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COMPLEX KRAENKEL-MANNA-MERLE SYSTEM IN A FERRITE: *N*-FOLD DARBOUX TRANSFORMATION, GENERALIZED DARBOUX TRANSFORMATION AND SOLITONS

Yuan Shen, Bo Tian*, Tian-Yu Zhou and Chong-Dong Cheng

Abstract. Ferromagnetic materials such as the ferrites are used in the electronic and energy industries. Here, we concentrate on a complex Kraenkel-Manna-Merle system in a ferrite, under some coefficient constraints. An N-fold Darboux transformation of that system is presented *via* an existing Lax pair, where N is a positive integer. An N-fold generalized Darboux transformation, which admits one spectral parameter, is proposed through a limit procedure. One-, two- and three-soliton solutions of that system are determined *via* that N-fold Darboux transformation. The second-order and third-order degenerate soliton solutions of that system are derived *via* that N-fold generalized Darboux transformation. Those solitons are graphically represented for the magnetization and external magnetic field related to a ferrite.

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1. Introduction

A ferromagnetic material has been considered as a material that possesses a spontaneous magnetization [3, 13, 28]. Ferromagnetic materials such as the ferrites have been utilized in the electronic and energy industries [4, 14, 33]. Inductive components containing the ferromagnetic cores, like the transformers, filters, chokes, resonant circuits, *etc.*, have been seen in almost all the modern electronic devices [6, 11, 14].

In magnetism, optics, fluid mechanics, plasma physics and other fields, there have been some nonlinear phenomena, including the solitons, breathers and rogue waves [1, 2, 7, 8, 12, 16, 25, 27]. Such methods as the Darboux transformation (DT) [29], Riemann-Hilbert approach [24], Hirota method [9], Bäcklund transformation [5] and similarity reduction [10] have been presented to explore those nonlinear phenomena. The DT has been applied to study the nonlinear waves in some physical systems, such as the soliton resonances of the transient stimulated Raman scattering system [26], soliton resonances and soliton molecules of the Lakshmanan-Porsezian-Daniel system in nonlinear optics [21], interactions between the rogue waves and breathers of the Manakov system describing the pulse propagation in a stationary self-focusing electromagnetic system [20], as well as rogue waves and phase transitions of the complex short pulse system in optical-fiber communication [22].

Keywords and phrases: Ferrite, complex Kraenkel-Manna-Merle system, N-fold Darboux transformation, N-fold generalized Darboux transformation, soliton.

State Key Laboratory of Information Photonics and Optical Communications, and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China.

* Corresponding author: tian_bupt@163.com

For studying some microwave ferrites, Ref. [15] has considered a ferromagnetic slab insulator of 0.5 mm thickness submitted to an external transverse magnetic field. Interaction process of an electromagnetic wave perturbation with the slab has been investigated through the Landau-Lifshitz-Gilbert equation with the Maxwell's equations [15]. Accordingly, Refs. [15, 17, 34] have discussed the complex Kraenkel-Manna-Merle system that reads

$$A_{xt} - AB_x + \kappa A_x - \mu A_{xx} = 0, \tag{1.1a}$$
$$A^*_{xt} - A^* B_x + \kappa A^*_x - \mu A^*_{xx} = 0, \tag{1.1b}$$
$$B_{xt} + \frac{1}{2}\left(A^* A_x + A A^*_x\right) = 0, \tag{1.1c}$$

which can describe certain short waves in a ferrite, where the complex differentiable function $A(x,t)$ represents the magnetization related to the ferrite, the real differentiable function $B(x,t)$ denotes the external magnetic field related to the ferrite, the constants κ and μ refer to the damping and inhomogeneous parameters, respectively, the superscript "$*$" means the complex conjugation, while the subscripts stand for the partial derivatives with respect to the normalized space variable x and time variable t. Painlevé integrability, Bäcklund transformation, bilinear form and one-soliton solutions of System (1.1) have been presented [15, 17]. Damping and inhomogeneous exchange effects of System (1.1) have been studied [34].

When $A(x,t)$ is a real differentiable function and $\mu = 0$, System (1.1) has been reduced to the Kraenkel-Manna-Merle system [30],

$$A_{xt} - AB_x + \kappa A_x = 0, \tag{1.2a}$$
$$B_{xt} + AA_x = 0, \tag{1.2b}$$

which describes certain short waves in a saturated ferromagnetic material. For System (1.2), hump solitons, cusp solitons, loop solitons and kink solitons have been derived *via* the generalized G'/G-expansion method [18], loop-like periodic waves and loop-like solitons have been presented *via* the auxiliary equation method [19], soliton twining behaviors have been studied *via* the Hirota method [32], and oscillation rogue waves have been obtained *via* the truncated Painlevé method [23].

Under the coefficient constraints

$$\kappa = \mu = 0, \tag{1.3}$$

Ref. [35] has given the following Lax pair of System (1.1):

$$\Phi_x = U\Phi, \qquad U = \begin{bmatrix} -iB_x\lambda & A_x\lambda \\ -A^*_x\lambda & iB_x\lambda \end{bmatrix}, \tag{1.4a}$$
$$\Phi_t = V\Phi, \qquad V = \begin{bmatrix} \frac{i}{4}\lambda^{-1} & -\frac{i}{2}A \\ -\frac{i}{2}A^* & -\frac{i}{4}\lambda^{-1} \end{bmatrix}, \tag{1.4b}$$

where $\Phi = (\phi, \psi)^T$, ϕ and ψ are the complex differentiable functions of x and t, spectral parameter λ is a complex constant, $i = \sqrt{-1}$, the superscript "T" means the transpose for a vector/matrix; System (1.1) has been reproduced through the compatibility condition $U_t - V_x + UV - VU = 0$ [35]; inverse scattering transform and N-soliton solutions of System (1.1) have been worked out [35].

However, to our knowledge, N-fold DT and N-fold generalized DT of System (1.1), as well as the soliton solutions different from those in Refs. [15, 17, 34, 35] have not been reported, where N is a positive integer. Under Coefficient Constraints (1.3), in Section 2, an N-fold DT of System (1.1) will be constructed *via* Lax Pair (1.4); in Section 3, an N-fold generalized DT of System (1.1), which admits one spectral parameter, will be given through a limit procedure; in Section 4, one-, two- and three-soliton solutions of System (1.1) will be obtained *via* that N-fold DT, while the second-order and third-order degenerate soliton solutions of System (1.1) will be derived *via* that generalized N-fold DT. Our conclusions will be addressed in Section 5.

2. N-fold DT of System (1.1) under coefficient constraints (1.3)

In this section, we shall provide an N-fold DT of System (1.1) with the aid of Lax Pair (1.4). We begin by introducing a gauge transformation [31]

$$\Phi^{[N]} = T^{[N]}\Phi, \tag{2.1}$$

where $T^{[N]}$ is a reversible matrix and $\Phi^{[N]}$ is required to satisfy

$$\Phi^{[N]}_x = U^{[N]}\Phi^{[N]}, \qquad U^{[N]} = \left(T^{[N]}_x + T^{[N]}U\right){T^{[N]}}^{-1}, \tag{2.2a}$$

$$\Phi^{[N]}_t = V^{[N]}\Phi^{[N]}, \qquad V^{[N]} = \left(T^{[N]}_t + T^{[N]}V\right){T^{[N]}}^{-1}, \tag{2.2b}$$

while $U^{[N]}$ and $V^{[N]}$ have the same forms as U and V, respectively, except that the old potentials A, B have been replaced with new ones $A^{[N]}$, $B^{[N]}$, the superscript "$[N]$" represents the Nth-order iteration, and the superscript "-1" denotes the inverse of a matrix. We suppose that the structure of the N-fold Darboux matrix $T^{[N]}$ is as follow:

$$T^{[N]} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \lambda^{-N} + \sum\limits_{j=0}^{N-1} a^{(j)}\lambda^{-j} & \sum\limits_{j=0}^{N-1} b^{(j)}\lambda^{-j} \\ -\sum\limits_{j=0}^{N-1} {b^{(j)}}^*\lambda^{-j} & \lambda^{-N} + \sum\limits_{j=0}^{N-1} {a^{(j)}}^*\lambda^{-j} \end{bmatrix}, \tag{2.3}$$

which indicates that $T^{[N]}$ has the form of a polynomial matrix of λ, where $a^{(j)}$'s and $b^{(j)}$'s are some to-be-determined functions of x and t. Then, we assume that $\lambda_\imath$'s and $\lambda_\imath^*$'s ($\imath = 1, 2, \ldots, N$) are the $2N$ different roots of $\det T^{[N]} = 0$. We have $T^{[N]}\Phi = 0$ when λ takes $\lambda_\imath$, which leads to the linear algebraic system shown below:

$$\sum_{j=0}^{N-1} a^{(j)}\lambda_\imath^{-j} + \delta_\imath \sum_{j=0}^{N-1} b^{(j)}\lambda_\imath^{-j} = -\lambda_\imath^{-N}, \tag{2.4a}$$

$$\delta_\imath \sum_{j=0}^{N-1} {a^{(j)}}^*\lambda_\imath^{-j} - \sum_{j=0}^{N-1} {b^{(j)}}^*\lambda_\imath^{-j} = -\delta_\imath\lambda_\imath^{-N}, \tag{2.4b}$$

where

$$\delta_\imath = \frac{\psi_\imath(\lambda_\imath)}{\phi_\imath(\lambda_\imath)}, \tag{2.5}$$

with $\Phi_\imath(\lambda_\imath) = \left(\phi_\imath(\lambda_\imath), \psi_\imath(\lambda_\imath)\right)^T = \left(\phi_\imath, \psi_\imath\right)^T$ being a column vector solution of Lax Pair (1.4).

In what follows, we shall demonstrate that $U^{[N]}$ and $V^{[N]}$ have the same forms as U and V, respectively.

Proposition 2.1. *The matrix $U^{[N]}$ defined via Equation (2.2a) has the same form as U, i.e.,*

$$U^{[N]} = \begin{bmatrix} -iB^{[N]}_x\lambda & A^{[N]}_x\lambda \\ -{A^{[N]}_x}^*\lambda & iB^{[N]}_x\lambda \end{bmatrix},$$

where the transformations from the old potential functions A, B into the new ones $A^{[N]}$, $B^{[N]}$ are given by

$$\begin{aligned}A^{[N]} &= A + b^{(N-1)},\\ B^{[N]} &= B + ia^{(N-1)} + \beta(t),\end{aligned}$$

and $\beta(t)$ is a real differentiable function of t.

Proof. Let

$$\left(T_x^{[N]} + T^{[N]}U\right)T^{[N]*} = \begin{bmatrix} p_{11}(\lambda) & p_{12}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda)\end{bmatrix}, \tag{2.6}$$

where $p_{11}(\lambda)$, $p_{12}(\lambda)$, $p_{21}(\lambda)$ and $p_{22}(\lambda)$ are the polynomials of λ with order $-2N+1$. Equations (1.4a), (2.4) and (2.5) are utilized to arrive at

$$T_{11}(\lambda_\imath) = -\delta_\imath T_{12}(\lambda_\imath), \quad T_{21}(\lambda_\imath) = -\delta_\imath T_{22}(\lambda_\imath), \quad \delta_{\imath,x} = -u_x^*\lambda_\imath + 2iv_x\lambda_\imath\delta_\imath - u_x\lambda_\imath\delta_\imath^2. \tag{2.7}$$

Then, it is possible to check that $\lambda_\imath$'s are the roots of $p_{11}(\lambda)$, $p_{12}(\lambda)$, $p_{21}(\lambda)$ and $p_{22}(\lambda)$. Likewise, $\lambda_\imath^*$'s are the roots of $p_{11}(\lambda)$, $p_{12}(\lambda)$, $p_{21}(\lambda)$ and $p_{22}(\lambda)$.

Noting that

$$\det T^{[N]} = \lambda^{-2N}\prod_{\imath=1}^{N}(\lambda-\lambda_\imath)(\lambda-\lambda_\imath^*), \tag{2.8}$$

we can confirm the existence of a matrix F such that

$$\left(T_x^{[N]} + T^{[N]}U\right)T^{[N]*} = \det T^{[N]}\cdot F, \tag{2.9}$$

where

$$F = \begin{bmatrix} f_{11}^{(1)}\lambda + f_{11}^{(0)} & f_{12}^{(1)}\lambda + f_{12}^{(0)} \\ f_{21}^{(1)}\lambda + f_{21}^{(0)} & f_{22}^{(1)}\lambda + f_{22}^{(0)}\end{bmatrix},$$

while $f_{11}^{(1)}$, $f_{11}^{(0)}$, $f_{12}^{(1)}$, $f_{12}^{(0)}$, $f_{21}^{(1)}$, $f_{21}^{(0)}$, $f_{22}^{(1)}$ and $f_{22}^{(0)}$ are some to-be-determined functions of x and t. To determine F, we rewrite Equation (2.9) as

$$T_x^{[N]} + T^{[N]}U = FT^{[N]}. \tag{2.10}$$

Equating the same powers of λ in Equation (2.10) yields the following outcomes:

$$\begin{aligned}
f_{11}^{(1)} &= -i\left(B_x + ia_x^{(N-1)}\right) = -iB_x^{[N]},\\
f_{12}^{(1)} &= A_x + b_x^{(N-1)} = A_x^{[N]},\\
f_{21}^{(1)} &= -A_x^* - b_x^{(N-1)*} = -A_x^{[N]*},\\
f_{22}^{(1)} &= i\left(B_x + ia_x^{(N-1)}\right) = iB_x^{[N]},\\
f_{11}^{(0)} &= f_{12}^{(0)} = f_{21}^{(0)} = f_{22}^{(0)} = 0.
\end{aligned} \tag{2.11}$$

The conclusion of Proposition 2.1 is reached from Equations (2.2a) and (2.11). The proof is completed.

Proposition 2.2. *The matrix $V^{[N]}$ defined via Equation (2.2b) has the same form as V, i.e.,*

$$V^{[N]} = \begin{bmatrix} \frac{i}{4}\lambda^{-1} & -\frac{i}{2}A^{[N]} \\ -\frac{i}{2}A^{[N]^*} & -\frac{i}{4}\lambda^{-1} \end{bmatrix},$$

where the transformation from the old potential function A into the new one $A^{[N]}$ is given by

$$A^{[N]} = A + b^{(N-1)}.$$

Proof. Let

$$\left(T_t^{[N]} + T^{[N]}V\right)T^{[N]^*} = \begin{bmatrix} q_{11}(\lambda) & q_{12}(\lambda) \\ q_{21}(\lambda) & q_{22}(\lambda) \end{bmatrix},$$

where $q_{11}(\lambda)$ and $q_{22}(\lambda)$ are the polynomials of λ with order $-2N-1$, whereas $q_{12}(\lambda)$ and $q_{21}(\lambda)$ are the polynomials of λ with order $-2N$. By virtue of Equation (2.5), we have

$$\delta_{\imath,t} = -\frac{i}{2}u^* - \frac{i}{2}\lambda_\imath^{-1}\delta_\imath + \frac{i}{2}u\delta_\imath^2. \tag{2.12}$$

After that, we can confirm that $\lambda_\imath$'s and $\lambda_\imath{}^*$'s are the roots of $q_{11}(\lambda)$, $q_{12}(\lambda)$, $q_{21}(\lambda)$ and $q_{22}(\lambda)$ as well.

With Equation (2.8), it can be demonstrated that there exists a matrix G such that

$$\left(T_t^{[N]} + T^{[N]}V\right)T^{[N]^*} = \det T^{[N]} \cdot G, \tag{2.13}$$

where

$$G = \begin{bmatrix} g_{11}^{(1)}\lambda^{-1} + g_{11}^{(0)} & g_{12}^{(0)} \\ g_{21}^{(0)} & g_{22}^{(1)}\lambda^{-1} + g_{22}^{(0)} \end{bmatrix},$$

while $g_{11}^{(1)}$, $g_{11}^{(0)}$, $g_{12}^{(0)}$, $g_{21}^{(0)}$, $g_{22}^{(1)}$ and $g_{22}^{(0)}$ are some to-be-determined functions of x and t. To calculate G, we rewrite Equation (2.13) as

$$T_t^{[N]} + T^{[N]}V = GT^{[N]}. \tag{2.14}$$

We reach the following findings by equating the same powers of λ in Equation (2.14):

$$\begin{aligned} g_{12}^{(0)} &= -\frac{i}{2}\left(A + b^{(N-1)}\right) = -\frac{i}{2}A^{[N]}, \\ g_{21}^{(0)} &= -\frac{i}{2}\left(A^* + b^{(N-1)^*}\right) = -\frac{i}{2}A^{[N]^*}, \\ g_{11}^{(1)} &= \frac{i}{4}, \qquad g_{22}^{(1)} = -\frac{i}{4}, \qquad g_{11}^{(0)} = g_{22}^{(0)} = 0. \end{aligned} \tag{2.15}$$

Equations (2.2b) and (2.15) allow us to derive the conclusion of Proposition 2.2. The proof is completed.

Further, we provide the following theorem in light of Propositions 2.1 and 2.2:

Theorem 2.3. *Let A and B be the initial solutions of System (1.1), $\Phi_\imath(\lambda_\imath) = (\phi_\imath(\lambda_\imath), \psi_\imath(\lambda_\imath))^T = (\phi_\imath, \psi_\imath)^T$ be the column vector solution of Lax Pair (1.4), and we see that the N-Fold DT of System (1.1) is shown as*

$$A^{[N]} = A + b^{(N-1)}, \tag{2.16a}$$

$$B^{[N]} = B + ia^{(N-1)} + \beta(t), \tag{2.16b}$$

where

$$a^{(N-1)} = \frac{\Delta a^{(N-1)}}{\Delta_N}, \qquad b^{(N-1)} = \frac{\Delta b^{(N-1)}}{\Delta_N},$$

$$\Delta_N = \begin{vmatrix} \phi_1 & \lambda_1^{-1}\phi_1 & \cdots & \lambda_1^{-N+1}\phi_1 & \psi_1 & \lambda_1^{-1}\psi_1 & \cdots & \lambda_1^{-N+1}\psi_1 \\ \phi_2 & \lambda_2^{-1}\phi_2 & \cdots & \lambda_2^{-N+1}\phi_2 & \psi_2 & \lambda_2^{-1}\psi_2 & \cdots & \lambda_2^{-N+1}\psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_N & \lambda_N^{-1}\phi_N & \cdots & \lambda_N^{-N+1}\phi_N & \psi_N & \lambda_N^{-1}\psi_N & \cdots & \lambda_N^{-N+1}\psi_N \\ \psi_1^* & \lambda_1^{*(-1)}\psi_1^* & \cdots & \lambda_1^{*(-N+1)}\psi_1^* & -\phi_1^* & -\lambda_1^{*(-1)}\phi_1^* & \cdots & -\lambda_1^{*(-N+1)}\phi_1^* \\ \psi_2^* & \lambda_2^{*(-1)}\psi_2^* & \cdots & \lambda_2^{*(-N+1)}\psi_2^* & -\phi_2^* & -\lambda_2^{*(-1)}\phi_2^* & \cdots & -\lambda_2^{*(-N+1)}\phi_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_N^* & {\lambda_N^*}^{(-1)}\psi_N^* & \cdots & {\lambda_N^*}^{(-N+1)}\psi_N^* & -\phi_N^* & -{\lambda_N^*}^{(-1)}\phi_N^* & \cdots & -{\lambda_N^*}^{(-N+1)}\phi_N^* \end{vmatrix},$$

while $\Delta a^{(N-1)}$ and $\Delta b^{(N-1)}$ are respectively produced from Δ_N by replacing the Nth and $2N$th columns with the column vector $(-\lambda_1^{-N}\phi_1, -\lambda_2^{-N}\phi_2, \ldots, -\lambda_N^{-N}\phi_N, -\lambda_1^{*(-N)}\psi_1^*, -\lambda_2^{*(-N)}\psi_2^*, \ldots, -{\lambda_N^*}^{(-N)}\psi_N^*)^T$.

3. N-fold generalized DT of System (1.1) under coefficient constraints (1.3)

In this section, we shall determine the N-fold generalized DT of System (1.1), which admits one spectral parameter. To construct the N-fold generalized DT of System (1.1), we redefine the N-fold Darboux matrix $T^{[N]}$ as

$$T^{[N]} = \begin{bmatrix} 1 + \sum\limits_{j=1}^{N} a^{(j)}\lambda^j & \sum\limits_{j=1}^{N} b^{(j)}\lambda^j \\ -\sum\limits_{j=1}^{N} {b^{(j)}}^*\lambda^j & 1 + \sum\limits_{j=1}^{N} {a^{(j)}}^*\lambda^j \end{bmatrix}. \tag{3.1}$$

Proceeding as the previous section, we obtain the following linear algebraic system:

$$\sum_{j=1}^{N} a^{(j)}\lambda_\imath^j + \delta_\imath \sum_{j=1}^{N} b^{(j)}\lambda_\imath^j = -1, \tag{3.2a}$$

$$\delta_\imath \sum_{j=1}^{N} {a^{(j)}}^*\lambda_\imath^j - \sum_{j=1}^{N} {b^{(j)}}^*\lambda_\imath^j = -\delta_\imath. \tag{3.2b}$$

We note that when $N > 1$, there are $2N$ functions $a^{(j)}$'s and $b^{(j)}$'s in the two algebraic equations of Equation (3.2). In order to obtain the additional $2N-2$ algebraic equations, we consider the Taylor expansion

$T^{[N]}\left(\lambda_1\right)\Phi_1\left(\lambda_1\right)|_{\{\lambda_1\to\lambda_1+\varepsilon\}}$ at $\varepsilon=0$ in the form of

$$T^{[N]}\left(\lambda_1+\varepsilon\right)\Phi_1\left(\lambda_1+\varepsilon\right)=\sum_{k=0}^{N-1}\sum_{j=0}^{k}T_j^{[N]}\left(\lambda_1\right)\Phi_1^{(k-j)}\left(\lambda_1\right)\varepsilon^k+O\left(\varepsilon^{N-1}\right), \tag{3.3}$$

where

$$T_0^{[N]}\left(\lambda_1\right)=\begin{bmatrix}1+\sum\limits_{j=1}^{N}a^{(j)}\lambda_1^j & \sum\limits_{j=1}^{N}b^{(j)}\lambda_1^j\\ -\sum\limits_{j=1}^{N}b^{(j)*}\lambda_1^j & 1+\sum\limits_{j=1}^{N}a^{(j)*}\lambda_1^j\end{bmatrix},$$

$$T_k^{[N]}\left(\lambda_1\right)=\begin{bmatrix}\sum\limits_{j=k}^{N}C_j^k a^{(j)}\lambda_1^{j-k} & \sum\limits_{j=k}^{N}C_j^k b^{(j)}\lambda_1^{j-k}\\ -\sum\limits_{j=k}^{N}C_j^k b^{(j)*}\lambda_1^{j-k} & \sum\limits_{j=k}^{N}C_j^k a^{(j)*}\lambda_1^{j-k}\end{bmatrix},$$

$$\Phi_1^{(k)}\left(\lambda_1\right)=\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\Phi_1\left(\lambda_1\right)=\left(\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\phi_1\left(\lambda_1\right),\frac{1}{k!}\frac{\partial^k}{\partial\lambda_1^k}\psi_1\left(\lambda_1\right)\right)^T,$$

ε is a small parameter, $C_j^k=\frac{j(j-1)\cdots(j-k+1)}{k!}$ and k is a non-negative integer.

Following that, we let

$$\lim_{\varepsilon\to0}\frac{T^{[N]}\left(\lambda_1+\varepsilon\right)\Phi_1\left(\lambda_1+\varepsilon\right)}{\varepsilon^k}=0, \tag{3.4}$$

which leads to the following linear algebraic system:

$$T_0^{[N]}\left(\lambda_1\right)\Phi_1^{(0)}\left(\lambda_1\right)=0, \tag{3.5a}$$

$$T_0^{[N]}\left(\lambda_1\right)\Phi_1^{(1)}\left(\lambda_1\right)+T_1^{[N]}\left(\lambda_1\right)\Phi_1^{(0)}\left(\lambda_1\right)=0, \tag{3.5b}$$

$$T_0^{[N]}\left(\lambda_1\right)\Phi_1^{(2)}\left(\lambda_1\right)+T_1^{[N]}\left(\lambda_1\right)\Phi_1^{(1)}\left(\lambda_1\right)+T_2^{[N]}\left(\lambda_1\right)\Phi_1^{(0)}\left(\lambda_1\right)=0, \tag{3.5c}$$

$$\ldots,$$

$$\sum_{j=0}^{N-1}T_j^{[N]}\left(\lambda_1\right)\Phi_1^{(N-1-j)}\left(\lambda_1\right)=0. \tag{3.5d}$$

The $2N$ functions $a^{(j)}$'s and $b^{(j)}$'s in Darboux Matrix (3.1) can be uniquely determined *via* Equation (3.5) when λ_1 is suitably chosen such that the coefficient determinant of Equation (3.5) is non-zero.

Using the previous analysis as a foundation, we present the following theorem:

Theorem 3.1. *Let A and B be the initial solutions of System (1.1), $\Phi_1\left(\lambda_1\right)=\left(\phi_1\left(\lambda_1\right),\psi_1\left(\lambda_1\right)\right)^T=\left(\phi_1,\psi_1\right)^T$ be the column vector solution of Lax Pair (1.4), and we see that the N-fold generalized DT of System (1.1) is shown as*

$$A^{[N]}=A+b^{(1)}, \tag{3.6a}$$

$$B^{[N]}=B+ia^{(1)}+\beta\left(t\right), \tag{3.6b}$$

where

$$a^{(1)}=\frac{\Delta^{\varepsilon}a^{(1)}}{\Delta_N^{\varepsilon}},\qquad b^{(1)}=\frac{\Delta^{\varepsilon}b^{(1)}}{\Delta_N^{\varepsilon}},$$

$$\Delta_N^{\varepsilon}=\begin{vmatrix}
\lambda_1\phi_1^{(0)} & \lambda_1^2\phi_1^{(0)} & \cdots & \lambda_1^N\phi_1^{(0)} & \lambda_1\psi_1^{(0)} & \lambda_1^2\psi_1^{(0)} & \cdots & \lambda_1^N\psi_1^{(0)}\\
\lambda_1\phi_1^{(1)}+\phi_1^{(0)} & \Delta_{2,2} & \cdots & \Delta_{2,N} & \lambda_1\psi_1^{(1)}+\psi_1^{(0)} & \Delta_{2,N+2} & \cdots & \Delta_{2,2N}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\lambda_1\phi_1^{(N-1)}+\phi_1^{(N-2)} & \Delta_{N,2} & \cdots & \Delta_{N,N} & \lambda_1\psi_1^{(N-1)}+\psi_1^{(N-2)} & \Delta_{N,N+2} & \cdots & \Delta_{N,2N}\\
\lambda_1^*\psi_1^{(0)*} & \lambda_1^{*2}\psi_1^{(0)*} & \cdots & \lambda_1^{*N}\psi_1^{(0)*} & -\lambda_1^*\phi_1^{(0)*} & -\lambda_1^{*2}\phi_1^{(0)*} & \cdots & -\lambda_1^{*N}\phi_1^{(0)*}\\
\lambda_1^*\psi_1^{(1)*}+\psi_1^{(0)*} & \Delta_{N+2,2} & \cdots & \Delta_{N+2,N} & -\lambda_1^*\phi_1^{(1)*}-\phi_1^{(0)*} & \Delta_{N+2,N+2} & \cdots & \Delta_{N+2,2N}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\lambda_1^*\psi_1^{(N-1)*}+\psi_1^{(N-2)*} & \Delta_{2N,2} & \cdots & \Delta_{2N,N} & -\lambda_1^*\phi_1^{(N-1)*}-\phi_1^{(N-2)*} & \Delta_{2N,N+2} & \cdots & \Delta_{2N,2N}
\end{vmatrix},$$

$$\Delta_{r,s}=\begin{cases}
\sum\limits_{k=0}^{r-1}C_s^k\lambda_1^{s-k}\phi_1^{(r-1-k)}, & for\ 1\le r,s\le N,\\
\sum\limits_{k=0}^{r-1}C_{s-N}^k\lambda_1^{s-k-N}\psi_1^{(r-1-k)}, & for\ 1\le r\le N,\ N+1\le s\le 2N,\\
\sum\limits_{k=0}^{r-N-1}C_s^k\lambda_1^{*(s-k)}\psi_1^{(r-N-1-k)*}, & for\ N+1\le r\le 2N,\ 1\le s\le N,\\
-\sum\limits_{k=0}^{r-N-1}C_{s-N}^k\lambda_1^{*(s-k-N)}\phi_1^{(r-N-1-k)*}, & for\ N+1\le r,s\le 2N,
\end{cases}$$

and $\Delta^{\varepsilon}a^{(1)}$ *and* $\Delta^{\varepsilon}b^{(1)}$ *are respectively produced from* Δ_N^{ε} *by replacing the first and* $(N+1)$*th columns with the column vector* $(b_1,b_2,\ldots,b_N,b_{N+1},b_{N+2},\ldots,b_{2N})^T$*, while*

$$b_r=\begin{cases}-\phi_1^{(r-1)}, & for\ 1\le r\le N,\\ -\psi_1^{(r-N-1)*}, & for\ N+1\le r\le 2N.\end{cases}$$

4. Soliton solutions of System (1.1) under coefficient constraints (1.3)

In this section, we shall use Theorem 2.3 to derive the one-, two- and three-soliton solutions of System (1.1), and Theorem 3.1 to derive the second-order and third-order degenerate soliton solutions of System (1.1). To derive those soliton solutions of System (1.1), we choose the seed solutions of System (1.1) as

$$A=0,\qquad B=\alpha x,\tag{4.1}$$

which result in the following solutions of Lax Pair (1.4):

$$\Phi_\imath(\lambda_\imath)=\begin{bmatrix}\phi_\imath(\lambda_\imath)\\ \psi_\imath(\lambda_\imath)\end{bmatrix}=\begin{bmatrix}e^{i\left(-\alpha\lambda_\imath x+\frac14\lambda_\imath^{-1}t\right)}\\ e^{i\left(\alpha\lambda_\imath x-\frac14\lambda_\imath^{-1}t\right)}\end{bmatrix},\tag{4.2}$$

where α is a real constant. Then, we let λ_i's be the purely imaginary numbers to ensure that $B^{[N]}$ is a real function.

According to Theorem 2.3, when $N=1$, one-soliton solutions of System (1.1) are obtained as

$$A^{[1]} = \frac{i}{\lambda_{1,I}}\operatorname{sech}\left(2\alpha\lambda_{1,I}x + \frac{1}{2\lambda_{1,I}}t\right), \tag{4.3a}$$

$$B^{[1]} = \alpha x + \beta(t) - \frac{1}{\lambda_{1,I}}\tanh\left(2\alpha\lambda_{1,I}x + \frac{1}{2\lambda_{1,I}}t\right), \tag{4.3b}$$

where the subscript "I" denotes the imaginary part. When $N=2$, two-soliton solutions of System (1.1) are as follows:

$$A^{[2]} = \frac{2i\left(\lambda_{1,I}^2 - \lambda_{2,I}^2\right)\left[\lambda_{1,I}\cosh(\xi_1) - \lambda_{2,I}\cosh(\xi_2)\right]}{\lambda_{1,I}\lambda_{2,I}\left[(\lambda_{1,I}-\lambda_{2,I})^2\cosh(\xi_1+\xi_2) + (\lambda_{1,I}+\lambda_{2,I})^2\cosh(\xi_1-\xi_2) - 4\lambda_{1,I}\lambda_{2,I}\right]}, \tag{4.4a}$$

$$B^{[2]} = \frac{\left(\lambda_{1,I}^2 - \lambda_{2,I}^2\right)\left[(\lambda_{1,I}+\lambda_{2,I})\sinh(\xi_1-\xi_2) - (\lambda_{1,I}-\lambda_{2,I})\sinh(\xi_1+\xi_2)\right]}{\lambda_{1,I}\lambda_{2,I}\left[(\lambda_{1,I}-\lambda_{2,I})^2\cosh(\xi_1+\xi_2) + (\lambda_{1,I}+\lambda_{2,I})^2\cosh(\xi_1-\xi_2) - 4\lambda_{1,I}\lambda_{2,I}\right]} + \alpha x + \beta(t), \tag{4.4b}$$

where

$$\xi_1 = 2\alpha\lambda_{1,I}x + \frac{1}{2\lambda_{1,I}}t, \qquad \xi_2 = 2\alpha\lambda_{2,I}x + \frac{1}{2\lambda_{2,I}}t.$$

When $N=3$, three-soliton solutions of System (1.1) can be expressed as the following determinant form:

$$A^{[3]} = \frac{\Delta b^{(2)}}{\Delta_3}, \tag{4.5a}$$

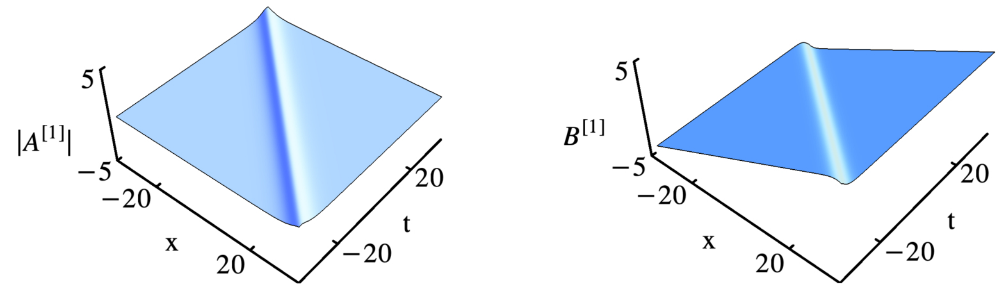
$$B^{[3]} = ax + \beta(t) + i\frac{\Delta a^{(2)}}{\Delta_3}, \tag{4.5b}$$

where

$$\Delta_3 = \begin{vmatrix} \phi_1 & \lambda_1^{-1}\phi_1 & \lambda_1^{-2}\phi_1 & \psi_1 & \lambda_1^{-1}\psi_1 & \lambda_1^{-2}\psi_1 \\ \phi_2 & \lambda_2^{-1}\phi_2 & \lambda_2^{-2}\phi_2 & \psi_2 & \lambda_2^{-1}\psi_2 & \lambda_2^{-2}\psi_2 \\ \phi_3 & \lambda_3^{-1}\phi_3 & \lambda_3^{-2}\phi_3 & \psi_3 & \lambda_3^{-1}\psi_3 & \lambda_3^{-2}\psi_3 \\ \psi_1^* & \lambda_1^{*(-1)}\psi_1^* & \lambda_1^{*(-2)}\psi_1^* & -\phi_1^* & -\lambda_1^{*(-1)}\phi_1^* & -\lambda_1^{*(-2)}\phi_1^* \\ \psi_2^* & \lambda_2^{*(-1)}\psi_2^* & \lambda_2^{*(-2)}\psi_2^* & -\phi_2^* & -\lambda_2^{*(-1)}\phi_2^* & -\lambda_2^{*(-2)}\phi_2^* \\ \psi_3^* & \lambda_3^{*(-1)}\psi_3^* & \lambda_3^{*(-2)}\psi_3^* & -\phi_3^* & -\lambda_3^{*(-1)}\phi_3^* & -\lambda_3^{*(-2)}\phi_3^* \end{vmatrix},$$

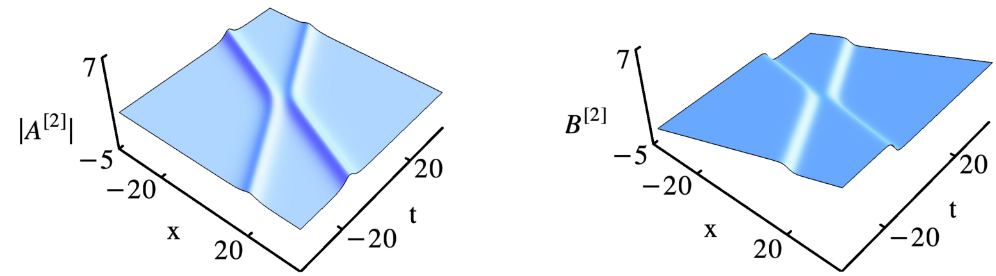
while $\Delta a^{(2)}$ and $\Delta b^{(2)}$ are respectively produced from Δ_3 by replacing the third and sixth columns with the column vector $(-\lambda_1^{-3}\phi_1, -\lambda_2^{-3}\phi_2, -\lambda_3^{-3}\phi_3, -\lambda_1^{*(-3)}\psi_1^*, -\lambda_2^{*(-3)}\psi_2^*, -\lambda_3^{*(-3)}\psi_3^*)^T$.

With respect to $A(x,t)$, the magnetization related to a ferrite, Figure 1(a) shows the one bell-shaped soliton, Figure 2(a) displays the interaction between the two bell-shaped solitons, and Figure 3(a) shows the interaction



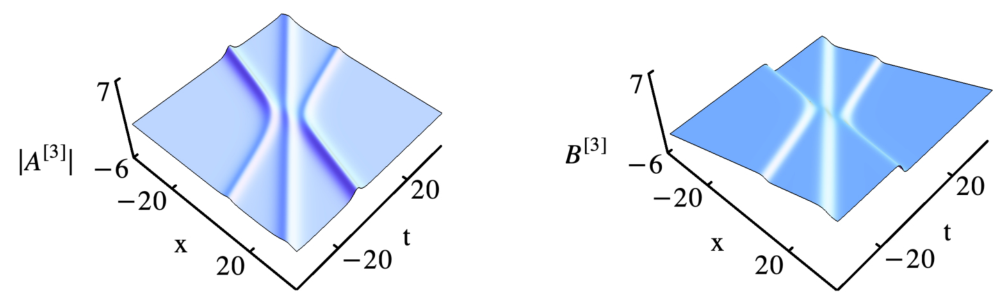
(a) (b)

FIGURE 1. One soliton *via* Solutions (4.3) with $\alpha = \frac{1}{8}$, $\beta\,(t) = 0$ and $\lambda_1 = -\frac{7}{5}i$.



(a) (b)

FIGURE 2. Interaction between the two solitons *via* Solutions (4.3) with $\alpha = \frac{1}{9}$, $\beta\,(t) = 0$, $\lambda_1 = -2i$ and $\lambda_2 = i$.



(a) (b)

FIGURE 3. Interaction among the three solitons *via* Solutions (4.5) with $\alpha = \frac{1}{8}$, $\beta\,(t) = 0$, $\lambda_1 = -\frac{5}{2}i$, $\lambda_2 = \frac{3}{2}i$ and $\lambda_3 = -\frac{4}{5}i$.

among the three bell-shaped solitons. With respect to $B(x,t)$, the external magnetic field related to a ferrite, Figure 1(b) shows the one kink-shaped soliton, Figure 2(b) displays the interaction between the two kink-shaped solitons, and Figure 3(b) shows the interaction among the three kink-shaped solitons.

According to Theorem 3.1, when $N = 2$, we obtain the second-order degenerate soliton solutions of System (1.1) as

$$A^{[2]} = \frac{8i\lambda_{1,I}\cosh(\xi_1) - 4i\left(t - 4\alpha\lambda_{1,I}^2 x\right)\sinh(\xi_1)}{2\lambda_{1,I}^2 + \left(t - 4\alpha\lambda_{1,I}^2 x\right)^2 + 2\lambda_{1,I}^2\cosh(2\xi_1)}, \tag{4.6a}$$

$$B^{[2]} = \alpha x + \beta(t) - \frac{4\left[t - 4\alpha\lambda_{1,I}^2 x + \sinh(2\xi_1)\right]}{2\lambda_{1,I}^2 + \left(t - 4\alpha\lambda_{1,I}^2 x\right)^2 + 2\lambda_{1,I}^2\cosh(2\xi_1)}. \tag{4.6b}$$

When $N = 3$, the third-order degenerate soliton solutions of System (1.1) can be expressed as the following determinant form:

$$A^{[3]} = \frac{\Delta^{\varepsilon} b^{(1)}}{\Delta_3^{\varepsilon}}, \tag{4.7a}$$

$$B^{[3]} = \alpha x + \beta(t) + i\frac{\Delta^{\varepsilon} a^{(1)}}{\Delta_3^{\varepsilon}}, \tag{4.7b}$$

where

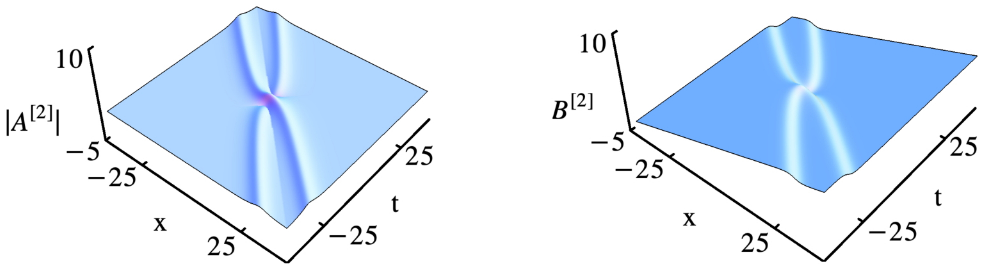
$$\Delta_3^{\varepsilon} = \begin{vmatrix} \lambda_1\phi_1^{(0)} & \lambda_1^2\phi_1^{(0)} & \lambda_1^3\phi_1^{(0)} & \lambda_1\psi_1^{(0)} & \lambda_1^2\psi_1^{(0)} & \lambda_1^3\psi_1^{(0)} \\ \lambda_1\phi_1^{(1)} + \phi_1^{(0)} & \Delta_{2,2} & \Delta_{2,3} & \lambda_1\psi_1^{(1)} + \psi_1^{(0)} & \Delta_{2,5} & \Delta_{2,6} \\ \lambda_1\phi_1^{(2)} + \phi_1^{(1)} & \Delta_{3,2} & \Delta_{3,3} & \lambda_1\psi_1^{(2)} + \psi_1^{(1)} & \Delta_{3,5} & \Delta_{3,6} \\ \lambda_1^*\psi_1^{(0)^*} & \lambda_1^{*2}\psi_1^{(0)^*} & \lambda_1^{*3}\psi_1^{(0)^*} & -\lambda_1^*\phi_1^{(0)^*} & -\lambda_1^{*2}\phi_1^{(0)^*} & -\lambda_1^{*3}\phi_1^{(0)^*} \\ \lambda_1^*\psi_1^{(1)^*} + \psi_1^{(0)^*} & \Delta_{5,2} & \Delta_{5,3} & -\lambda_1^*\phi_1^{(1)^*} - \phi_1^{(0)^*} & \Delta_{5,5} & \Delta_{5,6} \\ \lambda_1^*\psi_1^{(2)^*} + \psi_1^{(1)^*} & \Delta_{6,2} & \Delta_{6,3} & -\lambda_1^*\phi_1^{(2)^*} - \phi_1^{(1)^*} & \Delta_{6,5} & \Delta_{6,6} \end{vmatrix},$$

and $\Delta_{2,2} = \lambda_1^2\phi_1^{(1)} + 2\lambda_1\phi_1^{(0)}$, $\Delta_{2,3} = \lambda_1^3\phi_1^{(1)} + 3\lambda_1^2\phi_1^{(0)}$, $\Delta_{2,5} = \lambda_1^2\psi_1^{(1)} + 2\lambda_1\psi_1^{(0)}$, $\Delta_{2,6} = \lambda_1^3\psi_1^{(1)} + 3\lambda_1^2\psi_1^{(0)}$, $\Delta_{3,2} = \lambda_1^2\phi_1^{(2)} + 2\lambda_1\phi_1^{(1)} + \phi_1^{(0)}$, $\Delta_{3,3} = \lambda_1^3\phi_1^{(2)} + 3\lambda_1^2\phi_1^{(1)} + 3\lambda_1\phi_1^{(0)}$, $\Delta_{3,5} = \lambda_1^2\psi_1^{(2)} + 2\lambda_1\psi_1^{(1)} + \psi_1^{(0)}$, $\Delta_{3,6} = \lambda_1^3\psi_1^{(2)} + 3\lambda_1^2\psi_1^{(1)} + 3\lambda_1\psi_1^{(0)}$, $\Delta_{5,2} = \lambda_1^{*2}\psi_1^{(1)^*} + 2\lambda_1^*\psi_1^{(0)^*}$, $\Delta_{5,3} = \lambda_1^{*3}\psi_1^{(1)^*} + 3\lambda_1^{*2}\psi_1^{(0)^*}$, $\Delta_{5,5} = -\lambda_1^{*2}\phi_1^{(1)^*} - 2\lambda_1^*\phi_1^{(0)^*}$, $\Delta_{5,6} = -\lambda_1^{*3}\phi_1^{(1)^*} - 3\lambda_1^{*2}\phi_1^{(0)^*}$, $\Delta_{6,2} = \lambda_1^{*2}\psi_1^{(2)^*} + 2\lambda_1^*\psi_1^{(1)^*} + \psi_1^{(0)^*}$, $\Delta_{6,3} = \lambda_1^{*3}\psi_1^{(2)^*} + 3\lambda_1^{*2}\psi_1^{(1)^*} + 3\lambda_1^*\psi_1^{(0)^*}$, $\Delta_{6,5} = -\lambda_1^{*2}\phi_1^{(2)^*} - 2\lambda_1^*\phi_1^{(1)^*} - \phi_1^{(0)^*}$, $\Delta_{6,6} = -\lambda_1^{*3}\phi_1^{(2)^*} - 3\lambda_1^{*2}\phi_1^{(1)^*} - 3\lambda_1^*\phi_1^{(0)^*}$, while $\Delta^{\varepsilon}a^{(1)}$ and $\Delta^{\varepsilon}b^{(1)}$ are respectively produced from Δ_3^{ε} by replacing the first and fourth columns with the column vector $(-\phi_1^{(0)}, -\phi_1^{(1)}, -\phi_1^{(2)}, -\psi_1^{(0)^*}, -\psi_1^{(1)^*}, -\psi_1^{(2)^*})^T$.

For the magnetization $A(x,t)$ related to a ferrite, and the external magnetic field $B(x,t)$ related to a ferrite, Figures 4 and 5 show the second-order degenerate soliton and third-order degenerate soliton, respectively.

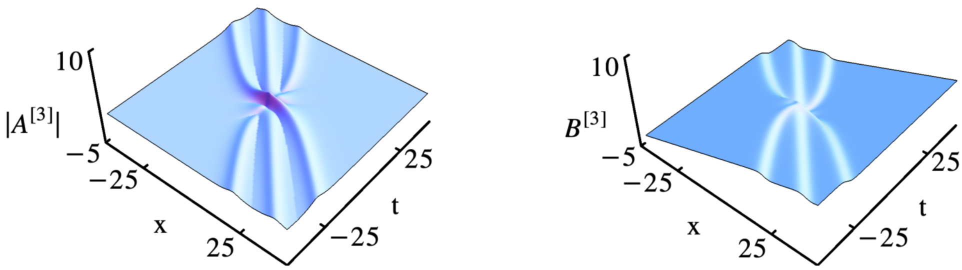
5. Conclusions

Ferromagnetic materials such as the ferrites have been used in the electronic and energy industries. In this paper, as for certain short waves in a ferrite, we have investigated a complex Kraenkel-Manna-Merle system, *i.e.*, System (1.1), under Coefficient Constraints (1.3). With Lax Pair (1.4), we have proofed Propositions 2.1 and 2.2, and then provided N-Fold DT (2.16) of System (1.1). Based on N-Fold DT (2.16), we have presented N-Fold Generalized DT (3.6) of System (1.1), which admits one spectral parameter. By virtue of N-Fold DT (2.16), we have derived One-Soliton Solutions (4.3), Two-Soliton Solutions (4.4) and Three-Soliton Solutions (4.5) of System (1.1). For the magnetization $A(x,t)$ related to a ferrite, and the external magnetic field $B(x,t)$ related to a ferrite, Figures 1, 2 and 3 have shown the one soliton, interaction between the two solitons, and interaction



(*a*) (*b*)

FIGURE 4. The second-order degenerate soliton *via* Solutions (4.6) with $\alpha = \frac{1}{9}$, $\beta(t) = 0$ and $\lambda_1 = -\frac{3}{2}i$.



(*a*) (*b*)

FIGURE 5. The third-order degenerate soliton *via* Solutions (4.7) with $\alpha = \frac{1}{9}$, $\beta(t) = 0$ and $\lambda_1 = \frac{8}{5}i$.

among the three solitons, respectively. With N-Fold Generalized DT (3.6), we have obtained Second-Order Degenerate Soliton Solutions (4.6) and Third-Order Degenerate Soliton Solutions (4.7). Figures 4 and 5 have displayed the second-order degenerate soliton and third-order degenerate soliton, respectively.

Declaration of competing interest. The authors declare that they have no conflict of interest.

Data availability. Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

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