Math. Model. Nat. Phenom.
Vol. 9, No. 5, 2014, pp. 39–43
DOI: 10.1051/mmnp/20149503

Spectral Theory of the Hermite Operator on $L^p(\mathbb{R}^n)$

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Abstract. We prove that the minimal operator and the maximal operator of the Hermite operator are the same on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$. The domain and the spectrum of the minimal operator (=maximal operator) of the Hermite operator on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$, are computed. In addition, we can give an estimate for the L^p-norm of the solution to the initial value problem for the heat equation governed by the minimal (maximal) operator for 4/3¡p¡4.

Keywords and phrases: Hermite operator, Hermite functions, minimal (maximal) operator, spectrum, the heat equation

Mathematics Subject Classification: 35K05, 47A10

1. Introduction

The Hermite operator plays an important role in both mathematics and physics. It is also known as the simple harmonic oscillator [5],[6], [7], and is given by

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^n.$$

where $\Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}$. It maps the Schwartz space $\mathcal{S}$ into $\mathcal{S}$. The goal of this paper is to prove that the minimal and the maximal operator of the Hermite operator agree with each other on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$. Throughout the paper, we denote the minimal operator on $L^p(\mathbb{R}^n)$ by $H_{0,p}$, and the maximal operator by $H_{1,p}$. The extension of H from $\mathcal{S}$ to $L^2(\mathbb{R}^n)$ has been well understood, and in fact, $H_{0,2} = H_{1,2}$. This means that H is essentially self-adjoint. The spectrum $\Sigma(H_{0,2})$ of $H_{0,2}$ is given by

$$\Sigma(H_{0,2}) = \{2|\alpha| + 1 : \alpha \in \mathbb{N}_0^n\},$$

where $\mathbb{N}_0^n$ is the set of all multi-indices. However, we are interested in knowing the spectrum of $H_{0,p}$ and $H_{1,p}$ for general p. In this paper we show that the spectrum of the Hermite operator is equal to that of its minimal (maximal) operator for $4/3 < p < 4$. Moreover, the two operators are the same for $4/3 < p < 4$, or in other words, the Hermite operator is essentially self-adjoint for p between 4/3 and 4.
We begin in Section 2 by proving that the Hermite operator H is closable on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. And it follows that $H_{0,p}$ is the minimal operator, (i.e., the smallest closed extension) of H on $L^p(\mathbb{R}^n)$. In order to

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prove our desired result, we first raise the Hermite operator to its Nth power, for some positive integer N, which enables us to compute explicitly the spectrum for $4/3 < p < 4$. Secondly, we compute explicitly the spectrum of the minimal and maximal operator of H on $L^p(\mathbb{R}^n)$, and finally via the functional calculus, we prove that the two operators are the same on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$. In addition, we give an estimate on the L^p norm of the solution to the initial value problem for the heat equation governed by the minimal (maximal) operator for $4/3 < p < 4$ in the last section.

Related results on the heat equation for the Hermite operator can be found in [2–4, 10, 11].

2. The Minimal Operator and the Maximal Operator

We first give the definition of a closable operator in general, which can be found in the book [12]. Let A be a linear operator from a Banach space X into a Banach space Y with dense domain $D(A)$.

Definition 2.1. The operator $A : X \to Y$ is said to be closable if for any sequence $\{x_k\}$ in $\mathcal{D}(A)$ such that $x_k \to 0$ in X and $Ax_k \to y$ in Y as $k \to \infty$, then we have $y = 0$.

To see that the Hermite operator is closable, we let $\{\phi_k\}$ be a sequence in $\mathcal{S}$ such that $\phi_k \to 0$ in $L^p(\mathbb{R}^n)$ and $H\phi_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Then for all $\psi \in \mathcal{S}$, we have

$$(f, \psi) = \lim_{k\to\infty} (H\phi_k, \psi) = \lim_{k\to\infty} (\phi_k, H\psi) = 0,$$

where

$$(g, h) = \int_{\mathbb{R}^n} g(x)\overline{h(x)}dx, \quad g \in L^p(\mathbb{R}^n),\, h \in L^{p'}(\mathbb{R}^n).$$

This implies that $f = 0$. Therefore H is closable in $L^p(\mathbb{R}^n)$.
Now we introduce a proposition in book [12].

Proposition 2.2. *A has a closed extension if and only if A is closable.*

In view of the above proposition, having proved the fact that the Hermite operator is closable, we can define the minimal operator $H_{0,p}$ of H on $L^p(\mathbb{R}^n)$ to be the smallest closed extension, or the closure of H, on $L^p(\mathbb{R}^n)$. We end this section with the definition of the maximal operator of the Hermite operator on $L^p(\mathbb{R}^n)$.

Definition 2.3. Let u and f be functions in $L^p(\mathbb{R}^n)$. We say that $u \in \mathcal{D}(H_{1,p})$ and $H_{1,p}u = f$ if and only if $(u,\, H\phi) = (f,\, \phi)$, $\phi \in \mathcal{S}$.

We note that $\mathcal{D}(H_{1,p}) = \{u :\, H_{1,p}u \in L^p(\mathbb{R}^n)\}$ and

$$H_{1,p}u = Hu,\, u \in \mathcal{D}(H_{1,p}).$$

3. The Spectrum of $H^N_{0,p}$ and $H^N_{1,p}$, $4/3 < p < 4$

In this section, we give the spectrum of the operator $H^N_{1,p}$ for some $N \in \mathbb{N}$.

Proposition 3.1. *For $4/3 < p < 4$ and N large enough, the spectrum $\Sigma(H_1^{1,p})$ of the operator $H^N_{1,p}$ is given by*

$$\Sigma(H^N_{1,p}) = \{(2|\alpha| + 1)^N :\, \alpha \in \mathbb{N}_0^n\}.$$

Proof. For each $N \in \mathbb{N}$, let $S_N = \{(2|\alpha|+1)^N : \alpha \in \mathbb{N}_0^n\}$. We now show that the resolvent set of the operator $H_{1,p}^N$ is $\mathbb{C} - S_N$. In other words, the spectrum $\Sigma(H_{1,p}^N)$ is S_N. For all complex numbers $\lambda \notin S_N$, we claim that the operator $H_{1,p}^N - \lambda$ is bijective. Indeed, for injectivity, let I be the identity operator on $L^p(\mathbb{R}^n)$ and suppose that $(H_{1,p}^N - \lambda I)u = 0$ for some $u \in L^p(\mathbb{R}^n)$. Then

$$((H_{1,p}^N - \lambda I)u)(\phi) = 0, \quad \phi \in \mathcal{S}.$$

Treating u as a distribution, we have

$$((H_{1,p}^N - \lambda I)u)(\phi) = u((H^N - \lambda I)\phi) = 0, \quad \phi \in \mathcal{S}.$$

On the other hand, let $\psi \in \mathcal{S}$. Then we show that there exists $\phi \in \mathcal{S}$ such that $(H^N - \lambda I)\phi = \psi$. Indeed, we define ϕ by

$$\phi = \sum_\alpha \frac{1}{(2|\alpha|+1)^N - \lambda}(\psi, e_\alpha)e_\alpha,$$

where e_α, $\alpha \in \mathbb{N}_0^n$, is the eigenfunction corresponding to the eighenvalue $2|\alpha|+1$. Then it is clear that $\phi \in \mathcal{S}$, and since

$$(H^N - \lambda I)\phi = \psi,$$

we have

$$u(\psi) = 0, \quad \psi \in \mathcal{S},$$

and $u = 0$, as desired. To see that $H_{1,p}^N - \lambda I$ is surjective, let $f \in L^p(\mathbb{R}^n)$. Then we need to prove that there exists $u \in L^p(\mathbb{R}^n)$ such that $(H_{1,p}^N - \lambda I)u = f$. By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned} ||u||_p = ||(H_{1,p}^N - \lambda I)^{-1} f||_p &\le \sum_\alpha \frac{1}{|(2|\alpha|+1)^N - \lambda|} |(f, e_\alpha)|\, ||e_\alpha||_p \\ &\le \sum_\alpha \frac{1}{|(2|\alpha|+1)^N - \lambda|} ||f||_p ||e_\alpha||_{p'} ||e_\alpha||_p, \end{aligned} \tag{3.1}$$

where p' is the conjugate index of p. And for N large enough (depending on p), by a result from the paper [1], we have

$$||e_\alpha||_p ||e_\alpha||_{p'} = O(1), \quad 4/3 < p < 4, \tag{3.2}$$

or equivalently, this product is bounded for all α, which makes the right hand side of (3.1) finite, given the range for p.

□

Remark 3.2. The range of p for which (3.2) holds is sharp according to the estimate in the paper [1].

4. $H_{0,p}^N = H_{1,p}^N$, $4/3 < p < 4$

Before proving the main result of this section, we need two lemmas.

Lemma 4.1. *For each $N \in \mathbb{N}$, $H_{0,p}^N \subseteq H_{1,p}^N$.*

Proof. We prove the lemma by induction. For $N = 1$, it is clearly true. Suppose it holds for N. Let $u \in \mathcal{D}(H_{0,p}^{N+1})$. Then $H_{0,p}u \in \mathcal{D}(H_{0,p}^N)$. Since $H_{0,p}u = H_{1,p}u$ and $\mathcal{D}(H_{0,p}^N) \subseteq \mathcal{D}(H_{1,p}^N)$, we have $H_{1,p}u \in \mathcal{D}(H_{1,p}^N)$, and it follows that $u \in \mathcal{D}(H_{1,p}^{N+1})$. Furthermore, we have

$$H_{1,p}^{N+1}u = H_{1,p}^N H_{1,p}u = H_{0,p}^N H_{0,p}u = H_{0,p}^{N+1}u.$$

Thus, $H_{0,p}^N \subseteq H_{1,p}^N$. □

Lemma 4.2. $(H_{0,p}^N)^{-1} = (H_{1,p}^N)^{-1}$.

Proof. By definition, $H^N = H_{1,p}^N$ in distribution sense, and $0 \notin \Sigma(H^N)$, so $(H_{1,p}^N)^{-1}$ exists. By the spectral mapping theorem, the spectrum of the operator $(H_{1,p}^N)^{-1}$ is given by

$$\Sigma((H_{1,p}^N)^{-1}) = \left\{ \frac{1}{(2|\alpha|+1)^N} : \alpha \in \mathbb{N}_0^n \right\}^c .$$

On the other hand, $(H_{0,p}^N)^{-1}$ clearly exists because 0 is not in the set of eigenvalues of $H_{0,p}^N$. Moreover, $(H_{0,p}^N)^{-1}$ and $(H_{1,p}^N)^{-1}$ are bounded linear operators on $L^p(\mathbb{R}^n)$. Now, let $v \in L^p(\mathbb{R}^n)$. Suppose $(H_{0,p}^N)^{-1}v = f$ and $(H_{1,p}^N)^{-1}v = g$ for some f and $g \in L^p(\mathbb{R}^n)$. Then we have $H_{0,p}^N f = v$ and $H_{1,p}^N g = v$. But we have shown in the previous lemma that $H_{0,p}^N \subseteq H_{1,p}^N$, so, in particular,

$$H_{0,p}^N f = H_{1,p}^N f = v.$$

Also, since $H_{1,p}^N$ is injective, we conclude that $f = g$. And it follows that $(H_{0,p}^N)^{-1} = (H_{1,p}^N)^{-1}$. □

Now, we prove our main theorem of this section.

Theorem 4.3. $H_{0,p}^N = H_{1,p}^N$, $4/3 < p < 4$.

Proof. Let $u \in \mathcal{D}(H_{1,p}^N)$ and $H_{1,p}^N u = f$ for some $f \in L^p(\mathbb{R}^n)$. Then $(H_{1,p}^N)^{-1} f = u$. But by Lemma 4.2, $(H_{0,p}^N)^{-1} f = (H_{1,p}^N)^{-1} f = u$. Thus, $u \in \mathcal{D}(H_{0,p}^N)$ and $H_{0,p}^N u = f$. It follows that $H_{0,p}^N = H_{1,p}^N$. □

5. The Spectrum of $H_{1,p}$, $4/3 < p < 4$

The goal in this section is to show that $\Sigma(H_{0,p}) = \Sigma(H_{1,p})$, $4/3 < p < 4$. To this end, we use the following result of Taylor [8], [9].

Theorem 5.1. *Let A be a closed linear operator and f be a holomorphic function on a neighbourhood of $\Sigma(A)$. Then the spectrum $\Sigma(f(A))$ of the operator $f(A)$ is given by*

$$\Sigma(f(A)) = \{f(\lambda) : \lambda \in \Sigma(A)\}.$$

In view of the theorem, we define a function f on $\mathbb{C} - (-\infty, 0]$ by

$$f(\lambda) = \lambda^{1/N}, \quad N \in \mathbb{N}, \quad \lambda \in \mathbb{C} - (-\infty, 0],$$

where the principal branch is taken. Secondly, we let the operator A be given by

$$A = H_{0,p}^N = H_{1,p}^N.$$

Then we have

$$\Sigma(A) = \left\{(2|\alpha|+1)^N : \alpha \in \mathbb{N}_0^n\right\}^c .$$

Lastly, since A^N is closed, and f is holomorphic on a neighbourhood of the spectrum of A, we apply Theorem 5.1 to the function f and operator A, and get

$$\Sigma(f(A)) = \{f(\lambda) : \lambda \in \Sigma(A)\}.$$

Therefore

$$\begin{aligned} \Sigma(H_{1,p}) &= \Sigma(H_{0,p}) \\ &= \left\{ \lambda^{1/N} : \lambda = (2|\alpha|+1)^N, \quad \alpha \in \mathbb{N}_0^n \right\}^c \\ &= \{2|\alpha|+1 : \alpha \in \mathbb{N}_0^n\}^c . \end{aligned}$$

Having computed explicitly the spectrum for $H_{0,p}$ and $H_{1,p}$, we can apply Taylor's theorem to the two operators, and by functional calculus, we see easily that

$$H_{0,p} = A^{1/N} = H_{1,p}, \quad 4/3 < p < 4.$$

So, this means that the Hermite operator is essentially self-adjoint on $L^p(\mathbb{R}^n)$, for p between 4/3 and 4.

6. An Initial Value Problem

In this last section, we give the L^p-estimate of the solution to the initial value problem for the heat eqation governed by $H_{0,p}$, which is equal to that of $H_{1,p}$, for p between 4/3 and 4, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = (-H_{0,p}u)(x,t), & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}^n,\ f \in L^p(\mathbb{R}^n),\ 4/3 < p < 4. \end{cases}$$

We have

$$u(\cdot,\, t) = e^{-H_{0,p}t} f, \quad t > 0,$$

and therefore

$$u(\cdot,\, t) = \sum_{\alpha} e^{-(2|\alpha|+1)t} (f,\, e_\alpha) e_\alpha, \quad t > 0,$$

where the convergence is in the space $\mathcal{S}'$ of tempered distributions [13]. So, for $t > 0$,

$$||u(\cdot,\, t)||_p \le \sum_{\alpha} e^{-(2|\alpha|+1)t} ||f||_p ||e_\alpha||_p ||e_\alpha||_{p'}.$$

Let K_p be defined by

$$K_p = \sup_{\alpha \in \mathbb{N}_0^n} \|e_\alpha\|_p \|e_\alpha\|_{p'}.$$

Since

$$||e_\alpha||_p ||e_\alpha||_{p'} = O(1) \le K, \quad 4/3 < p < 4,$$

where K is some positive constant, and

$$\sum_{\alpha} e^{-(2|\alpha|+1)t} = e^{-t} \left(\sum_{j=0}^{\infty} e^{-2jt} \right)^n = \frac{e^{(n-1)t}}{2^n \sinh^n t},$$

we see that

$$u(\cdot,\, t) \in L^p(\mathbb{R}^n), \quad 4/3 < p < 4,$$

and that

$$||u(\cdot,\, t)||_p \le \frac{K_p e^{(n-1)t}}{2^n \sinh^n t} ||f||_p, \quad t > 0,\ 4/3 < p < 4.$$

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