $2^{-\frac{1}{2}}$, so that, by (3.10), $x_1, x_2 \to \frac{1}{\sqrt{\lambda\rho}}\log(3+2\sqrt{2})$. (3.33) then follows from Lemma 1.1.

To prove (3.34) we just notice that, from the first identity in (3.27) and the rescaling (3.24), one has $\gamma\omega \to 2$. Then (3.12) shows that $\bar{x} \to \infty$. □

Remark 3.15. It is immediate to observe that the stationary state $\varphi_{\rho,+}^{0,0}$ is untouched as γ varies, while $\varphi_{\rho,-}^{x_1,x_2}$ ceases to exist as $\gamma \leq \frac{8}{\lambda\rho}(\sqrt{2}-1)$.

Theorem 3.16. *For $\gamma \to +\infty$, the following limits hold:*

$$\varphi_{\rho,+}^{x_1,x_2} \longrightarrow \chi_{(-\infty,0]}\,\phi\left(\sqrt{\frac{\lambda}{2}}\frac{\rho}{3}, \frac{\lambda\rho}{6}, 0\right), \text{ in the weak topology of } Q_{\delta'} \tag{3.35}$$

$$\varphi_{\rho,-}^{\bar{x},-\bar{x}} \longrightarrow \operatorname{sign}(x)\phi\left(\sqrt{\frac{\lambda}{2}}\frac{\rho}{2}, \frac{\lambda\rho}{4}, 0\right), \text{ in the strong topology of } Q_{\delta'} \tag{3.36}$$

$$\varphi_{\rho,-}^{x_1,x_2} \longrightarrow \chi_{[0,+\infty)}\phi\left(\sqrt{\frac{\lambda}{2}}\rho, \frac{\lambda\rho}{2}, 0\right), \text{ in the strong topology of } Q_{\delta'} \tag{3.37}$$

Proof. To prove the limit (3.35), notice first that, from the first identity in (3.26) and the scaling law (3.24), one has $\sqrt{\omega} \to \frac{\lambda\rho}{6}$. Then, (3.2) shows that $t_1 \to 0$ as $t_2 \to 1$, so it must be $x_1 \to 0$ and $x_2 \to \infty$. Lemma 1.1 shows that in the left half line the convergence is strong in the space $H^1(\mathbb{R}^-)$, while in the positive half line there is a soliton that runs away and weakly converges to zero.

Limit (3.36) is an immediate consequence of (3.27) and (3.12) through Lemma (1.1).

Finally, limit (3.37) is proven through the first of (3.28) that shows $\sqrt{\omega} \to \frac{\rho}{2}$, (3.7) that gives $t_{1,+} \to 1$ and $t_{2,+} \to 0$, and Lemma (1.1). □

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