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GLOBAL HOPF BIFURCATION OF A DELAYED DIFFUSIVE GAUSE-TYPE PREDATOR–PREY SYSTEM WITH THE FEAR EFFECT AND HOLLING TYPE III FUNCTIONAL RESPONSE

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Abstract. In this paper, a delayed diffusive predator–prey system with the fear effect and Holling type III functional response is considered, and Neumann boundary condition is imposed on this system. First, we explore the stability of the unique positive constant steady state and the existence of local Hopf bifurcation. Then the global attraction domain G_* of system (1.4) is obtained by the comparison principle and the iterative method. Through constructing the Lyapunov function, we investigate uniform boundedness of periodic solutions' periods. Finally, we prove the global continuation of periodic solutions by the global Hopf bifurcation theorem of Wu. Moreover, some numerical simulations that support the analysis results are given.

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1. Introduction

It is well known that the dynamics analysis of predator–prey systems has a significant role in biology. In order to study dynamical behavior between predator and prey, Lotka [1] and Volterra [2] first proposed a mathematical predator–prey system. Nevertheless, the system supposed that the population density of prey is going to become infinitely large in absence of predator, but that is unrealistic in nature. Therefore, many scholars have progressively revised predator–prey systems according to the actual situation. The Gause-type predator–prey system is one of systems that has been progressively revised, and its classical form is shown below [3–5]:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = uf_1(u) - g_1(u)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v(g_2(u) - d), \end{cases} \tag{1.1}$$

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where $u = u(t)$ and $v = v(t)$ are the population densities of prey and predator at time t respectively. $d > 0$ is the death rate of predator. $f_1(u)$ is the growth rate of prey in absence of predator and satisfies $f_1(0) > 0$, $f_1(K) = 0$ and $f_1'(u) < 0$ for all $u > 0$, where K represents the environmental carrying capacity of prey. $g_1(u)$ is the functional response of predator to prey, which describes the change in the rate of predation by a single predator per unit time as a function of prey density. $g_2(u)$ is the numerical response of predator to prey, which describes the change in reproduction rate with changing prey density. Through those years of research, $g_1(u)$ and $g_2(u)$ have been derived in multiple types, like

Holling II-IV type: $\frac{bu}{a+u}$, $\frac{bu^2}{a+u^2}$, $\frac{bu}{a+u^2}$,

Beddington-DeAngelis type: $\frac{bu}{a+m_1u+m_2v}$,

ratio-dependent type: $\frac{bu}{m_1u+m_2v}$,

see [6–12]. Up to now, scholars have done a large number of work related to Gause-type predator–prey systems. Yang *et al.* [13] investigated the sliding bifurcation and global stability for a non-smooth Gause predator–prey system. The predator and prey are usually influenced by the spatial factor, that is, the populations are abundant in space with different densities at different positions. Based on this consideration, for the Gause-type predator–prey system, Ko *et al.* [14] studied the existence/non-existence of non-constant positive solutions, the asymptotic behavior of spatially inhomogeneous solutions and the local existence of periodic solutions, and Liu *et al.* [15] proved the existence of a unique globally stable periodic solution. Further considering the quadratic mortality is suitable for intermediate predators, some scholars [16–19] have introduced the quadratic mortality into the predator–prey system to analysis. In in most real ecological systems, one biological population does not respond immediately to the interactions of populations of other species. Considering that the reproduction of predator after predating the prey is not instantaneous but will be mediated by some discrete time delay required for gestation of the predator, many scholars have incorporated the gestation time of predator into the predator–prey system, and studied the stability of equilibrium, Hopf bifurcation, Bogdanov-Takens bifurcation and so on, see [20–24]. The time delay may greatly affect the dynamic behavior of biological systems, so we suppose that the gestation time $\tau > 0$ of predator occurs in the predator response term. Then one delayed reaction-diffusion Gause-type predator–prey system can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + ru(1 - \dfrac{u}{K}) - \dfrac{bu^2v}{a+u^2}, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + \dfrac{cu_\tau^2}{a+u_\tau^2}v - s_0v^2. \end{cases} \tag{1.2}$$

Here $u = u(x,t)$, $u_\tau = u(x, t-\tau)$ and $v = (x,t)$. The parameters $r > 0$ and $a > 0$ represent the growth rate and the half-saturation constant of prey, respectively. b is positive constant and represents the maximum value which per capita reduction rate of prey can attain. $c > 0$ indicates the conversion efficiency of consumed prey into new predator. $-s_0v^2$ is the quadratic mortality for predator population. d_1 and d_2 are the diffusion coefficients of prey and predator, respectively.

In recent years, scholars have found that the fear of predator drives prey to carry out various anti-predator behavior (foraging activity, habitat changes, vigilance and different physiological changes, *etc.*) to escape direct predation, which can increase the survival of prey in a short term, but it is detrimental to the reproduction of prey population for the long term [25, 26]. The experiments of some birds [27, 28] and wolves [29] have shown that even in absence of predation behavior, the population of prey diminished substantially due to the presence of predator. In practically the fear of predator has a greater effect on prey than direct predation, so the fear effect should be included in ecosystem as a form of reproductive reduction. The general function indicating the fear effect is $f_2(\beta, v)$, which $f_2(\beta, v)$ satisfies

(i) $f_2(0, v) = f_2(\beta, 0) = 1$;

(ii) $\lim_{v\to\infty} f_2(\beta,v) = \lim_{\beta\to\infty} f_2(\beta,v) = 0$;

(iii) $\frac{\partial f_2(\beta,v)}{\partial\beta} < 0$, $\frac{\partial f_2(\beta,v)}{\partial v} < 0$,

where $\beta > 0$ is the fear level caused by predator. In 2016, Wang *et al.* [30] introduced $f_2(\beta,v) = \frac{1}{1+\beta v}$ into a predator–prey system, studying the existence of limit cycles and Hopf bifurcation. Since then, more and more scholars have focused on the issue. Under the fear effect, the authors [31] discussed the asymptotic stability for a prey-predator system with prey refuge and gestation delay. In [32], Halder *et al.* added both the fear and Allee effect to a predator–prey system, investigating the indirect interaction of prey and predator. Liu and Jiang constructed a stochastic predator–prey system with the fear effect and obtained some results about the existence and uniqueness of an ergodic stationary distribution of positive solutions in [33].

In this paper, we consider system (1.2) with the fear effect as follow:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + \dfrac{r}{1+\beta v}u - \dfrac{r}{K}u^2 - \dfrac{bu^2v}{a+u^2}, & \\ & x\in\Omega,\quad t>0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + \dfrac{cu_\tau^2}{a+u_\tau^2}v - s_0v^2, & \\ \dfrac{\partial u}{\partial \mathbf{v}} = \dfrac{\partial v}{\partial \mathbf{v}} = 0, \qquad x\in\partial\Omega,\quad t>0, & \\ u(x,t)\ge 0, v(x,t)\ge 0, \qquad x\in\bar{\Omega},\quad t\in[-\tau,0], & \end{cases} \tag{1.3}$$

where $\Omega = (0, l\pi)(l > 0)$ and $\mathbf{v}$ is the outward unit normal vector of the boundary $\partial\Omega$. Then, we define

$$\bar{u} = \frac{u}{K},\quad \bar{u}_\tau = \frac{u_\tau}{K},\quad \bar{v} = s_0 v,\quad \bar{b} = \frac{b}{Ks_0},\quad \bar{a} = \frac{a}{K^2},\quad \bar{\beta} = \frac{\beta}{s_0},\quad \bar{d_1} = d_1K,\quad \bar{d_2} = \frac{d_2}{s_0}.$$

Dropping the bars for simplification, system (1.3) can transform into

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + \dfrac{r}{1+\beta v}u - ru^2 - \dfrac{bu^2v}{a+u^2}, & \\ & x\in\Omega,\quad t>0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + \dfrac{cu_\tau^2}{a+u_\tau^2}v - v^2, & \\ \dfrac{\partial u}{\partial \mathbf{v}} = \dfrac{\partial v}{\partial \mathbf{v}} = 0, \qquad x\in\partial\Omega,\quad t>0, & \\ u(x,t)\ge 0, v(x,t)\ge 0, \qquad x\in\bar{\Omega},\quad t\in[-\tau,0]. & \end{cases} \tag{1.4}$$

In this paper, we will focus on proving the extended existence of periodic solutions of system (1.4) by the global Hopf bifurcation theorem of Wu [34]. Hopf bifurcation is commonly applied to explain the phenomenon of periodic fluctuations in predator and prey populations. For practical modeling, appropriate simplifications and assumptions are indispensable. Due to some neglected factors, the parameter values of the system are usually variable within a certain range, and this range of variation may be larger, so global Hopf bifurcation might be a better explanation for periodic fluctuations than local Hopf bifurcation. As far as I know, there have been numerous research on the global Hopf bifurcation of delay differential equations (DDEs), see [35–38]. However, owing to the influence of spatial factor, studying the global Hopf bifurcation for partial functional differential equations (PFDEs) is quite difficult, thus research on this aspect is relatively few. Su *et al.* [39] and Xu *et al.* [40] investigated the local and global Hopf bifurcation of one-dimensional PFDEs. In [41], Xu and Liu discussed a reaction-diffusion predator–prey system with stage structures, proving the global continuation of periodic solutions. Liu *et al.* studied the global Hopf bifurcation of a phytoplankton-zooplankton system with delay and diffusion [42]. To apply the global Hopf bifurcation theorem of Wu, we need obtain uniform boundedness of periodic solutions and their periods. Unlike previous studies, we first get the permanence properties of system (1.4) by using the comparison principle. Then, to expand the scope of the theorem, we use the iterative method to compress the upper and lower boundedness of periodic solutions and get the global attraction domain of system (1.4). Additionally, because system (1.4) is monotonous with respect to the time-delay term, then the upper and lower solutions method is no longer applicable to this system. We have to carry out a Lyapunov function to verify uniform boundedness of periodic solutions' periods.

The paper is organized as follows. In Section 2, the stability of unique positive constant steady state for system (1.4) is proved, then we study the existence of local Hopf bifurcation. In Section 3, the existence of global Hopf bifurcation is demonstrated by the global Hopf bifurcation theorem of Wu [34]. In Section 4, we carry out some numerical simulations supporting the analysis results. In Section 5, a conclusion section completes the paper.

2. Local Hopf bifurcation

There are two boundary steady states $E_a(0,0)$ and $E_b(1,0)$ in system (1.4). For positive constant steady states of system (1.4), we have the following result.

Theorem 2.1. *System (1.4) has one unique positive constant steady state $E_*(u_*,v_*)$ if the assumption (H_0) (see Appendix A) holds, where $v_* = \frac{cu_*^2}{a+u_*^2}$.*

Due to the complexity of the proof process of Theorem 2.1, we put the proof in the Appendix A.

We denote the real-valued Sobolev space

$$X = \{(u,v) \in H^2(0,l\pi) \times H^2(0,l\pi) | u_x = v_x = 0, x = 0, l\pi\},$$

and the abstract space

$$\mathcal{C} = C([-\tau,0],X).$$

The following equation is the linearized form for system (1.4) at $E_*(u_*,v_*)$ in the phase space $\mathcal{C}$,

$$\frac{\partial U}{\partial t} = D\Delta U(x,t) + AU(x,t) + BU(x,t-\tau), \tag{2.1}$$

where $U(x,t) = (u(x,t),v(x,t))^T$, $D = \mathrm{diag}(d_1,d_2)$ and

$$A = \begin{pmatrix} \frac{r}{1+\beta v_*} - 2ru_* - \frac{2abu_*v_*}{(a+u_*^2)^2} & -\frac{r\beta u_*}{(1+\beta v_*)^2} - \frac{bu_*^2}{a+u_*^2} \\ 0 & -v_* \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 0 \\ \frac{2acu_*v_*}{(a+u_*^2)^2} & 0 \end{pmatrix}.$$

Then, the corresponding characteristic equation of equation (2.1) is

$$\Delta_n(\lambda) = \lambda^2 + T_n\lambda + D_n + P\mathrm{e}^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{2.2}$$

where

$$\begin{aligned}
\alpha(u_*) &= -\frac{r}{1+\beta v_*} + 2ru_* + \frac{2abu_*v_*}{(a+u_*^2)^2},\\
T_n &= \frac{(d_1+d_2)n^2}{l^2} + v_* + \alpha(u_*),\\
D_n &= \left(d_1\frac{n^2}{l^2} + \alpha(u_*)\right)\left(d_2\frac{n^2}{l^2} + v_*\right),\\
P &= \frac{2acu_*v_*}{(a+u_*^2)^2}\left(\frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2}\right).
\end{aligned}$$

Moreover, denote

$$(H_1):\ \alpha(u_*) > \Gamma, \quad where \quad \Gamma = \max\left\{-v_*,\ -\frac{d_1}{d_2}v_*,\ -\frac{P}{v_*}\right\}.$$

Lemma 2.2. *Under assumptions* (H_0) *and* (H_1)*, the positive constant steady state* $E_*(u_*, v_*)$ *is locally asymptotically stable when* $\tau = 0$.

Proof. In absence of time delay, $\Delta_n(\lambda) = 0$ becomes

$$\lambda^2 + T_n\lambda + D_n + P = 0. \tag{2.3}$$

For any $n \in \mathbb{N}_0$, it is easy to know that T_n and $D_n + P$ are monotonically increasing in terms of n. If (H_0) and (H_1) hold, we obtain $T_0 > 0$ and $D_0 + P > 0$. Then $T_n > 0$ and $D_n + P > 0$. Hence, all roots of equation (2.3) have negative real parts. The proof is completed. □

Lemma 2.3. *Suppose* (H_0) *and* (H_1) *hold. For any* $\tau \geq 0$, $\Delta_n(\lambda) = 0$ *has no zero characteristic roots.*

Next, we assume there is a purely imaginary root $\lambda = i\omega(\omega > 0)$ for $\Delta_n(\lambda) = 0$. Then substituting $\lambda = i\omega$ into $\Delta_n(\lambda) = 0$ as shown below,

$$-\omega^2 + iT_n\omega + D_n + P(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.4}$$

Separating the real and imaginary parts, we get

$$-\omega^2 + D_n = -P\cos\omega\tau, \quad T_n\omega = P\sin\omega\tau. \tag{2.5}$$

Hence

$$\sin\omega\tau = \frac{T_n\omega}{P}, \quad \cos\omega\tau = \frac{\omega^2 - D_n}{P}. \tag{2.6}$$

Let $\chi = \omega^2$, it follows from equation (2.5) that

$$\chi^2 + (T_n^2 - 2D_n)\chi + D_n^2 - P^2 = 0. \tag{2.7}$$

Note that

$$T_n^2 - 2D_n = \left(d_1\frac{n^2}{l^2} + \alpha(u_*)\right)^2 + \left(d_2\frac{n^2}{l^2} + v_*\right)^2 > 0.$$

Calculating shows equation (2.7) has two real roots $\chi_n^{\pm}$, where

$$\chi_n^{\pm} = \frac{-(T_n^2 - 2D_n) \pm \sqrt{(T_n^2 - 2D_n)^2 - 4(D_n^2 - P^2)}}{2}.$$

Through analysis, we know

(i) If $D_n^2 - P^2 \geq 0$ holds, then equation (2.7) has no positive roots, which implies $\Delta_n(\lambda) = 0$ has no purely imaginary roots.
(ii) If $D_n^2 - P^2 < 0$ holds, then equation (2.7) has one positive root χ_n^+, which implies $\Delta_n(\lambda) = 0$ has a pair of purely imaginary roots $\pm i\omega_n = \pm i\sqrt{\chi_n^+}$.

Suppose that (H_0) and (H_1) hold. According to the proof of Lemma 2.2, there is $D_n + P > 0$ for any $n \in \mathbb{N}_0$. On the other hand, for any $n \in \mathbb{N}_0$, $D_n - P$ is a monotonically increasing function with respect to n. Obviously, there is

$$\begin{cases} D_0 - P < 0, & \alpha(u_*)v_* < P, \\ D_0 - P \geq 0, & \alpha(u_*)v_* \geq P. \end{cases} \tag{2.8}$$

Therefore, we clearly obtain $D_n - P < 0$ for $n < n_*$ and $D_n - P \geq 0$ for $n \geq n_*$, where

$$n_* = l\left(\frac{-d_1 v_* - d_2\alpha(u_*) + \sqrt{(d_1 v_* - d_2\alpha(u_*))^2 + 4d_1 d_2 P}}{2d_1 d_2}\right)^{\frac{1}{2}}.$$

As a result, if

$$(H_2): \ \alpha(u_*) < \frac{P}{v_*}, \quad 0 \leq n < n_*, \quad n \in \mathbb{N}_0,$$

holds, then $D_n^2 - P^2 < 0$. For convenience, let $I^* = \{n | \ (H_2) \ \ holds\}$. Then we have the following result.

Lemma 2.4. *Under assumptions (H_0)–(H_2), $\Delta_n(\lambda) = 0$ has a pair of purely imaginary roots $\pm i\omega_n$ when $\tau = \tau_{n,j}$, where*

$$\begin{aligned} \omega_n &= \sqrt{\frac{-(T_n^2 - 2D_n) + \sqrt{(T_n^2 - 2D_n)^2 - 4(D_n^2 - P^2)}}{2}}, \\ \tau_{n,j} &= \frac{1}{\omega_n}\left(\arccos\frac{\omega_n^2 - D_n}{P} + 2j\pi\right), \qquad n \in I^*, \quad j \in \mathbb{N}_0. \end{aligned}$$

Lemma 2.5. *Suppose that (H_0)–(H_2) hold. For any $n \in I^*$, letting $\lambda(\tau) = \xi_1(\tau) + i\xi_2(\tau)$ is a root of equation (2.2) satisfying $\xi_1(\tau_{n,j}) = 0$ and $\xi_2(\tau_{n,j}) = \omega_n$ when $\tau = \tau_{n,j}$, then the transversality condition holds.*

Proof. Taking the derivative of $\Delta_n(\lambda) = 0$ with respect to τ, one has

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{T_n + 2i\omega}{T_n\omega^2 + i\omega(\omega^2 - D_n)} - \frac{\tau}{i\omega}. \tag{2.9}$$

The real part of the above equation at $\tau = \tau_{n,j}$ is

$$\Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_{n,j}} = \frac{\sqrt{(T_n^2-2D_n)^2-4(D_n^2-P^2)}}{T_n^2\omega_n^2+(\omega_n^2-D_n)^2} > 0. \tag{2.10}$$

Then

$$\text{Sign } \xi_1^{'}(\tau_{n,j}) = \text{Sign } \Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\Big|_{\tau=\tau_{n,j}} = 1. \tag{2.11}$$

□

On the basis of the above analysis, we get the following theorem.

Theorem 2.6. *Assume that* (H_0)–(H_2) *hold. For system (1.4), we define* $\tau_0 = \min_{n\in I^*}\{\tau_{n,0}\}$ *and have the following statements:*

(i) If $0 \le \tau < \tau_0$*, then the positive constant steady state* $E_*(u_*, v_*)$ *of system (1.4) is locally asymptotically stable.*
(ii) If $\tau > \tau_0$*, then the positive constant steady state* $E_*(u_*, v_*)$ *of system (1.4) is unstable.*
(iii) If $\tau = \tau_{n,j}$ *for* $n \in I^*$ *and* $j \in \mathbb{N}_0$*, there is one Hopf bifurcation occurring at* $E_*(u_*, v_*)$ *for system (1.4).*

3. Global Hopf bifurcation

In this section, we are going to investigate the existence of global Hopf bifurcation for system (1.4) by the global Hopf bifurcation theorem in Wu [34].

Lemma 3.1. *All solutions of system (1.4) with* $u(x,0) \not\equiv 0$ *and* $v(x,0) \not\equiv 0$ *are ultimately uniformly bounded.*

Proof. The proof is mainly based on the comparison principle. From the maximal principle, we obtain that every solution $(u(x,t), v(x,t))$ of system (1.4) with $u(x,0) \not\equiv 0$ and $v(x,0) \not\equiv 0$ is positive for $x \in \Omega$ and all $t > 0$.

According to the first equation of system (1.4), one has

$$\frac{\partial u}{\partial t} \le d_1 \Delta u + ru\,(1-u)\,, \quad x \in \Omega, \quad t > 0. \tag{3.1}$$

Let $\tilde{Y_0}(x,t)$ be the solution of the following equation,

$$\begin{cases} \dfrac{\partial \tilde{Y_0}}{\partial t} = d_1\Delta\tilde{Y_0} + r\tilde{Y_0}\left(1-\tilde{Y_0}\right), \quad x\in\Omega, \quad t>0, \\ \dfrac{\partial \tilde{Y_0}}{\partial \mathbf{v}} = 0, \quad x\in\partial\Omega, \quad t>0, \\ \tilde{Y_0}(x,0) = u(x,0), \quad x\in\bar{\Omega}. \end{cases} \tag{3.2}$$

Applying the comparison principle, it is easy to obtain $u(x,t) \le \tilde{Y_0}(x,t)$ in $\bar{\Omega}\times[0,+\infty)$. For all $x \in \bar{\Omega}$, there is $\lim_{t\to\infty} \tilde{Y_0}(x,t) = 1$. Then

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}}\; u(x,t) \le 1 := \tilde{u}_0.$$

Consequently, for sufficiently small $\varepsilon_1 > 0$, there is $T_1 > 0$ such that $u(x,t) \le \tilde{u}_0 + \varepsilon_1$ for all $t > T_1$.

According to the second equation of system (1.4), we get

$$\frac{\partial v}{\partial t} \le d_2 \Delta v + \frac{c(\tilde{u}_0+\varepsilon_1)^2}{a+(\tilde{u}_0+\varepsilon_1)^2} v - v^2, \quad x \in \Omega, \quad t > T_1. \tag{3.3}$$

Let $\tilde{Z}_0(x,t)$ be the solution of equation (3.4),

$$\begin{cases} \dfrac{\partial \tilde{Z}_0}{\partial t} = d_2 \Delta \tilde{Z}_0 + \dfrac{c(\tilde{u}_0+\varepsilon_1)^2}{a+(\tilde{u}_0+\varepsilon_1)^2} \tilde{Z}_0 - \tilde{Z}_0^2, \quad x \in \Omega, \quad t > T_1, \\ \dfrac{\partial \tilde{Z}_0}{\partial \mathbf{v}} = 0, \quad x \in \partial\Omega, \quad t > T_1, \\ \tilde{Z}_0(x,T_1) = v(x,T_1), \quad x \in \bar{\Omega}. \end{cases} \tag{3.4}$$

Well, it is clear to get $v(x,t) \le \tilde{Z}_0(x,t)$ in $\bar{\Omega} \times [T_1, +\infty)$. Furthermore,

$$\lim_{t\to\infty} \tilde{Z}_0(x,t) = \frac{c(\tilde{u}_0+\varepsilon_1)^2}{a+(\tilde{u}_0+\varepsilon_1)^2}, \quad x \in \bar{\Omega}.$$

Owing to the arbitrariness of $\varepsilon_1 > 0$, then

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} \ v(x,t) \le \frac{c\tilde{u}_0^2}{a+\tilde{u}_0^2} := \tilde{v}_0.$$

If $\varepsilon_2 > 0$ is enough small, there is $T_2 > T_1 > 0$ such that $v(x,t) \le \tilde{v}_0 + \varepsilon_2$ for all $t > T_2$.

According to the first equation of system (1.4), we have

$$\frac{\partial u}{\partial t} \ge d_1 \Delta u + u\left(\frac{r}{1+\beta(\tilde{v}_0+\varepsilon_2)} - ru - \frac{b(\tilde{v}_0+\varepsilon_2)}{a} u\right), \quad x \in \Omega, \quad t > T_2. \tag{3.5}$$

There is $u(x,t) \ge \hat{Y}_0(x,t)$ in $\bar{\Omega} \times [T_2, +\infty)$, where $\hat{Y}_0(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \hat{Y}_0}{\partial t} = d_1 \Delta \hat{Y}_0 + \hat{Y}_0 \left(\dfrac{r}{1+\beta(\tilde{v}_0+\varepsilon_2)} - r\hat{Y}_0 - \dfrac{b(\tilde{v}_0+\varepsilon_2)}{a} \hat{Y}_0\right), \quad x \in \Omega, \quad t > T_2, \\ \dfrac{\partial \hat{Y}_0}{\partial \mathbf{v}} = 0, \quad x \in \partial\Omega, \quad t > T_2, \\ \hat{Y}_0(x,T_2) = u(x,T_2), \quad x \in \bar{\Omega}. \end{cases} \tag{3.6}$$

Besides, for sufficiently small $\varepsilon_2 > 0$, we have

$$\lim_{t\to\infty} \hat{Y}_0(x,t) = \frac{r}{1+\beta(\tilde{v}_0+\varepsilon_2)} \left(r + \frac{b(\tilde{v}_0+\varepsilon_2)}{a}\right)^{-1}, \quad x \in \bar{\Omega}.$$

Hence,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} \ u(x,t) \ge \frac{r}{1+\beta\tilde{v}_0} \left(r + \frac{b\tilde{v}_0}{a}\right)^{-1} := \hat{u}_0.$$

For sufficiently small $\varepsilon_3 > 0$ and all $t > T_3$, there is $T_3 > T_2$ such that $u(x,t) \ge \hat{u}_0 - \varepsilon_3$.

According to the second equation of system (1.4), we have

$$\frac{\partial v}{\partial t} \geq d_2 \Delta v + \frac{c(\hat{u}_0 - \varepsilon_3)^2}{a + (\hat{u}_0 - \varepsilon_3)^2} v - v^2, \quad x \in \Omega, \quad t > T_3. \tag{3.7}$$

Considering the following equation:

$$\begin{cases} \dfrac{\partial \hat{Z}_0}{\partial t} = d_2 \Delta \hat{Z}_0 + \dfrac{c(\hat{u}_0 - \varepsilon_3)^2}{a + (\hat{u}_0 - \varepsilon_3)^2} \hat{Z}_0 - \hat{Z}_0^2, \quad x \in \Omega, \quad t > T_3, \\ \dfrac{\partial \hat{Z}_0}{\partial \mathbf{v}} = 0, \quad x \in \partial\Omega, \quad t > T_3, \\ \hat{Z}_0(x, T_3) = v(x, T_3), \quad x \in \bar{\Omega}, \end{cases} \tag{3.8}$$

we denote $\hat{Z}_0(x,t)$ as the solution of equation (3.8). Obviously, for any $x \in \bar{\Omega}$ and all $t \geq T_3$, there are $v(x,t) \geq \hat{Z}_0(x,t)$ and

$$\lim_{t \to \infty} \hat{Z}_0(x,t) = \frac{c(\hat{u}_0 - \varepsilon_3)^2}{a + (\hat{u}_0 - \varepsilon_3)^2}.$$

Thus for sufficiently small ε_3,

$$\liminf_{t \to \infty} \min_{x \in \bar{\Omega}} \ v(x,t) \geq \frac{c\hat{u}_0^2}{a + \hat{u}_0^2} := \hat{v}_0.$$

The proof is complete. □

In order to avoid contradictions with the conditions for the existence of local Hopf bifurcation, we compress the upper and lower boundedness of perdioc solutions of system (1.4) by the iterative method.

Lemma 3.2. *System (1.4) with $u(x,0) \not\equiv 0$ and $v(x,0) \not\equiv 0$ has a global attraction domain $G_* = [\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}]$, where $\tilde{u}$, $\hat{u}$, $\tilde{v}$ and $\hat{v}$ are the limits of sequences $\{\tilde{u}_n\}$, $\{\hat{u}_n\}$, $\{\tilde{v}_n\}$ and $\{\hat{v}_n\}$ for $n \in \mathbb{N}_0$, respectively. The four sequences are as follow,*

$$\begin{aligned} \tilde{u}_{n+1} &= \frac{r}{1+\beta\hat{v}_n}\left(r + \frac{b\hat{v}_n}{a+\tilde{u}_n^2}\right)^{-1}, \qquad \tilde{v}_{n+1} = \frac{c\tilde{u}_{n+1}^2}{a+\tilde{u}_{n+1}^2}, \\ \hat{u}_{n+1} &= \frac{r}{1+\beta\tilde{v}_{n+1}}\left(r + \frac{b\tilde{v}_{n+1}}{a+\hat{u}_n^2}\right)^{-1}, \quad \hat{v}_{n+1} = \frac{c\hat{u}_{n+1}^2}{a+\hat{u}_{n+1}^2}, \end{aligned} \tag{3.9}$$

where

$$\tilde{u}_0 = 1, \quad \tilde{v}_0 = \frac{c\tilde{u}_0^2}{a+\tilde{u}_0^2}, \quad \hat{u}_0 = \frac{r}{1+\beta\tilde{v}_0}\left(r + \frac{b\tilde{v}_0}{a}\right)^{-1}, \quad \hat{v}_0 = \frac{c\hat{u}_0^2}{a+\hat{u}_0^2}.$$

Proof. From Lemma 3.1, if $\varepsilon_4 > 0$ is sufficiently small, and $T_4 > T_3$, one has $u(x,t) \leq \tilde{u}_0 + \varepsilon_4$ and $v(x,t) \geq \hat{v}_0 - \varepsilon_4$ for all $t > T_4$. According to the first equation of system (1.4), we obtain

$$\frac{\partial u}{\partial t} \leq d_1 \Delta u + u\left(\frac{r}{1+\beta(\hat{v}_0 - \varepsilon_4)} - ru - \frac{b(\hat{v}_0 - \varepsilon_4)}{a + (\tilde{u}_0 + \varepsilon_4)^2} u\right), \quad x \in \Omega, \ t > T_4. \tag{3.10}$$

By the comparison principle, we know $u(x,t) \le \tilde{Y}_1(x,t)$ for $(x,t) \in \bar{\Omega} \times [T_4, +\infty)$, where $\tilde{Y}_1(x,t)$ is the solution of the following equation,

$$\begin{cases} \dfrac{\partial \tilde{Y}_1}{\partial t} = d_1 \Delta \tilde{Y}_1 + \tilde{Y}_1 \left(\dfrac{r}{1+\beta(\hat{v}_0 - \varepsilon_4)} - r\tilde{Y}_1 - \dfrac{b(\hat{v}_0 - \varepsilon_4)}{a + (\tilde{u}_0 + \varepsilon_4)^2} \tilde{Y}_1 \right), \ x \in \Omega, \ t > T_4, \\ \dfrac{\partial \tilde{Y}_1}{\partial \mathbf{v}} = 0, \quad x \in \partial\Omega, \quad t > T_4, \\ \tilde{Y}_1(x, T_4) = u(x, T_4), \quad x \in \bar{\Omega}. \end{cases} \tag{3.11}$$

Additionally,

$$\lim_{t\to\infty} \tilde{Y}_1(x,t) = \frac{r}{1+\beta(\hat{v}_0 - \varepsilon_4)} \left(r + \frac{b(\hat{v}_0 - \varepsilon_4)}{a + (\tilde{u}_0 + \varepsilon_4)^2} \right)^{-1}, \quad x \in \bar{\Omega}.$$

Then,

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} \ u(x,t) \le \frac{r}{1+\beta\hat{v}_0} \left(r + \frac{b\hat{v}_0}{a + \tilde{u}_0^2} \right)^{-1} := \tilde{u}_1.$$

Hence, there is any $\varepsilon_5 > 0$ and $T_5 > T_4$ such that $u(x,t) \le \tilde{u}_1 + \varepsilon_5$ for all $t > T_5$.

According to the second equation of system (1.4), we have

$$\frac{\partial v}{\partial t} \le d_2 \Delta v + \frac{c(\tilde{u}_1 + \varepsilon_5)^2}{a + (\tilde{u}_1 + \varepsilon_5)^2} v - v^2, \quad x \in \Omega, \quad t > T_5. \tag{3.12}$$

Then we consider the following equation,

$$\begin{cases} \dfrac{\partial \tilde{Z}_1}{\partial t} = d_2 \Delta \tilde{Z}_1 + \dfrac{c(\tilde{u}_1 + \varepsilon_5)^2}{a + (\tilde{u}_1 + \varepsilon_5)^2} \tilde{Z}_1 - \tilde{Z}_1^2, \quad x \in \Omega, \quad t > T_5, \\ \dfrac{\partial \tilde{Z}_1}{\partial \mathbf{v}} = 0, \quad x \in \partial\Omega, \quad t > T_5, \\ \tilde{Z}_1(x, T_5) = v(x, T_5), \quad x \in \bar{\Omega}. \end{cases} \tag{3.13}$$

Defining $\tilde{Z}_1(x,t)$ is the solution of equation (3.13) and satisfies

$$\lim_{t\to\infty} \tilde{Z}_1(x,t) = \frac{c(\tilde{u}_1 + \varepsilon_5)^2}{a + (\tilde{u}_1 + \varepsilon_5)^2}, \quad x \in \bar{\Omega}.$$

Similarly, we have $v(x,t) \le \tilde{Z}_1(x,t)$ in $\bar{\Omega} \times [T_5, +\infty)$, then

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} \ v(x,t) \le \frac{c\tilde{u}_1^2}{a + \tilde{u}_1^2} := \tilde{v}_1.$$

Clearly, if $\varepsilon_6 > 0$ is enough small, there is $T_6 > T_5$ such that $u(x,t) \ge \hat{u}_0 - \varepsilon_6$ and $v(x,t) \le \tilde{v}_1 + \varepsilon_6$ for all $t > T_6$.

For $x \in \Omega$ and $t > T_6$, one has the following equation from system (1.4),

$$\frac{\partial u}{\partial t} \ge d_1 \Delta u + u \left(\frac{r}{1+\beta(\tilde{v}_1 + \varepsilon_6)} - ru - \frac{b(\tilde{v}_1 + \varepsilon_6)}{a + (\hat{u}_0 - \varepsilon_6)^2} u \right). \tag{3.14}$$

Letting $\hat{Y}_1(x,t)$ is the solution of equation (3.15), where equation (3.15) is

$$\begin{cases} \dfrac{\partial \hat{Y}_1}{\partial t} = d_1\Delta\hat{Y}_1 + \hat{Y}_1\left(\dfrac{r}{1+\beta(\tilde{v}_1+\varepsilon_6)} - r\hat{Y}_1 - \dfrac{b(\tilde{v}_1+\varepsilon_6)}{a+(\hat{u}_0-\varepsilon_6)^2}\hat{Y}_1\right),\ x\in\Omega,\ t>T_6, \\ \dfrac{\partial \hat{Y}_1}{\partial \mathbf{v}} = 0, \quad x\in\partial\Omega, \quad t>T_6, \\ \hat{Y}_1(x,T_6) = u(x,T_6), \quad x\in\bar{\Omega}. \end{cases} \tag{3.15}$$

For $x\in\bar{\Omega}$ and $t\in[T_6,+\infty)$, we know $u(x,t)\geq \hat{Y}_1(x,t)$ and

$$\lim_{t\to\infty}\hat{Y}_1(x,t) = \frac{r}{1+\beta(\tilde{v}_1+\varepsilon_6)}\left(r+\frac{b(\tilde{v}_1+\varepsilon_6)}{a+(\hat{u}_0-\varepsilon_6)^2}\right)^{-1}.$$

Therefore,

$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}}\ u(x,t) \geq \frac{r}{1+\beta\tilde{v}_1}\left(r+\frac{b\tilde{v}_1}{a+\hat{u}_0^2}\right)^{-1} := \hat{u}_1.$$

As a result, for enough small $\varepsilon_7>0$, and all $t>T_7>T_6$, we have $u(x,t)\geq \hat{u}_1-\varepsilon_7$.

According to the second equation of system (1.4), there is

$$\frac{\partial v}{\partial t} \geq d_2\Delta v + \frac{c(\hat{u}_1-\varepsilon_7)^2}{a+(\hat{u}_1-\varepsilon_7)^2}v - v^2, \quad x\in\Omega, \quad t>T_7. \tag{3.16}$$

It is easy to see $v(x,t)\geq \hat{Z}_1(x,t)$ in $\bar{\Omega}\times[T_7,+\infty)$, where $\hat{Z}_1(x,t)$ satisfies the following equation,

$$\begin{cases} \dfrac{\partial \hat{Z}_1}{\partial t} = d_2\Delta\hat{Z}_1 + \dfrac{c(\hat{u}_1-\varepsilon_7)^2}{a+(\hat{u}_1-\varepsilon_7)^2}\hat{Z}_1 - \hat{Z}_1^2, \quad x\in\Omega, \quad t>T_7, \\ \dfrac{\partial \hat{Z}_1}{\partial \mathbf{v}} = 0, \quad x\in\partial\Omega, \quad t>T_7, \\ \hat{Z}_1(x,T_7) = v(x,T_7), \quad x\in\bar{\Omega}. \end{cases} \tag{3.17}$$

Moreover,

$$\lim_{t\to\infty}\hat{Z}_1(x,t) = \frac{c(\hat{u}_1-\varepsilon_7)^2}{a+(\hat{u}_1-\varepsilon_7)^2}, \quad x\in\bar{\Omega}.$$

Then

$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}}\ v(x,t) \geq \frac{c\hat{u}_1^2}{a+\hat{u}_1^2} := \hat{v}_1.$$

Furthermore, due to

$$\begin{aligned}
\tilde{u}_1 &= \frac{r}{1+\beta\hat{v}_0}\left(r+\frac{b\hat{v}_0}{a+\tilde{u}_0^2}\right)^{-1} < r\left(r+\frac{b\hat{v}_0}{a+\tilde{u}_0^2}\right)^{-1} < 1 = \tilde{u}_0, \qquad \tilde{v}_1 < \tilde{v}_0,\\
\hat{u}_0 &< \frac{r}{1+\beta\tilde{v}_0}\left(r+\frac{b\tilde{v}_0}{a+\hat{u}_0^2}\right)^{-1} < \frac{r}{1+\beta\tilde{v}_1}\left(r+\frac{b\tilde{v}_1}{a+\hat{u}_0^2}\right)^{-1} = \hat{u}_1, \quad \hat{v}_0 < \hat{v}_1,\\
\hat{u}_0 &< \frac{r}{1+\beta\tilde{v}_0}\left(r+\frac{b\tilde{v}_0}{a+\tilde{u}_0^2}\right)^{-1} < \frac{r}{1+\beta\hat{v}_0}\left(r+\frac{b\hat{v}_0}{a+\tilde{u}_0^2}\right)^{-1} = \tilde{u}_1, \quad \hat{v}_0 < \tilde{v}_1,
\end{aligned}$$

then $\hat{u}_0 < \hat{u}_1 < \tilde{u}_1 < \tilde{u}_0$ and $\hat{v}_0 < \hat{v}_1 < \tilde{v}_1 < \tilde{v}_0$.

In this way, system (1.4) have a smaller global attraction domain $G_1 = [\hat{u}_1, \tilde{u}_1] \times [\hat{v}_1, \tilde{v}_1]$ than $G_0 = [\hat{u}_0, \tilde{u}_0] \times [\hat{v}_0, \tilde{v}_0]$. By continuing to iterate with this method, we can get the smaller and smaller global attraction domain, *i.e.* $G_n = [\hat{u}_n, \tilde{u}_n] \times [\hat{v}_n, \tilde{v}_n]$, $n \in \mathbb{N}_0$, where four sequences $\{\tilde{u}_n\}$, $\{\hat{u}_n\}$, $\{\tilde{v}_n\}$ and $\{\hat{v}_n\}$ satisfy equation (3.9) and

$$\begin{aligned}
&\hat{u}_0 < \hat{u}_1 < \cdots < \hat{u}_{n+1} < \cdots < \tilde{u}_{n+1} < \cdots < \tilde{u}_1 < \tilde{u}_0,\\
&\hat{v}_0 < \hat{v}_1 < \cdots < \hat{v}_{n+1} < \cdots < \tilde{v}_{n+1} < \cdots < \tilde{v}_1 < \tilde{v}_0.
\end{aligned}$$

By analyzing, it is easy to know $\{\tilde{u}_n\}$, $\{\hat{u}_n\}$, $\{\tilde{v}_n\}$ and $\{\hat{v}_n\}$ are uniformly convergent, *i.e.*

$$\begin{aligned}
\tilde{u} &= \lim_{n\to\infty}\frac{r}{1+\beta\hat{v}_n}\left(r+\frac{b\hat{v}_n}{a+\tilde{u}_n^2}\right)^{-1}, \qquad \tilde{v} = \lim_{n\to\infty}\frac{c\tilde{u}_{n+1}^2}{a+\tilde{u}_{n+1}^2},\\
\hat{u} &= \lim_{n\to\infty}\frac{r}{1+\beta\tilde{v}_{n+1}}\left(r+\frac{b\tilde{v}_{n+1}}{a+\hat{u}_n^2}\right)^{-1}, \quad \hat{v} = \lim_{n\to\infty}\frac{c\hat{u}_{n+1}^2}{a+\hat{u}_{n+1}^2}.
\end{aligned}\tag{3.18}$$

In conclusion, for any positive initial values, system (1.4) exists the global attraction domain $G_* = [\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}]$. The proof is completed.

□

Lemma 3.3. *Denote*

$$\begin{aligned}
(H_3):\ \ & r + \frac{ab\hat{v}}{(a+\tilde{u}^2)(a+u_*^2)} - \zeta_{\max} > 0, \quad \frac{\tilde{H}_{\max}}{2} < L,\\
& \textit{where}\ \ L = \min\left\{ r + \frac{ab\hat{v}}{(a+\tilde{u}^2)(a+u_*^2)} - \zeta_{\max},\ \ \frac{b}{ac}u_*^2 \right\},
\end{aligned}$$

and $\zeta_{\max}$ *and* $\tilde{H}_{\max}$ *are defined in the following proof. Under assumptions* (H_0) *and* (H_3)*, the positive constant steady state* $E_*(u_*, v_*)$ *is global attractivity for system (1.4) with* $u(x,0) \not\equiv 0$ *and* $v(x,0) \not\equiv 0$ *when* $\tau = 0$*.*

Proof. On the global attraction domain G_*, define a Lyapunov function as follows:

$$\mathcal{V}(t) = \int_\Omega V(u(x,t), v(x,t))\mathrm{d}x, \tag{3.19}$$

where

$$\begin{aligned}
V(u,v) &= (u-u_*) - u_*\ln\left(\frac{u}{u_*}\right) + \rho\left((v-v_*) - v_*\ln\left(\frac{v}{v_*}\right)\right)\\
&= V_1(u) + V_2(v),
\end{aligned}\tag{3.20}$$

with

$$\rho = \frac{b}{ac}u_*^2.$$

It follows that

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} = \int_\Omega \left(\frac{u-u_*}{u} d_1 \triangle u + \rho \frac{v-v_*}{v} d_2 \triangle v \right) \mathrm{d}x + \int_\Omega \frac{\mathrm{d}V(u,v)}{\mathrm{d}t} \mathrm{d}x. \tag{3.21}$$

Clearly, $V(u,v)$ is a continuous function defined on G_*, and is corresponding to the ordinary differential equation (ODE) of system (1.4),

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{r}{1+\beta v}u - ru^2 - \dfrac{bu^2 v}{a+u^2}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{cu^2}{a+u^2}v - v^2, \\ u(0) \geq 0, \quad v(0) \geq 0. \end{cases} \qquad t > 0, \tag{3.22}$$

There exists one unique positive equilibrium $E_*(u_*, v_*)$ in system (3.22) if (H_0) holds. Then computing along the solution of system (3.22), the derivative of $V(u,v)$ is shown below:

$$\begin{aligned} \frac{\mathrm{d}V(u,v)}{\mathrm{d}t} =& (u-u_*) \left(\frac{r}{1+\beta v} - ru - \frac{buv}{a+u^2} \right) + \rho(v-v_*) \left(\frac{cu^2}{a+u^2} - v \right) \\ =& \left(-r - \frac{abv - bu_* v_* u}{(a+u^2)(a+u_*^2)} \right)(u-u_*)^2 - \frac{b}{ac}u_*^2 (v-v_*)^2 + \tilde{H}(u,v)(u-u_*)(v-v_*) \\ \leq& \left(-r - \frac{abv - bu_* v_* u}{(a+u^2)(a+u_*^2)} \right)(u-u_*)^2 - \frac{b}{ac}u_*^2 (v-v_*)^2 + \left| \tilde{H}(u,v) \right| \frac{(u-u_*)^2 + (v-v_*)^2}{2} \\ =& P_1(u,v)(u-u_*)^2 + P_2(u,v)(v-v_*)^2, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \tilde{H}(u,v) &= -\frac{r\beta}{(1+\beta v)(1+\beta v_*)} + \frac{bu_*^3 - abu_*}{(a+u^2)(a+u_*^2)}, \\ P_1(u,v) &= -r - \frac{abv}{(a+u^2)(a+u_*^2)} + \zeta(u) + \frac{\left|\tilde{H}(u,v)\right|}{2}, \quad \zeta(u) = \frac{bu_* v_* u}{(a+u^2)(a+u_*^2)}, \\ P_2(u,v) &= \frac{\left|\tilde{H}(u,v)\right|}{2} - \frac{b}{ac}u_*^2. \end{aligned}$$

On the one hand, from $\tilde{H}(u,v)$, one has

$$\frac{\partial \tilde{H}(u,v)}{\partial v} = \frac{r\beta^2}{(1+\beta v)^2(1+\beta v_*)} > 0. \tag{3.24}$$

Consequently, $\tilde{H}(u,v)$ has the maximum or minimum value at the boundary of G_*, then the maximum value $\tilde{H}_{\max}$ of $|\tilde{H}(u,v)|$ is taken at $(u,\tilde{v})$ or $(u,\hat{v})$, *i.e.*

$$\tilde{H}_{\max} = \max\left\{|\tilde{H}(u,\tilde{v})|, \quad |\tilde{H}(u,\hat{v})|\right\}, \quad u \in G_*. \tag{3.25}$$

On the other hand, $\zeta(u)$ is continuous on G_*, and have

$$\frac{\mathrm{d}\zeta(u)}{\mathrm{d}u} = \frac{bu_*v_*(a-u^2)}{(a+u^2)^2(a+u_*^2)}. \tag{3.26}$$

Through the analysis of the above formula, we have the following results:

(i) If $0 < a \le \hat{u}^2$ holds, then $\frac{\mathrm{d}\zeta(u)}{\mathrm{d}u} \le 0$, thus $\zeta(u)$ exists the maximum value $\zeta_{\max} = \zeta(\hat{u})$.
(ii) If $\hat{u}^2 < a < \tilde{u}^2$ holds, then $\frac{\mathrm{d}\zeta(u)}{\mathrm{d}u} \ge 0$ in $(\hat{u},\sqrt{a}\,]$ and $\frac{\mathrm{d}\zeta(u)}{\mathrm{d}u} < 0$ in $(\sqrt{a},\tilde{u})$, thus $\zeta(u)$ exists the maximum value $\zeta_{\max} = \zeta(\sqrt{a})$.
(iii) If $a \ge \tilde{u}^2$ holds, then $\frac{\mathrm{d}\zeta(u)}{\mathrm{d}u} \ge 0$, thus $\zeta(u)$ exists the maximum value $\zeta_{\max} = \zeta(\tilde{u})$.

Under assumption (H_3), we know

$$\begin{aligned} P_1(u,v) &\le -r - \frac{ab\hat{v}}{(a+\tilde{u}^2)(a+u_*^2)} + \zeta_{\max} + \frac{\tilde{H}_{\max}}{2} \\ &< -r - \frac{ab\hat{v}}{(a+\tilde{u}^2)(a+u_*^2)} + \zeta_{\max} + L \\ &< 0, \\ P_2(u,v) &\le \frac{\tilde{H}_{\max}}{2} - \frac{b}{ac}u_*^2 < L - \frac{b}{ac}u_*^2 < 0. \end{aligned} \tag{3.27}$$

Then,

$$\frac{\mathrm{d}V(u,v)}{\mathrm{d}t} \le P_1(u,v)(u-u_*)^2 + P_2(u,v)(v-v_*)^2 \le 0. \tag{3.28}$$

In addition, $V(u,v)$ satisfies

$$V(u,v) = V_1(u) + V_2(v), \quad \frac{\partial^2 V_1(u)}{\partial u^2} = \frac{u_*}{u^2} > 0, \quad \frac{\partial^2 V_2(v)}{\partial v^2} = \rho\frac{v_*}{v^2} > 0.$$

Applying some results of Hsu [43], it is easy to obtain

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \le 0.$$

The proof is completed. □

Remark 3.4. Assume (H_0) and (H_3) hold. Owing to the global attractivity of steady state $E_*(u_*,v_*)$ of system (1.4) without time delay, it is clear to know that there are no positive nonconstant periodic solutions of period τ in this system.

The boundedness of positive periodic solutions and their periods have been verified by Lemma 3.2 and Lemma 3.3, next we are going to state the global Hopf bifurcation theorem. From [34], define

(i) $E = C(S^1, X)$ is a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C} : |z| = 1\}$;
(ii) $E^G := \{x \in E : gx = x \text{ for all } g \in G\}$, then $E^G = X$. E exists an isotypical direct sum decomposition $E = E^G \bigoplus_{k=1}^{\infty} E_k$, where $E_k = \{\mathrm{e}^{ikt}x : x \in X\}$ for $k \geq 1$.

System (1.4) can be written as an integral equation, and it satisfies continuous differentiability, complete continuity and G-invariant property.

Suppose the conditions (H_0)–(H_3) hold. From Lemma 3.3, it is easy to see system (1.4) has no nonconstant steady states, which shows the system has a unique positive steady state $E_*(u_*, v_*)$. For any $\tau \geq 0$, the characteristic equation at $E_*(u_*, v_*)$ has no zero eigenvalues from Lemma 2.3. Besides, there exist one unique pair of purely imaginary eigenvalues $\lambda = \pm i\omega_n$ for $\Delta_n(\lambda) = 0$ if and only if $\tau = \tau_{n,j}$ and $\omega = \omega_n$. Obviously, the assumptions $(H1)$–$(H3)$ in Section 6.5 of [34] are satisfied for system (1.4). Then denoting $\mathbb{R}_+ = (0, +\infty)$, we define the local steady state manifold

$$
\begin{aligned}
&M = \{(E_*, \tau, \omega) : |\tau - \tau_{n,j}| < \epsilon_0, |\omega - \omega_n| < \varsigma_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+, \\
&(\tau, \omega) \in [\tau_{n,j} - \epsilon_0, \tau_{n,j} + \epsilon_0] \times [\omega_n - \varsigma_0, \omega_n + \varsigma_0],
\end{aligned}
$$

where ϵ_0 and ς_0 are positive and sufficiently small. Therefore, $(E_*, \tau_{n,j}, \omega_n)$ is an isolated singular point in M.

From Section 6.5 of [34], we call $\mu_k(E_*, \tau_{n,j}, \omega_n)$ the generalized crossing number, $k = 1, 2, \cdots$, and know $\mu_1(E_*, \tau_{n,j}, \omega_n) = 1$ from the transversality condition. Define

$$
\begin{aligned}
S = \mathrm{Cl}\Big\{&(z, \tau, \omega) \in E \times \mathbb{R} \times \mathbb{R}_+ : z = (z_1(\cdot, \omega t), z_2(\cdot, \omega t)) = (u(\cdot, t), v(\cdot, t)) \\
&\text{is a nontrivial } \frac{2\pi}{\omega} \text{ periodic solution of system (1.4)}\Big\}.
\end{aligned}
$$

By the local bifurcation theorem, it is easy to conclude that $(E_*, \tau_{n,j}, \omega_n) \in S$. Then, we denote the following form as the complete steady state manifold,

$$
M^* = \{(E_*, \tau) : \tau \in \mathbb{R}\} \subset E^G \times \mathbb{R},
$$

and let $\mathfrak{C}_{n,j} = \mathfrak{C}_{n,j}(E_*, \tau_{n,j}, \omega_n)$ be the connected component of $(E_*, \tau_{n,j}, \omega_n)$ in S. Hence,

$$
\sum_{(E_*, \tau_{n,j}, \omega_n) \in \mathfrak{C}_{n,j} \cap M^* \times \mathbb{R}_+} \mu_1(E_*, \tau_{n,j}, \omega_n) > 0. \tag{3.29}
$$

Theorem 3.5. *Assume that (H_0)-(H_3) hold. System (1.4) has at least one positive periodic solution for $\tau \in (\tau_1, +\infty)$, where $\tau_1 = \min_{n \in I^*}\{\tau_{n,1}\}$.*

Proof. At first, we know each connected component $\mathfrak{C}_{n,j}$ is unbounded from equation (3.29). Then from Lemma 3.2, it is easy to see the projection of $\mathfrak{C}_{n,j}$ onto the z-space is bounded. Moreover, one has

$$
2j\pi < \omega_n \tau_{n,j} < 2(j+1)\pi, \quad j \in \mathbb{N}.
$$

It follows that

$$
\frac{1}{j+1} < \frac{2\pi}{\omega_n \tau_{n,j}} < \frac{1}{j}, \quad j \in \mathbb{N}.
$$

By Lemma 3.3 and Remark 3.4, we get $\frac{\tau}{j+1} < \frac{2\pi}{\omega} < \frac{\tau}{j}$ if $(z, \tau, \omega) \in \mathfrak{C}_{n,j}$ for $j \in \mathbb{N}$ holds. This and Lemma 3.2 illustrate that if τ is bounded, then the projection of $\mathfrak{C}_{n,j}$ onto the T-space must be bounded. Consequently, the projection of $\mathfrak{C}_{n,j}$ onto the τ-space includes $[\tau_{n,j}, \infty)$. Therefore, (i) of Theorem 6.6.5 of [34] is satisfied. This proof is completed. $\square$

4. Numerical example

In this section, we will carry out some numerical simulations to support our analysis results. To start with, we choose the parameter values as follow:

$$d_1 = 0.2, \quad d_2 = 0.2, \quad r = 0.2, \quad \beta = 0.01, \quad a = 0.18, \quad b = 0.81, \quad c = 0.35, \quad l = 16.$$

From Appendix A, we calculate

$$\Delta = -5.1812 < 0, \quad u_2^6 = 0.0597, \quad \varphi^{(3)}(u_2^6) = -0.3476 < 0, \quad u_4^{33} = 0.1057,$$
$$\varphi'(u_4^{33}) = 8.594 \times 10^{-4} > 0, \quad u_5^{79} = 0.0370, \quad u_5^{80} = 0.1683, \quad \varphi(u_5^{79}) = 0.0011 > 0,$$

which shows the condition (16) of (H_0) holds. System (1.4) has one unique positive steady state $E_*(0.3457, 0.1397)$. Under this set of parameters, we obtain

$$\alpha(u_*) = 0.0955, \quad T_0 = 0.2352 > 0, \quad D_0 + P = 0.0353 > 0,$$

then

$$T_n > 0, \quad D_n + P > 0.$$

The characteristic equation $\Delta_0(\lambda) = 0$ has a pair of purely imaginary eigenvalues $\pm 0.0909i$. Moreover, we also get

$$\rho = 1.5367, \quad G_* = [0.1513, 0.8488] \times [0.0395, 0.2800], \quad P_1(u, v) = -0.0025 < 0, \quad P_2(u, v) = -1.3961 < 0.$$

Clearly, the assumptions (H_0)–(H_3) are satisfied with this set of parameter values. Following the formulas in Appendix B, one has

$$C_1(0) = -0.9322 - 2.0901i, \quad \mu_2 = 0.04695 > 0, \quad \beta_2 = -1.8643 < 0.$$

By Theorem 2.6 and Theorem B.1, system (1.4) undergoes the Hopf bifurcation at $E_*(0.3457, 0.1397)$, and the Hopf bifurcation is supercritical. For the bifurcation parameter τ, we let $n = 0, 1, 2, \cdots$ and get

$$\tau_{0,0} = 19.8556, \quad \tau_{0,1} = 89.0008, \quad \tau_{0,2} = 158.1459, \quad \cdots$$
$$\tau_{1,0} = 20.2593, \quad \tau_{1,1} = 90.1550, \quad \tau_{1,2} = 160.0507, \quad \cdots$$
$$\tau_{2,0} = 21.5642, \quad \tau_{2,1} = 93.9009, \quad \tau_{2,2} = 166.2375, \quad \cdots$$

By the definition of τ_0 in Theorem 2.6, it is easy to know $\tau_0 = \min\limits_{n \in I^*}\{\tau_{n,0}\} = 19.8556$. According to (a) and (e) of Figure 1, the fact shows the positive steady state $E_*(0.3457, 0.1397)$ of system (1.4) is asymptotically stable for $\tau = 16 < \tau_0$. Clearly, when $\tau = 19.86$ lies in a small right-side neighborhood of τ_0, system (1.4) has one stable spatial homogeneous periodic solution, see (b) and (f) of Figure 1. In order to study the influence of spatial factor on system (1.4), we choose the time delay $\tau = 21.5 > \tau_{1,0} = 20.2593$ to simulate the dynamics behavior of predator and prey, (c) and (g) of Figure 1 indicate that system (1.4) has one spatial inhomogeneous periodic solution. These simulation results support Theorem 2.6 and Theorem B.1. Lastly, the simulation results ((d)

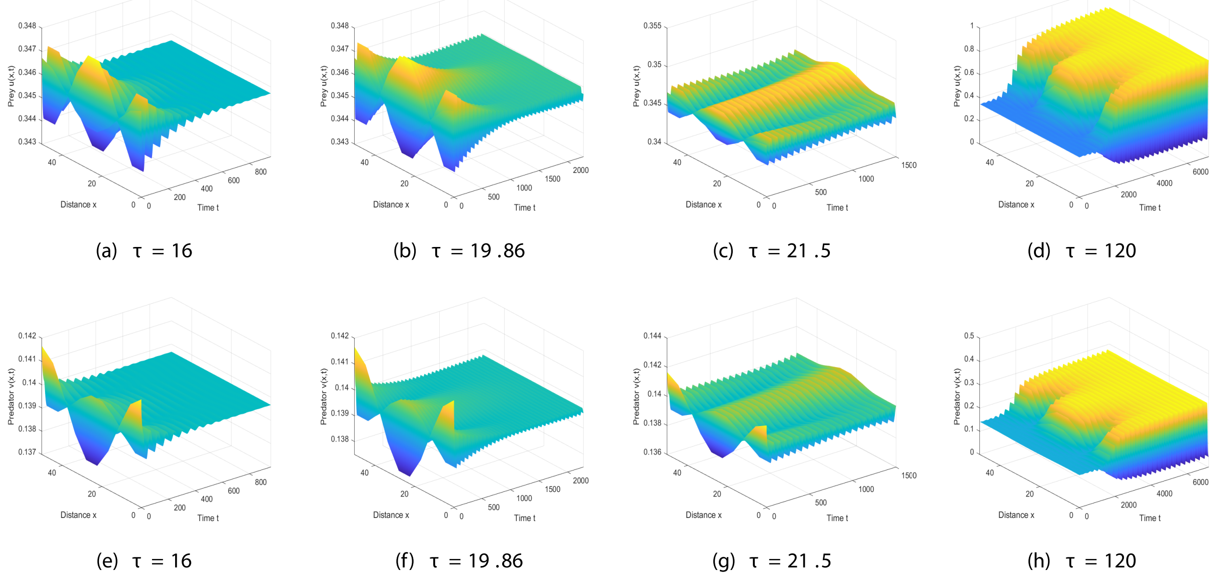

FIGURE 1. we use the initial values $u(x, 0) = 0.3457 + 0.001 \cos x$ and $v(x, 0) = 0.1397 + 0.002 \cos x$. (a)–(d) show the dynamical behavior of the prey with $\tau = 16, 19.86, 21.5, 120$, respectively; (e)–(h) show the dynamical behavior of the predator with $\tau = 16, 19.86, 21.5, 120$, respectively.

and (h) of Figure 1) of system (1.4) at $\tau = 120 > \tau_1$ show that there is at least one positive periodic solution for system (1.4), where $\tau_1 = \min_{n \in I^*} \{\tau_{n,1}\} = 89.0008$.

5. Conclusion

In this paper, we investigate a delayed diffusive predator–prey system with the fear effect and Holling type III functional response under Neumann boundary condition. By analysing, we give the conditions for the existence of one unique positive constant steady state $E_*(u_*, v_*)$ of system (1.4). Next taking the time delay τ as a bifurcation parameter, we get that if the assumptions (H_0)–(H_2) hold, there is a Hopf bifurcation occurring at $\tau = \tau_{n,j}$. Unlike previous research methods, we use the new research method to prove the existence of periodic solutions of system (1.4) away from bifurcation points $\tau_{n,j}$. First we use the iterative method to obtain the global attraction domain G_* of system (1.4), which shows that periodic solutions of this system are uniformly bounded. Then, by proving the global attractivity for system (1.4) without delay at positive steady state $E_*(u_*, v_*)$, we have the uniform boundedness of periods of periodic solutions. At last, applying the global Hopf bifurcation theorem of Wu, we know system (1.4) has at least one positive periodic solution for $\tau > \tau_1$. Moreover, we also give μ_2 that determines the direction of local Hopf bifurcation and β_2 that determines the stability of bifurcating periodic solutions by the center manifold theorem and normal form method. It is clear from the simulation results that our analysis results are reasonable.

Appendix A. The proof of Theorem 2.1

In this Appendix, we investigate the conditions that the unique positive constant steady state of system (1.4) exists. Positive constant steady states of system (1.4) always satisfy

$$\frac{r}{1+\beta v} - ru - \frac{buv}{a+u^2} = 0, \quad \frac{cu^2}{a+u^2} - v = 0. \tag{A.1}$$

Sorting out the above equations, we get

$$\varphi(u) = \bar{A}u^7 + \bar{B}u^6 + \bar{C}u^5 + \bar{D}u^4 + \bar{E}u^3 + \bar{F}u^2 + \bar{G}u + \bar{H} = 0, \tag{A.2}$$

where

$$\begin{aligned} &\bar{A} = -r(1 + c\beta), \quad \bar{B} = r, \quad \bar{C} = -ar(3 + 2c\beta) - bc(1 + c\beta), \quad \bar{D} = 3ar, \\ &\bar{E} = -a^2r(3 + c\beta) - abc, \quad \bar{F} = 3a^2r, \quad \bar{G} = -a^3r, \quad \bar{H} = a^3r. \end{aligned}$$

Obviously, the unique positive constant steady state $E_*(u_*, v_*)$ of system (1.4) exists if and only if equation (A.2) have one positive root. Moreover, thanks to $\varphi(+\infty) = -\infty$ and $\varphi(0) = \bar{H} > 0$, if $\varphi(u)$ is a monotonically decreasing function, then there is only one positive root for $\varphi(u) = 0$.

It follows from equation (A.2) that

$$\begin{aligned} &\varphi'(u) = 7\bar{A}u^6 + 6\bar{B}u^5 + 5\bar{C}u^4 + 4\bar{D}u^3 + 3\bar{E}u^2 + 2\bar{F}u + \bar{G}, \\ &\varphi^{''}(u) = 2(21\bar{A}u^5 + 15\bar{B}u^4 + 10\bar{C}u^3 + 6\bar{D}u^2 + 3\bar{E}u + \bar{F}), \\ &\varphi^{(3)}(u) = 6(35\bar{A}u^4 + 20\bar{B}u^3 + 10\bar{C}u^2 + 4\bar{D}u + \bar{E}), \\ &\varphi^{(4)}(u) = 24(35\bar{A}u^3 + 15\bar{B}u^2 + 5\bar{C}u + \bar{D}), \\ &\varphi^{(5)}(u) = 120(21\bar{A}u^2 + 6\bar{B}u + \bar{C}). \end{aligned} \tag{A.3}$$

First of all, the discriminant $\Delta = 36\bar{B}^2 - 84\bar{A}\bar{C}$ has two cases as follows.

Case 1: $\Delta > 0$

When $\Delta > 0$, $\varphi^{(5)}(u) = 0$ has two positive roots,

$$u_1^{\pm} = \frac{-3\bar{B} \pm \sqrt{9\bar{B}^2 - 21\bar{A}\bar{C}}}{21\bar{A}},$$

where $u_1^+ < u_1^-$. Therefore, $\varphi^{(4)}(u)$ is decreasing monotonically in $(0, u_1^+) \cup (u_1^-, +\infty)$ and increasing monotonically in $[u_1^+, u_1^-]$, then we have the following results:

(i) If $\varphi^{(4)}(u_1^+) \geq 0$, then $\varphi^{(4)}(u) = 0$ exists at least a positive root u_2^1, where $u_2^1 > u_1^-$.
(ii) If $\varphi^{(4)}(u_1^+) < 0$ and $\varphi^{(4)}(u_1^-) > 0$, then $\varphi^{(4)}(u) = 0$ exists three positive roots u_2^2, u_2^3 and u_2^4, where $u_2^2 < u_2^3 < u_2^4$.
(iii) If $\varphi^{(4)}(u_1^-) \leq 0$, then $\varphi^{(4)}(u) = 0$ exists at least a positive root u_2^5, where $u_2^5 < u_1^+$.

The analysis of (i) and (iii) is similar. For (i) (respectively, (iii)), $\varphi^{(3)}(u)$ increases in $(0, u_2^1]$ (respectively, $(0, u_2^5]$) and decreases in $(u_2^1, +\infty)$ (respectively, $(u_2^5, +\infty)$). Then for $\varphi^{(3)}(u)$, there exist the analytical results $(a1)$ and $(a2)$.

$(a1)$ Suppose $\varphi^{(3)}(u_2^1) \leq 0$ or $\varphi^{(3)}(u_2^5) \leq 0$, then $\varphi''(u)$ is a monotonically decreasing function and $\varphi''(u) = 0$ has a positive root u_4^1. Hence, $\varphi^{'}(u)$ first increases monotonically and then decreases monotonically. As a result, when $\varphi'(u_4^1) \leq 0$, then $\varphi'(u) \leq 0$, which implies $\varphi(u)$ is monotonically decreasing for all $u > 0$. When $\varphi'(u_4^1) > 0$, then $\varphi'(u) = 0$ has two positive roots u_5^1 and u_5^2, where $u_5^1 < u_5^2$, which implies $\varphi(u)$ is monotonically decreasing in $(0, u_5^1) \cup (u_5^2, +\infty)$ and increasing in $[u_5^1, u_5^2]$. Only $\varphi(u_5^1) > 0$ or $\varphi(u_5^2) < 0$ holds, $\varphi(u) = 0$ has one positive root.

$(a2)$ Suppose $\varphi^{(3)}(u_2^1) > 0$ or $\varphi^{(3)}(u_2^5) > 0$, then $\varphi^{(3)}(u) = 0$ has two positive roots u_3^1 and u_3^2, where $u_3^1 < u_3^2$. Therefore, $\varphi''(u)$ is a monotonically decreasing function for $u \in (0, u_3^1) \cup (u_3^2, +\infty)$ and increasing function for $u \in [u_3^1, u_3^2]$. Then, the analysis results $(b1)$,$(b2)$ and $(b3)$ of $\varphi^{''}(u)$ are shown below:

($b1$) If $\varphi^{''}(u_3^1) \geq 0$, then $\varphi^{'}(u) = 0$ has at least a positive root u_4^2, where $u_4^2 > u_3^2$.
($b2$) If $\varphi^{''}(u_3^1) < 0$ and $\varphi^{''}(u_3^2) > 0$, then $\varphi'(u) = 0$ has three positive roots u_4^3, u_4^4 and u_4^5, where $u_4^3 < u_4^4 < u_4^5$.
($b3$) If $\varphi^{''}(u_3^2) \leq 0$, then $\varphi^{'}(u) = 0$ has at least a positive root u_4^6, where $u_4^6 < u_3^1$.

On the one hand, for ($b1$) (respectively, ($b3$)), $\varphi'(u)$ is increasing monotonically for $0 < u \leq u_4^2$ (respectively, $0 < u \leq u_4^6$) and decreasing monotonically for $u > u_4^2$ (respectively, $u > u_4^6$). Now assuming $\varphi'(u_4^2) \leq 0$ or $\varphi'(u_4^6) \leq 0$, $\varphi(u)$ is a decreasing function for $u \in \mathbb{R}_+$. Assuming $\varphi'(u_4^2) > 0$ or $\varphi'(u_4^6) > 0$, there are two positive roots $u_5^3 < u_5^4$ for $\varphi'(u) = 0$, which means $\varphi(u)$ first decreases in $(0, u_5^3) \cup (u_5^4, +\infty)$ and then increases in $[u_5^3, u_5^4]$. By analysing, we know $\varphi(u) = 0$ exists a positive root if $\varphi(u_5^3) > 0$ or $\varphi(u_5^4) < 0$. On the other hand, for ($b2$), we have the five results: Firstly, suppose $\varphi'(u_4^4) \geq 0$, then $\varphi(u)$ decreases in $(0, u_5^5) \cup (u_5^6, +\infty)$ and increases in $[u_5^5, u_5^6]$, where u_5^5 and u_5^6 are positive roots of $\varphi^{'}(u) = 0$. Obviously, $\varphi(u) = 0$ has a positive root under assumption $\varphi(u_5^5) > 0$ or $\varphi(u_5^6) < 0$. Secondly, suppose $\varphi'(u_4^3) > 0$, $\varphi'(u_4^4) < 0$ and $\varphi'(u_4^5) > 0$, then $\varphi'(u) = 0$ has four positive roots u_5^7, u_5^8, u_5^9 and u_5^{10}, where $u_5^7 < u_5^8 < u_5^9 < u_5^{10}$. It is easy to see that $\varphi(u)$ is decreasing monotonically in $(0, u_5^7) \cup (u_5^8, u_5^9) \cup (u_5^{10}, +\infty)$ and increasing monotonically in $[u_5^7, u_5^8] \cup [u_5^9, u_5^{10}]$. Hence, if A_1 or A_2 is satisfied, then $\varphi(u) = 0$ exists a positive root, where $A_1 = \{\varphi(u_5^7) > 0, \varphi(u_5^9) > 0 \text{ } or \text{ } \varphi(u_5^{10}) < 0\}$ and $A_2 = \{\varphi(u_5^8) < 0, \varphi(u_5^{10}) < 0\}$. Thirdly, suppose $\varphi'(u_4^3) \leq 0$ and $\varphi'(u_4^5) > 0$, there are at least two positive roots u_5^{11} and u_5^{12} for $\varphi^{'}(u) = 0$, where $u_4^3 < u_5^{11} < u_5^{12}$. Through analysis, we obtain $\varphi(u) = 0$ has a positive root if and only if $\varphi(u_5^{11}) > 0$ or $\varphi(u_5^{12}) < 0$ holds. Fourthly, suppose $\varphi'(u_4^3) > 0$ and $\varphi'(u_4^5) \leq 0$, then $\varphi'(u) = 0$ has at least two positive roots $u_5^{13} < u_5^{14}$ on the left-side of u_4^5. Similarly, if $\varphi(u_5^{13}) > 0$ or $\varphi(u_5^{14}) < 0$, then there is a positive root for $\varphi(u) = 0$. Fifthly, suppose $\varphi'(u_4^3) \leq 0$ and $\varphi'(u_4^5) \leq 0$, then $\varphi(u)$ is a monotonic decreasing function for all $u > 0$.

For (ii), $\varphi^{(3)}(u)$ is a monotonic increasing function for $u \in (0, u_2^2) \cup (u_2^3, u_2^4)$ and decreasing function for $u \in [u_2^2, u_2^3] \cup [u_2^4, +\infty)$. Based on analysis, we obtain the following results:

($c1$) Supposing $\varphi^{(3)}(u_2^2) > 0$ and $\varphi^{(3)}(u_2^3) \geq 0$, then there are at least two positive roots u_3^3 and u_3^4 in $\varphi^{(3)}(u) = 0$, where $u_3^3 < u_2^3 < u_3^4$.
($c2$) Supposing $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$ and $\varphi^{(3)}(u_2^4) > 0$, then there are four positive roots u_3^5, u_3^6, u_3^7 and u_3^8 in $\varphi^{(3)}(u) = 0$, where $u_3^5 < u_3^6 < u_3^7 < u_3^8$.
($c3$) Supposing $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$ and $\varphi^{(3)}(u_2^4) \leq 0$, then there are at least two positive roots u_3^9 and u_3^{10} in $\varphi^{(3)}(u) = 0$, where $u_3^9 < u_3^{10} < u_2^4$.
($c4$) Supposing $\varphi^{(3)}(u_2^2) \leq 0$ and $\varphi^{(3)}(u_2^4) > 0$, then there are at least two positive roots u_3^{11} and u_3^{12} in $\varphi^{(3)}(u) = 0$, where $u_2^2 < u_3^{11} < u_3^{12}$.
($c5$) Supposing $\varphi^{(3)}(u_2^2) \leq 0$ and $\varphi^{(3)}(u_2^4) \leq 0$, then $\varphi^{(3)}(u) \leq 0$, which means $\varphi^{''}(u)$ is decreasing monotonically in $\mathbb{R}_+$.

Because the analysis processes of ($c1$), ($c3$) and ($c4$) are similar, we can just analyze one case. For ($c1$), we have the following results:

($d1$) If $\varphi^{''}(u_3^3) \geq 0$, then $\varphi^{''}(u) = 0$ exists at least one positive root u_4^7 on the right-side of u_3^3, which indicates $\varphi'(u)$ first increases for $u > u_4^7$ and then decreases for $u \leq u_4^7$.
($d2$) If $\varphi^{''}(u_3^3) < 0$ and $\varphi^{''}(u_3^4) > 0$, then $\varphi^{''}(u) = 0$ exists three positive roots u_4^8, u_4^9 and u_4^{10} with $u_4^8 < u_4^9 < u_4^{10}$, which indicates $\varphi'(u)$ first increases for $0 < u < u_4^8$ and $u_4^9 \leq u \leq u_4^{10}$ and then decreases for $u_4^8 \leq u < u_4^9$ and $u > u_4^{10}$.
($d3$) If $\varphi^{''}(u_3^4) \leq 0$, then $\varphi^{''}(u) = 0$ exists at least one positive root u_4^{11} on the left-side of u_3^4, which indicates $\varphi'(u)$ first increases for $u > u_4^{11}$ and then decreases for $u \leq u_4^{11}$.

Therefore, for ($d1$) or ($d3$), if $\varphi'(u_4^7) \leq 0$ or $\varphi'(u_4^{11}) \leq 0$ is satisfied, then $\varphi(u)$ is monotonic and decreasing for $u \in \mathbb{R}_+$. If $\varphi'(u_4^7) > 0$ or $\varphi'(u_4^{11}) > 0$ is satisfied, then $\varphi'(u) = 0$ has two positive roots u_5^{15} and u_5^{16}, where $u_5^{15} < u_5^{16}$. Then, under assumption $\varphi(u_5^{15}) > 0$ or $\varphi(u_5^{16}) < 0$, $\varphi(u) = 0$ has only one positive root. As for ($d2$), we can obtain five possible results: under $\varphi'(u_4^9) \geq 0$, $\varphi(u)$ is decreasing monotonically in $(0, u_5^{17}) \cup (u_5^{18}, +\infty)$ and increasing monotonically in $[u_5^{17}, u_5^{18}]$, where u_5^{17} and u_5^{18} are positive roots of $\varphi'(u) = 0$ with $u_5^{17} < u_4^9 < u_5^{18}$. So $\varphi(u) = 0$ has a positive root if $\varphi(u_5^{17}) > 0$ or $\varphi(u_5^{18}) < 0$. Under $\varphi'(u_4^8) > 0$, $\varphi'(u_4^9) < 0$ and $\varphi'(u_4^{10}) > 0$, $\varphi(u)$ is decreasing monotonically in $(0, u_5^{19}) \cup (u_5^{20}, u_5^{21}) \cup (u_5^{22}, +\infty)$ and increasing monotonically in $[u_5^{19}, u_5^{20}] \cup [u_5^{21}, u_5^{22}]$, where u_5^{19}, u_5^{20}, u_5^{21} and u_5^{22} are positive roots of $\varphi'(u) = 0$. Thereby, if $A_3 = \{\varphi(u_5^{19}) > 0, \varphi(u_5^{21}) > 0 \ or \ \varphi(u_5^{22}) < 0\}$ or $A_4 = \{\varphi(u_5^{20}) < 0, \varphi(u_5^{22}) < 0\}$ holds, there is one positive root for $\varphi(u) = 0$. Under $\varphi'(u_4^8) \leq 0$ and $\varphi'(u_4^{10}) > 0$, $\varphi(u)$ is decreasing monotonically in $(0, u_5^{23}) \cup (u_5^{24}, +\infty)$ and increasing monotonically in $[u_5^{23}, u_5^{24}]$, where u_5^{23} and u_5^{24} are positive roots of $\varphi'(u) = 0$ with $u_4^8 < u_5^{23} < u_5^{24}$. Then when $\varphi(u_5^{23}) > 0$ or $\varphi(u_5^{24}) < 0$, there is one positive root for $\varphi(u) = 0$. Under $\varphi'(u_4^8) > 0$ and $\varphi'(u_4^{10}) \leq 0$, $\varphi(u)$ is decreasing monotonically for $u \in (0, u_5^{25}) \cup (u_5^{26}, +\infty)$ and increasing monotonically for $u \in [u_5^{25}, u_5^{26}]$, where u_5^{25} and u_5^{26} are positive roots of $\varphi'(u) = 0$ and $u_5^{25} < u_5^{26} < u_4^{10}$. Similarly, under $\varphi(u_5^{25}) > 0$ or $\varphi(u_5^{26}) < 0$, $\varphi(u) = 0$ has a positive root. Finally, under $\varphi'(u_4^8) \leq 0$ and $\varphi'(u_4^{10}) \leq 0$, for all $u > 0$, $\varphi(u)$ is monotonic and decreasing.

For $(c2)$, we get the analysis results indicated below:

$(e1)$ If $\varphi''(u_3^5) \geq 0$ and $\varphi''(u_3^7) \geq 0$, then $\varphi''(u) = 0$ exists at least one positive root u_4^{12}, and $u_3^7 < u_4^{12}$. It implies that $\varphi'(u)$ first increases in $(0, u_4^{12})$ and then decreases in $[u_4^{12}, +\infty)$.

$(e2)$ If $\varphi''(u_3^5) \geq 0$, $\varphi''(u_3^7) < 0$ and $\varphi''(u_3^8) > 0$, then $\varphi''(u) = 0$ exists at least three positive roots u_4^{13}, u_4^{14} and u_4^{15}, and $u_3^5 < u_4^{13} < u_4^{14} < u_4^{15}$. It implies that $\varphi'(u)$ increases monotonically in $(0, u_4^{13}) \cup (u_4^{14}, u_4^{15})$ and decreases monotonically in $[u_4^{13}, u_4^{14}] \cup [u_4^{15}, +\infty)$.

$(e3)$ If $\varphi''(u_3^5) \geq 0$ and $\varphi''(u_3^8) \leq 0$, then $\varphi''(u) = 0$ exists at least one positive root u_4^{16}, and $u_3^5 < u_4^{16} < u_3^8$. It implies that $\varphi'(u)$ first increases in $(0, u_4^{16})$ and then decreases in $[u_4^{16}, +\infty)$.

$(e4)$ If $\varphi''(u_3^5) < 0$ and $\varphi''(u_3^7) \geq 0$, then $\varphi''(u) = 0$ exists at least three positive roots u_4^{17}, u_4^{18} and u_4^{19}, and $u_4^{17} < u_4^{18} < u_3^7 < u_4^{19}$. It implies that $\varphi'(u)$ increases monotonically in $(0, u_4^{17}) \cup (u_4^{18}, u_4^{19})$ and decreases monotonically in $[u_4^{17}, u_4^{18}] \cup [u_4^{19}, +\infty)$.

$(e5)$ If $\varphi''(u_3^5) < 0$, $\varphi''(u_3^6) > 0$, $\varphi''(u_3^7) < 0$ and $\varphi''(u_3^8) > 0$, then $\varphi''(u) = 0$ exists five positive roots u_4^{20}, u_4^{21}, u_4^{22}, u_4^{23} and u_4^{24}, and $u_4^{20} < u_4^{21} < u_4^{22} < u_4^{23} < u_4^{24}$. It implies that $\varphi'(u)$ increases monotonically in $(0, u_4^{20}) \cup (u_4^{21}, u_4^{22}) \cup (u_4^{23}, u_4^{24})$ and decreases monotonically in $[u_4^{20}, u_4^{21}] \cup [u_4^{22}, u_4^{23}] \cup [u_4^{24}, +\infty)$.

$(e6)$ If $\varphi''(u_3^5) < 0$, $\varphi''(u_3^6) > 0$ and $\varphi''(u_3^8) \leq 0$, then $\varphi''(u) = 0$ exists at least three positive roots u_4^{25}, u_4^{26} and u_4^{27}, and $u_4^{25} < u_4^{26} < u_4^{27} < u_3^8$. It implies that $\varphi'(u)$ increases monotonically in $(0, u_4^{25}) \cup (u_4^{26}, u_4^{27})$ and decreases monotonically in $[u_4^{25}, u_4^{26}] \cup [u_4^{27}, +\infty)$.

$(e7)$ If $\varphi''(u_3^6) \leq 0$ and $\varphi''(u_3^8) > 0$, then $\varphi''(u) = 0$ exists at least three positive roots u_4^{28}, u_4^{29} and u_4^{30}, and $u_4^{28} < u_3^6 < u_4^{29} < u_4^{30}$. It implies that $\varphi'(u)$ increases monotonically in $(0, u_4^{28}) \cup (u_4^{29}, u_4^{30})$ and decreases monotonically in $[u_4^{28}, u_4^{29}] \cup [u_4^{30}, +\infty)$.

$(e8)$ If $\varphi''(u_3^6) \leq 0$ and $\varphi''(u_3^8) \leq 0$, then $\varphi''(u) = 0$ exists at least one positive root u_4^{31}, and $u_4^{31} < u_3^6$. It implies that $\varphi'(u)$ first increases in $(0, u_4^{31})$ and then decreases in $[u_4^{31}, +\infty)$.

Then for $(e1)$ ($(e3)$ or $(e8)$), assuming $\varphi'(u_4^{12}) \leq 0$ ($\varphi'(u_4^{16}) \leq 0$ or $\varphi'(u_4^{31}) \leq 0$), then $\varphi'(u) \leq 0$, thus $\varphi(u)$ is a monotonically decreasing function. Assume $\varphi'(u_4^{12}) > 0$ ($\varphi'(u_4^{16}) > 0$ or $\varphi'(u_4^{31}) > 0$), then $\varphi'(u) = 0$ has two positive roots u_5^{27} and u_5^{28}, where $u_5^{27} < u_5^{28}$. By computation, when $\varphi(u_5^{27}) > 0$ or $\varphi(u_5^{28}) < 0$, we know $\varphi(u) = 0$ exists only one positive root. Next, the analytical results $(f1)$–$(f5)$ of $(e2)$, $(e4)$, $(e6)$ and $(e7)$ are shown below:

$(f1)$ If $\varphi'(u_4^{14}) \geq 0$ ($\varphi'(u_4^{18}) \geq 0$ or $\varphi'(u_4^{26}) \geq 0$ or $\varphi'(u_4^{29}) \geq 0$), then there are positive roots u_5^{29} and u_5^{30} for $\varphi'(u) = 0$ with $u_5^{29} < u_5^{30}$. Then, the equation $\varphi(u)$ is decreasing monotonically for $(0, u_5^{29}) \cup (u_5^{30}, +\infty)$ and increasing monotonically in $[u_5^{29}, u_5^{30}]$, so $\varphi(u) = 0$ has only one positive root under $\varphi(u_5^{29}) > 0$ or $\varphi(u_5^{30}) < 0$.

$(f2)$ If $B_1 = \{\varphi'(u_4^{13}) > 0, \varphi'(u_4^{14}) < 0, \varphi'(u_4^{15}) > 0\}$ ($B_2 = \{\varphi'(u_4^{17}) > 0, \varphi'(u_4^{18}) < 0, \varphi'(u_4^{19}) > 0\}$ or $B_3 = \{\varphi'(u_4^{25}) > 0, \varphi'(u_4^{26}) < 0, \varphi'(u_4^{27}) > 0\}$ or $B_4 = \{\varphi'(u_4^{28}) > 0, \varphi'(u_4^{29}) < 0, \varphi'(u_4^{30}) > 0\}$), then there are positive roots u_5^{31}, u_5^{32}, u_5^{33} and u_5^{34} for $\varphi'(u) = 0$ with $u_5^{31} < u_5^{32} < u_5^{33} < u_5^{34}$. It is easy to see that $\varphi(u)$ decreases in $(0, u_5^{31}) \cup (u_5^{32}, u_5^{33}) \cup (u_5^{34}, +\infty)$ and increases in $[u_5^{31}, u_5^{32}] \cup [u_5^{33}, u_5^{34}]$. By analyzing, if $A_5 = \{\varphi(u_5^{31}) > 0, \varphi(u_5^{33}) > 0 \ or \ \varphi(u_5^{34}) < 0\}$ or $A_6 = \{\varphi(u_5^{32}) < 0, \varphi(u_5^{34}) < 0\}$ is satisfied, then $\varphi(u) = 0$ has only one positive root.

($f3$) If $B_5 = \{\varphi'(u_4^{13}) \le 0, \varphi'(u_4^{15}) > 0\}$ ($B_6 = \{\varphi'(u_4^{17}) \le 0, \varphi'(u_4^{19}) > 0\}$ or $B_7 = \{\varphi'(u_4^{25}) \le 0, \varphi'(u_4^{27}) > 0\}$ or $B_8 = \{\varphi'(u_4^{28}) \le 0, \varphi'(u_4^{30}) > 0\}$), then $\varphi'(u) = 0$ has at least two positive roots u_5^{35} and u_5^{36}, where $u_5^{35} < u_5^{36}$. Similar to ($f1$), when $\varphi(u_5^{35}) > 0$ or $\varphi(u_5^{36}) < 0$, there is only one positive root for $\varphi(u) = 0$.

($f4$) If $B_9 = \{\varphi'(u_4^{13}) > 0, \varphi'(u_4^{15}) \le 0\}$ ($B_{10} = \{\varphi'(u_4^{17}) > 0, \varphi'(u_4^{19}) \le 0\}$ or $B_{11} = \{\varphi'(u_4^{25}) > 0, \varphi'(u_4^{27}) \le 0\}$ or $B_{12} = \{\varphi'(u_4^{28}) > 0, \varphi'(u_4^{30}) \le 0\}$), then $\varphi'(u) = 0$ has at least two positive roots u_5^{37} and u_5^{38}, where $u_5^{37} < u_5^{38}$. Similar to ($f1$), under $\varphi(u_5^{37}) > 0$ or $\varphi(u_5^{38}) < 0$, $\varphi(u) = 0$ has only one positive root.

($f5$) If $B_{13} = \{\varphi'(u_4^{13}) \le 0, \varphi'(u_4^{15}) \le 0\}$ ($B_{14} = \{\varphi'(u_4^{17}) \le 0, \varphi'(u_4^{19}) \le 0\}$ or $B_{15} = \{\varphi'(u_4^{25}) \le 0, \varphi'(u_4^{27}) \le 0\}$ or $B_{16} = \{\varphi'(u_4^{28}) \le 0, \varphi'(u_4^{30}) \le 0\}$), the equation $\varphi(u) = 0$ decreases monotonically for all $u > 0$.

As for ($e5$), the analysis results are as follow:

($g1$) Assume $\varphi'(u_4^{21}) \ge 0$ and $\varphi'(u_4^{23}) \ge 0$, then there are at least two positive roots u_5^{39} and u_5^{40} for $\varphi'(u) = 0$, where $u_5^{39} < u_5^{40}$. Hence, $\varphi(u)$ decreases monotonically in $(0, u_5^{39}) \cup (u_5^{40}, +\infty)$ and increases monotonically in $[u_5^{39}, u_5^{40}]$. Consequently, if $\varphi(u_5^{39}) > 0$ or $\varphi(u_5^{40}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

($g2$) Assume $\varphi'(u_4^{21}) \ge 0$, $\varphi'(u_4^{23}) < 0$ and $\varphi'(u_4^{24}) > 0$, then there are at least four positive roots u_5^{41}, u_5^{42}, u_5^{43} and u_5^{44} for $\varphi'(u) = 0$, where $u_5^{41} < u_5^{42} < u_5^{43} < u_5^{44}$. Hence, $\varphi(u)$ decreases monotonically in $(0, u_5^{41}) \cup (u_5^{42}, u_5^{43}) \cup (u_5^{44}, +\infty)$ and increases monotonically in $[u_5^{41}, u_5^{42}] \cup [u_5^{43}, u_5^{44}]$. Consequently, if $A_7 = \{\varphi(u_5^{41}) > 0, \varphi(u_5^{43}) > 0 \textit{ or } \varphi(u_5^{44}) < 0\}$ or $A_8 = \{\varphi(u_5^{42}) < 0, \varphi(u_5^{44}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

($g3$) Assume $\varphi'(u_4^{21}) \ge 0$ and $\varphi'(u_4^{24}) \le 0$, then there are at least two positive roots u_5^{45} and u_5^{46} for $\varphi'(u) = 0$, where $u_5^{45} < u_5^{46}$. Similar to ($g1$), if $\varphi(u_5^{45}) > 0$ or $\varphi(u_5^{46}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

($g4$) Assume $\varphi'(u_4^{20}) > 0$, $\varphi'(u_4^{21}) < 0$ and $\varphi'(u_4^{23}) \ge 0$, then there are at least four positive roots u_5^{47}, u_5^{48}, u_5^{49} and u_5^{50} for $\varphi'(u) = 0$, where $u_5^{47} < u_5^{48} < u_5^{49} < u_5^{50}$. Similar to ($g2$), if $A_9 = \{\varphi(u_5^{47}) > 0, \varphi(u_5^{49}) > 0 \textit{ or } \varphi(u_5^{50}) < 0\}$ or $A_{10} = \{\varphi(u_5^{48}) < 0, \varphi(u_5^{50}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

($g5$) Assume $\varphi'(u_4^{20}) > 0$, $\varphi'(u_4^{21}) < 0$, $\varphi'(u_4^{22}) > 0$, $\varphi'(u_4^{23}) < 0$ and $\varphi'(u_4^{24}) > 0$, then there are six positive roots u_5^{51}, u_5^{52}, u_5^{53}, u_5^{54}, u_5^{55} and u_5^{56} for $\varphi'(u) = 0$, where $u_5^{51} < u_5^{52} < u_5^{53} < u_5^{54} < u_5^{55} < u_5^{56}$. Hence, $\varphi(u)$ decreases monotonically in $(0, u_5^{51}) \cup (u_5^{52}, u_5^{53}) \cup (u_5^{54}, u_5^{55}) \cup (u_5^{56}, +\infty)$ and increases monotonically in $[u_5^{51}, u_5^{52}] \cup [u_5^{53}, u_5^{54}] \cup [u_5^{55}, u_5^{56}]$. Consequently, if $A_{11} = \{\varphi(u_5^{51}) > 0, \varphi(u_5^{53}) > 0, \varphi(u_5^{55}) > 0 \textit{ or } \varphi(u_5^{56}) < 0\}$ or $A_{12} = \{\varphi(u_5^{51}) > 0 \textit{ or } \varphi(u_5^{52}) < 0, \varphi(u_5^{54}) < 0, \varphi(u_5^{56}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

($g6$) Assume $\varphi'(u_4^{20}) > 0$, $\varphi'(u_4^{21}) < 0$, $\varphi'(u_4^{22}) > 0$ and $\varphi'(u_4^{24}) \le 0$, then there are at least four positive roots u_5^{57}, u_5^{58}, u_5^{59} and u_5^{60} for $\varphi'(u) = 0$, where $u_5^{57} < u_5^{58} < u_5^{59} < u_5^{60}$. Similar to ($g2$), if $A_{13} = \{\varphi(u_5^{57}) > 0, \varphi(u_5^{59}) > 0 \textit{ or } \varphi(u_5^{60}) < 0\}$ or $A_{14} = \{\varphi(u_5^{58}) < 0, \varphi(u_5^{60}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

($g7$) Assume $\varphi'(u_4^{20}) > 0$, $\varphi'(u_4^{22}) \le 0$ and $\varphi'(u_4^{24}) > 0$, then there are at least four positive roots u_5^{61}, u_5^{62}, u_5^{63} and u_5^{64} for $\varphi'(u) = 0$, where $u_5^{61} < u_5^{62} < u_5^{63} < u_5^{64}$. Similar to ($g2$), if $A_{15} = \{\varphi(u_5^{61}) > 0, \varphi(u_5^{63}) > 0 \textit{ or } \varphi(u_5^{64}) < 0\}$ or $A_{16} = \{\varphi(u_5^{62}) < 0, \varphi(u_5^{64}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

($g8$) Assume $\varphi'(u_4^{20}) > 0$, $\varphi'(u_4^{22}) \le 0$ and $\varphi'(u_4^{24}) \le 0$, then there are at least two positive roots u_5^{65} and u_5^{66} for $\varphi'(u) = 0$, where $u_5^{65} < u_5^{66}$. Similar to ($g1$), if $\varphi(u_5^{65}) > 0$ or $\varphi(u_5^{66}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

($g9$) Assume $\varphi'(u_4^{20}) \le 0$ and $\varphi'(u_4^{23}) \ge 0$, then there are at least two positive roots u_5^{67} and u_5^{68} for $\varphi'(u) = 0$, where $u_5^{67} < u_5^{68}$. Similar to ($g1$), if $\varphi(u_5^{67}) > 0$ or $\varphi(u_5^{68}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

($g10$) Assume $\varphi'(u_4^{20}) \le 0$, $\varphi'(u_4^{22}) > 0$, $\varphi'(u_4^{23}) < 0$ and $\varphi'(u_4^{24}) > 0$, then there are at least four positive roots u_5^{69}, u_5^{70}, u_5^{71} and u_5^{72} for $\varphi'(u) = 0$, where $u_5^{69} < u_5^{70} < u_5^{71} < u_5^{72}$. Similar to ($g2$), if $A_{17} = \{\varphi(u_5^{69}) > 0, \varphi(u_5^{71}) > 0 \textit{ or } \varphi(u_5^{72}) < 0\}$ or $A_{18} = \{\varphi(u_5^{70}) < 0, \varphi(u_5^{72}) < 0\}$, then there is only one positive root for $\varphi(u) = 0$.

$(g11)$ Assume $\varphi^{'}(u_4^{20}) \leq 0$, $\varphi'(u_4^{22}) > 0$ and $\varphi^{'}(u_4^{24}) \leq 0$, then there are at least two positive roots u_5^{73} and u_5^{74} for $\varphi^{'}(u) = 0$, where $u_5^{73} < u_5^{74}$. Similar to $(g1)$, if $\varphi(u_5^{73}) > 0$ or $\varphi(u_5^{74}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

$(g12)$ Assume $\varphi^{'}(u_4^{20}) \leq 0$, $\varphi'(u_4^{22}) \leq 0$ and $\varphi^{'}(u_4^{24}) > 0$, then there are at least two positive roots u_5^{75} and u_5^{76} for $\varphi^{'}(u) = 0$, where $u_5^{75} < u_5^{76}$. Similar to $(g1)$, if $\varphi(u_5^{75}) > 0$ or $\varphi(u_5^{76}) < 0$, then there is only one positive root for $\varphi(u) = 0$.

$(g13)$ Assume $\varphi^{'}(u_4^{20}) \leq 0$, $\varphi'(u_4^{22}) \leq 0$ and $\varphi^{'}(u_4^{24}) \leq 0$, then $\varphi(u)$ is decreasing monotonically in $\mathbb{R}_+$.

For $(c5)$, $\varphi^{''}(u) = 0$ has one positive root u_4^{32}. Then under assumption $\varphi'(u_4^{32}) \leq 0$, $\varphi(u)$ is decreasing monotonically for all $u > 0$. Under assumption $\varphi'(u_4^{32}) > 0$, $\varphi'(u) = 0$ has two positive roots u_5^{77} and u_5^{78} with $u_5^{77} < u_5^{78}$. Similarly, when $\varphi(u_5^{77}) > 0$ or $\varphi(u_5^{78}) < 0$, $\varphi(u) = 0$ has a unique positive root.

Case 2: $\Delta \leq 0$

If $\Delta \leq 0$ holds, then $\varphi^{(5)}(u) \leq 0$, which means that $\varphi^{(4)}(u)$ is decreasing and monotonous for any $u > 0$. There are $\varphi^{(4)}(u) > 0$ in $(0, u_2^6)$ and $\varphi^{(4)}(u) \leq 0$ in $[u_2^6, +\infty)$, where u_2^6 is positive root of $\varphi^{(4)}(u) = 0$. The fact shows $\varphi^{(3)}(u)$ first increases monotonically and then decreases monotonically. Then, the analysis of $\varphi^{(3)}(u)$ is as follow: Suppose $\varphi^{(3)}(u_2^6) \leq 0$, then $\varphi^{''}(u)$ is a monotone decreasing function. By now, there is one positive root u_4^{33} in the equation $\varphi^{''}(u) = 0$. Then if $\varphi'(u_4^{33}) \leq 0$ holds, then $\varphi(u)$ is decreasing and monotonic in $\mathbb{R}_+$. If $\varphi'(u_4^{33}) > 0$ holds, then $\varphi'(u) = 0$ has two positive roots u_5^{79} and u_5^{80}, where $u_5^{79} < u_5^{80}$. Obviously, $\varphi(u)$ increases monotonically in $[u_5^{79}, u_5^{80}]$ and decreases monotonically in $(0, u_5^{79}) \cup (u_5^{80}, +\infty)$. Therefore, when $\varphi(u_5^{79}) > 0$ or $\varphi(u_5^{80}) < 0$ is satisfied, $\varphi(u) = 0$ has only one positive root. Suppose $\varphi^{(3)}(u_2^6) > 0$, then $\varphi^{''}(u)$ is a monotone decreasing function in $(0, u_3^{13}) \cup (u_3^{14}, +\infty)$ and increasing function in $[u_3^{13}, u_3^{14}]$. After that, for $\varphi^{''}(u)$, the following results are indicated below:

$(h1)$ When $\varphi^{''}(u_3^{13}) \geq 0$, $\varphi^{''}(u) = 0$ exists at least one positive root u_4^{34}, where $u_4^{34} > u_3^{13}$. Then, $\varphi^{''}(u)$ increases monotonically in $(0, u_4^{34})$ and decreases monotonically in $[u_4^{34}, +\infty)$.

$(h2)$ When $\varphi^{''}(u_3^{13}) < 0$ and $\varphi^{''}(u_3^{14}) > 0$, $\varphi^{''}(u) = 0$ exists three positive roots u_4^{35}, u_4^{36} and u_4^{37}, where $u_4^{35} < u_4^{36} < u_4^{37}$. Then, $\varphi^{''}(u)$ increases monotonically in $(0, u_4^{35}) \cup (u_5^{36}, u_4^{37})$ and decreases monotonically in $[u_4^{35}, u_4^{36}] \cup [u_4^{37}, +\infty)$.

$(h3)$ When $\varphi^{''}(u_3^{14}) \leq 0$, $\varphi^{''}(u) = 0$ exists at least one positive root u_4^{38}, where $u_4^{38} < u_3^{14}$. Then, $\varphi^{''}(u)$ increases monotonically in $(0, u_4^{38})$ and decreases monotonically in $[u_4^{38}, +\infty)$.

For $(h1)$ or $(h3)$, if $\varphi'(u_4^{34}) \leq 0$ or $\varphi'(u_4^{38}) \leq 0$ holds, then $\varphi(u)$ decreases and monotonic for all $u > 0$. If $\varphi'(u_4^{34}) > 0$ or $\varphi'(u_4^{38}) > 0$ holds, then $\varphi'(u) = 0$ has two positive roots u_5^{81} and u_5^{82}, where $u_5^{81} < u_5^{82}$. It is clear to obtain that $\varphi(u)$ is decreasing monotonically for $0 < u < u_5^{81}$ and $u > u_5^{82}$ and increasing monotonically for $u_5^{81} \leq u \leq u_5^{82}$. Consequently, there is one unique positive root for $\varphi(u) = 0$ under $\varphi(u_5^{81}) > 0$ or $\varphi(u_5^{82}) < 0$. In the end, for $(h2)$, we have

$(l1)$ If $\varphi'(u_4^{36}) \geq 0$, then $\varphi'(u) = 0$ has at least two positive roots u_5^{83} and u_5^{84} with $u_5^{83} < u_4^{36} < u_5^{84}$. Obviously, $\varphi(u)$ decreases monotonically for $(0, u_5^{83}) \cup (u_5^{84}, +\infty)$ and increases monotonically for $[u_5^{83}, u_5^{84}]$. Hence, under assumption $\varphi(u_5^{83}) > 0$ or $\varphi(u_5^{84}) < 0$, $\varphi(u) = 0$ has a unique positive root.

$(l2)$ If $\varphi'(u_4^{35}) > 0$, $\varphi'(u_4^{36}) < 0$ and $\varphi'(u_4^{37}) > 0$, then $\varphi'(u) = 0$ has four positive roots u_5^{85}, u_5^{86}, u_5^{87} and u_5^{88} with $u_5^{85} < u_5^{86} < u_5^{87} < u_5^{88}$. Obviously, $\varphi(u)$ decreases monotonically for $(0, u_5^{85}) \cup (u_5^{86}, u_5^{87}) \cup (u_5^{88}, +\infty)$ and increases monotonically for $[u_5^{85}, u_5^{86}] \cup [u_5^{87}, u_5^{88}]$. Hence, under assumption $A_{19} = \{\varphi(u_5^{85}) > 0, \varphi(u_5^{87}) > 0 \ or \ \varphi(u_5^{88}) < 0\}$ or $A_{20} = \{\varphi(u_5^{86}) < 0, \varphi(u_5^{88}) < 0\}$, $\varphi(u) = 0$ has a unique positive root.

$(l3)$ If $\varphi'(u_4^{35}) > 0$ and $\varphi'(u_4^{37}) \leq 0$, then $\varphi'(u) = 0$ has at least two positive roots u_5^{89} and u_5^{90} with $u_5^{89} < u_5^{90} < u_4^{37}$. Similar to $(l1)$, under assumption $\varphi(u_5^{89}) > 0$ or $\varphi(u_5^{90}) < 0$, $\varphi(u) = 0$ has a unique positive root.

$(l4)$ If $\varphi'(u_4^{35}) \leq 0$ and $\varphi'(u_4^{37}) > 0$, then $\varphi'(u) = 0$ has at least two positive roots u_5^{91} and u_5^{92} with $u_4^{35} < u_5^{91} < u_5^{92}$. Similar to $(l1)$, under assumption $\varphi(u_5^{91}) > 0$ or $\varphi(u_5^{92}) < 0$, $\varphi(u) = 0$ has a unique positive root.

$(l5)$ If $\varphi'(u_4^{35}) \leq 0$ and $\varphi'(u_4^{37}) \leq 0$, then $\varphi(u)$ is a monotonic decreasing function in $\mathbb{R}_+$.

In order to facilitate, the above analysis can be summarized as: when (H_0) is satisfied, system (1.4) has one unique positive constant steady state $E_*(u_*, v_*)$, where
(H_0): any one of the conditions (1)–(19) holds,

(1) $\Delta > 0$, $\{\varphi^{(4)}(u_1^+) \geq 0, \varphi^{(3)}(u_2^1) \leq 0\}$ or $\{\varphi^{(4)}(u_1^-) \leq 0, \varphi^{(3)}(u_2^5) \leq 0\}$, $\varphi'(u_4^1) \leq 0$ or $\{\varphi'(u_4^1) > 0, \varphi(u_5^1) > 0$ or $\varphi(u_5^2) < 0\}$;
(2) $\Delta > 0$, $\{\varphi^{(4)}(u_1^+) \geq 0, \varphi^{(3)}(u_2^1) > 0\}$ or $\{\varphi^{(4)}(u_1^-) \leq 0, \varphi^{(3)}(u_2^5) > 0\}$, $\{\varphi''(u_3^1) \geq 0, \varphi'(u_4^2) \leq 0\}$ or $\{\varphi''(u_3^2) \leq 0, \varphi'(u_4^6) \leq 0\}$ or $\{\varphi''(u_3^1) \geq 0, \varphi'(u_4^2) > 0, \varphi(u_5^3) > 0$ or $\varphi(u_5^4) < 0\}$ or $\{\varphi''(u_3^2) \leq 0, \varphi'(u_4^6) > 0, \varphi(u_5^3) > 0$ or $\varphi(u_5^4) < 0\}$;
(3) $\Delta > 0$, $\{\varphi^{(4)}(u_1^+) \geq 0, \varphi^{(3)}(u_2^1) > 0\}$ or $\{\varphi^{(4)}(u_1^-) \leq 0, \varphi^{(3)}(u_2^5) > 0\}$, $\varphi''(u_3^1) < 0$, $\varphi''(u_3^2) > 0$, $\{\varphi'(u_4^4) \geq 0, \varphi(u_5^5) > 0$ or $\varphi(u_5^6) < 0\}$ or $\{\varphi'(u_4^3) > 0, \varphi'(u_4^4) < 0, \varphi'(u_4^5) > 0, A_1$ or $A_2\}$ or $\{\varphi'(u_4^3) \leq 0, \varphi'(u_4^5) > 0, \varphi(u_5^{11}) > 0$ or $\varphi(u_5^{12}) < 0\}$ or $\{\varphi'(u_4^3) > 0, \varphi'(u_4^5) \leq 0, \varphi(u_5^{13}) > 0$ or $\varphi(u_5^{14}) < 0\}$ or $\{\varphi'(u_4^3) \leq 0, \varphi'(u_4^5) \leq 0\}$;
(4) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) \geq 0, \varphi''(u_3^3) \geq 0\}$ or $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) < 0, \varphi^{(3)}(u_2^4) \leq 0, \varphi''(u_3^9) \geq 0\}$ or $\{\varphi^{(3)}(u_2^2) \leq 0, \varphi^{(3)}(u_2^4) > 0, \varphi''(u_3^{11}) \geq 0\}$, $\varphi'(u_4^7) \leq 0$ or $\{\varphi'(u_4^7) > 0, \varphi(u_5^{15}) > 0$ or $\varphi(u_5^{16}) < 0\}$;
(5) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) \geq 0, \varphi''(u_3^4) \leq 0\}$ or $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) < 0, \varphi^{(3)}(u_2^4) \leq 0, \varphi''(u_3^{10}) \leq 0\}$ or $\{\varphi^{(3)}(u_2^2) \leq 0, \varphi^{(3)}(u_2^4) > 0, \varphi''(u_3^{12}) \leq 0\}$, $\varphi'(u_4^{11}) \leq 0$ or $\{\varphi'(u_4^{11}) > 0, \varphi(u_5^{15}) > 0$ or $\varphi(u_5^{16}) < 0\}$;
(6) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) \geq 0, \varphi''(u_3^3) < 0, \varphi''(u_3^4) > 0\}$ or $\{\varphi^{(3)}(u_2^2) > 0, \varphi^{(3)}(u_2^3) < 0, \varphi^{(3)}(u_2^4) \leq 0, \varphi''(u_3^9) < 0, \varphi''(u_3^{10}) > 0\}$ or $\{\varphi^{(3)}(u_2^2) \leq 0, \varphi^{(3)}(u_2^4) > 0, \varphi''(u_3^{11}) < 0, \varphi''(u_3^{12}) > 0\}$, $\{\varphi'(u_4^9) \geq 0, \varphi(u_5^{17}) > 0$ or $\varphi(u_5^{18}) < 0\}$ or $\{\varphi'(u_4^8) > 0, \varphi'(u_4^9) < 0, \varphi'(u_4^{10}) > 0, A_3$ or $A_4\}$ or $\{\varphi'(u_4^8) \leq 0, \varphi'(u_4^{10}) > 0, \varphi(u_5^{23}) > 0$ or $\varphi(u_5^{24}) < 0\}$ or $\{\varphi'(u_4^8) > 0, \varphi'(u_4^{10}) \leq 0, \varphi(u_5^{25}) > 0$ or $\varphi(u_5^{26}) < 0\}$ or $\{\varphi'(u_4^8) \leq 0, \varphi'(u_4^{10}) \leq 0\}$;
(7) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) \geq 0, \varphi'(u_4^{12}) \leq 0\}$ or $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^8) \leq 0, \varphi'(u_4^{16}) \leq 0\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) \leq 0, \varphi'(u_4^{31}) \leq 0\}$;
(8) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) \geq 0, \varphi'(u_4^{12}) > 0\}$ or $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^8) \leq 0, \varphi'(u_4^{16}) > 0\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) \leq 0, \varphi'(u_4^{31}) > 0\}$, $\varphi(u_5^{27}) > 0$ or $\varphi(u_5^{28}) < 0$;
(9) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) < 0, \varphi''(u_3^8) > 0, \varphi'(u_4^{14}) \geq 0\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^7) \geq 0, \varphi'(u_4^{18}) \geq 0\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^6) > 0, \varphi''(u_3^8) \leq 0, \varphi'(u_4^{26}) \geq 0\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) > 0, \varphi'(u_4^{29}) \geq 0\}$, $\varphi(u_5^{29}) > 0$ or $\varphi(u_5^{30}) < 0$;
(10) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) < 0, \varphi''(u_3^8) > 0, B_1\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^7) \geq 0, B_2\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^6) > 0, \varphi''(u_3^8) \leq 0, B_3\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) > 0, B_4\}$, A_5 or A_6;
(11) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) < 0, \varphi''(u_3^8) > 0, B_5\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^7) \geq 0, B_6\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^6) > 0, \varphi''(u_3^8) \leq 0, B_7\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) > 0, B_8\}$, $\varphi(u_5^{35}) > 0$ or $\varphi(u_5^{36}) < 0$;
(12) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) < 0, \varphi''(u_3^8) > 0, B_9\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^7) \geq 0, B_{10}\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^6) > 0, \varphi''(u_3^8) \leq 0, B_{11}\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) > 0, B_{12}\}$, $\varphi(u_5^{37}) > 0$ or $\varphi(u_5^{38}) < 0$;
(13) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\{\varphi''(u_3^5) \geq 0, \varphi''(u_3^7) < 0, \varphi''(u_3^8) > 0, B_{13}\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^7) \geq 0, B_{14}\}$ or $\{\varphi''(u_3^5) < 0, \varphi''(u_3^6) > 0, \varphi''(u_3^8) \leq 0, B_{15}\}$ or $\{\varphi''(u_3^6) \leq 0, \varphi''(u_3^8) > 0, B_{16}\}$;
(14) $\Delta > 0$, $\varphi^{(4)}(u_1^+) < 0$, $\varphi^{(4)}(u_1^-) > 0$, $\varphi^{(3)}(u_2^2) > 0$, $\varphi^{(3)}(u_2^3) < 0$, $\varphi^{(3)}(u_2^4) > 0$, $\varphi''(u_3^5) < 0$, $\varphi''(u_3^6) > 0$, $\varphi''(u_3^7) < 0$, $\varphi''(u_3^8) > 0$, $\{\varphi'(u_4^{21}) \geq 0, \varphi'(u_4^{23}) \geq 0, \varphi(u_5^{39}) > 0$ or $\varphi(u_5^{40}) < 0\}$ or $\{\varphi'(u_4^{21}) \geq 0, \varphi'(u_4^{23}) < 0, \varphi'(u_4^{24}) > 0, A_7$ or $A_8\}$ or $\{\varphi'(u_4^{21}) \geq 0, \varphi'(u_4^{24}) \leq 0, \varphi(u_5^{45}) > 0$ or $\varphi(u_5^{46}) < 0\}$ or $\{\varphi'(u_4^{20}) > 0, \varphi'(u_4^{21}) < 0, \varphi'(u_4^{23}) \geq 0, A_9$ or $A_{10}\}$ or $\{\varphi'(u_4^{20}) > 0, \varphi'(u_4^{21}) < 0, \varphi'(u_4^{22}) > 0, \varphi'(u_4^{23}) < 0, \varphi'(u_4^{24}) > 0, A_{11}$ or $A_{12}\}$ or $\{\varphi'(u_4^{20}) > 0, \varphi'(u_4^{21}) < 0, \varphi'(u_4^{22}) > 0, \varphi'(u_4^{24}) \leq 0, A_{13}$ or $A_{14}\}$ or $\{\varphi'(u_4^{20}) > 0, \varphi'(u_4^{22}) \leq 0, \varphi'(u_4^{24}) >$

0, A_{15} or $A_{16}\}$ or $\{\varphi^{'}(u_4^{20}) > 0,\ \varphi'(u_4^{22}) \le 0,\ \varphi^{'}(u_4^{24}) \le 0,\ \varphi(u_5^{65}) > 0$ or $\varphi(u_5^{66}) < 0\}$ or $\{\varphi^{'}(u_4^{20}) \le 0,\ \varphi^{'}(u_4^{23}) \ge 0,\ \varphi(u_5^{67}) > 0$ or $\varphi(u_5^{68}) < 0\}$ or $\{\varphi^{'}(u_4^{20}) \le 0,\ \varphi'(u_4^{22}) > 0,\ \varphi^{'}(u_4^{23}) < 0,\ \varphi'(u_4^{24}) > 0,\ A_{17}$ or $A_{18}\}$ or $\{\varphi^{'}(u_4^{20}) \le 0,\ \varphi'(u_4^{22}) > 0,\ \varphi^{'}(u_4^{24}) \le 0,\ \varphi(u_5^{73}) > 0$ or $\varphi(u_5^{74}) < 0\}$ or $\{\varphi^{'}(u_4^{20}) \le 0,\ \varphi'(u_4^{22}) \le 0,\ \varphi^{'}(u_4^{24}) > 0,\ \varphi(u_5^{75}) > 0$ or $\varphi(u_5^{76}) < 0\}$ or $\{\varphi^{'}(u_4^{20}) \le 0,\ \varphi'(u_4^{22}) \le 0,\ \varphi^{'}(u_4^{24}) \le 0\}$;

(15) $\Delta > 0,\ \varphi^{(4)}(u_1^+) < 0,\ \varphi^{(4)}(u_1^-) > 0,\ \varphi^{(3)}(u_2^2) \le 0,\ \varphi^{(3)}(u_2^4) \le 0,\ \varphi'(u_4^{32}) \le 0$ or $\{\varphi'(u_4^{32}) > 0,\ \varphi(u_5^{77}) > 0$ or $\varphi(u_5^{78}) < 0\}$;

(16) $\Delta \le 0,\ \varphi^{(3)}(u_2^6) \le 0,\ \varphi^{'}(u_4^{33}) \le 0$ or $\{\varphi^{'}(u_4^{33}) > 0,\ \varphi(u_5^{79}) > 0$ or $\varphi(u_5^{80}) < 0\}$;

(17) $\Delta \le 0,\ \varphi^{(3)}(u_2^6) > 0,\ \{\varphi^{''}(u_3^{13}) \ge 0,\ \varphi'(u_4^{34}) \le 0\}$ or $\{\varphi^{''}(u_3^{14}) \le 0,\ \varphi'(u_4^{38}) \le 0\}$;

(18) $\Delta \le 0,\ \varphi^{(3)}(u_2^6) > 0,\ \{\varphi^{''}(u_3^{13}) \ge 0,\ \varphi'(u_4^{34}) > 0\}$ or $\{\varphi^{''}(u_3^{14}) \le 0,\ \varphi'(u_4^{38}) > 0\},\ \varphi(u_5^{81}) > 0$ or $\varphi(u_5^{82}) < 0$;

(19) $\Delta \le 0,\ \varphi^{(3)}(u_2^6) > 0,\ \varphi^{''}(u_3^{13}) < 0,\ \varphi^{''}(u_3^{14}) > 0,\ \{\varphi^{'}(u_4^{36}) \ge 0,\ \varphi(u_5^{83}) > 0$ or $\varphi(u_5^{84}) < 0\}$ or $\{\varphi^{'}(u_4^{35}) > 0,\ \varphi^{'}(u_4^{36}) < 0,\ \varphi^{'}(u_4^{37}) > 0,\ A_{19}$ or $A_{20}\}$ or $\{\varphi^{'}(u_4^{35}) > 0,\ \varphi^{'}(u_4^{37}) \le 0,\ \varphi(u_5^{89}) > 0$ or $\varphi(u_5^{90}) < 0\}$ or $\{\varphi^{'}(u_4^{35}) \le 0,\ \varphi^{'}(u_4^{37}) > 0,\ \varphi(u_5^{91}) > 0$ or $\varphi(u_5^{92}) < 0\}$ or $\{\varphi^{'}(u_4^{35}) \le 0,\ \varphi^{'}(u_4^{37}) \le 0\}$,

with

$$
\begin{aligned}
&A_1 = \{\varphi(u_5^7) > 0, \varphi(u_5^9) > 0 \textit{ or } \varphi(u_5^{10}) < 0\}, && A_2 = \{\varphi(u_5^8) < 0, \varphi(u_5^{10}) < 0\},\\
&A_3 = \{\varphi(u_5^{19}) > 0, \varphi(u_5^{21}) > 0 \textit{ or } \varphi(u_5^{22}) < 0\}, && A_4 = \{\varphi(u_5^{20}) < 0, \varphi(u_5^{22}) < 0\},\\
&A_5 = \{\varphi(u_5^{31}) > 0, \varphi(u_5^{33}) > 0 \textit{ or } \varphi(u_5^{34}) < 0\}, && A_6 = \{\varphi(u_5^{32}) < 0, \varphi(u_5^{34}) < 0\},\\
&A_7 = \{\varphi(u_5^{41}) > 0, \varphi(u_5^{43}) > 0 \textit{ or } \varphi(u_5^{44}) < 0\}, && A_8 = \{\varphi(u_5^{42}) < 0, \varphi(u_5^{44}) < 0\},\\
&A_9 = \{\varphi(u_5^{47}) > 0, \varphi(u_5^{49}) > 0 \textit{ or } \varphi(u_5^{50}) < 0\}, && A_{10} = \{\varphi(u_5^{48}) < 0, \varphi(u_5^{50}) < 0\},\\
&A_{11} = \{\varphi(u_5^{51}) > 0, \varphi(u_5^{53}) > 0, \varphi(u_5^{55}) > 0 \textit{ or } \varphi(u_5^{56}) < 0\}, &&\\
&A_{12} = \{\varphi(u_5^{51}) > 0 \textit{ or } \varphi(u_5^{52}) < 0, \varphi(u_5^{54}) < 0, \varphi(u_5^{56}) < 0\}, &&\\
&A_{13} = \{\varphi(u_5^{57}) > 0, \varphi(u_5^{59}) > 0 \textit{ or } \varphi(u_5^{60}) < 0\}, && A_{14} = \{\varphi(u_5^{58}) < 0, \varphi(u_5^{60}) < 0\},\\
&A_{15} = \{\varphi(u_5^{61}) > 0, \varphi(u_5^{63}) > 0 \textit{ or } \varphi(u_5^{64}) < 0\}, && A_{16} = \{\varphi(u_5^{62}) < 0, \varphi(u_5^{64}) < 0\},\\
&A_{17} = \{\varphi(u_5^{69}) > 0, \varphi(u_5^{71}) > 0 \textit{ or } \varphi(u_5^{72}) < 0\}, && A_{18} = \{\varphi(u_5^{70}) < 0, \varphi(u_5^{72}) < 0\},\\
&A_{19} = \{\varphi(u_5^{85}) > 0, \varphi(u_5^{87}) > 0 \textit{ or } \varphi(u_5^{88}) < 0\}, && A_{20} = \{\varphi(u_5^{86}) < 0, \varphi(u_5^{88}) < 0\},\\
&B_1 = \{\varphi'(u_4^{13}) > 0, \varphi'(u_4^{14}) < 0, \varphi'(u_4^{15}) > 0\}, && B_2 = \{\varphi'(u_4^{17}) > 0, \varphi'(u_4^{18}) < 0, \varphi'(u_4^{19}) > 0\},\\
&B_3 = \{\varphi'(u_4^{25}) > 0, \varphi'(u_4^{26}) < 0, \varphi'(u_4^{27}) > 0\}, && B_4 = \{\varphi'(u_4^{28}) > 0, \varphi'(u_4^{29}) < 0, \varphi'(u_4^{30}) > 0\},\\
&B_5 = \{\varphi'(u_4^{13}) \le 0,\ \varphi'(u_4^{15}) > 0\}, && B_6 = \{\varphi'(u_4^{17}) \le 0,\ \varphi'(u_4^{19}) > 0\},\\
&B_7 = \{\varphi'(u_4^{25}) \le 0,\ \varphi'(u_4^{27}) > 0\}, && B_8 = \{\varphi'(u_4^{28}) \le 0,\ \varphi'(u_4^{30}) > 0\},\\
&B_9 = \{\varphi'(u_4^{13}) > 0,\ \varphi'(u_4^{15}) \le 0\}, && B_{10} = \{\varphi'(u_4^{17}) > 0,\ \varphi'(u_4^{19}) \le 0\},\\
&B_{11} = \{\varphi'(u_4^{25}) > 0,\ \varphi'(u_4^{27}) \le 0\}, && B_{12} = \{\varphi'(u_4^{28}) > 0,\ \varphi'(u_4^{30}) \le 0\},\\
&B_{13} = \{\varphi'(u_4^{13}) \le 0,\ \varphi'(u_4^{15}) \le 0\}, && B_{14} = \{\varphi'(u_4^{17}) \le 0,\ \varphi'(u_4^{19}) \le 0\},\\
&B_{15} = \{\varphi'(u_4^{25}) \le 0,\ \varphi'(u_4^{27}) \le 0\}, && B_{16} = \{\varphi'(u_4^{28}) \le 0,\ \varphi'(u_4^{30}) \le 0\}.
\end{aligned}
$$

Appendix B. The direction and stability of local Hopf bifurcation

In this Appendix, we study the direction of local Hopf bifurcation and the stability of bifurcating periodic solutions by applying the center manifold theorem and normal form method [34, 44] for partial differential equations with time delay.

Let $\bar{u}(x,\bar{t}) = u(x,\tau t)$, $\bar{v}(x,\bar{t}) = v(x,\tau t)$, and $\bar{t} = \frac{t}{\tau}$. After dropping the bar, system (1.4) can become

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\tau\Delta u + \tau\left(\dfrac{r}{1+\beta v}u - ru^2 - \dfrac{bu^2v}{a+u^2}\right), \\ \dfrac{\partial v}{\partial t} = d_2\tau\Delta v + \tau\left(\dfrac{cu_1^2}{a+u_1^2}v - v^2\right), \end{cases} \tag{B.1}$$

where $u_1 = u(x,t-1)$. For convenience, denote $\tau_n^* = \tau_{n,j}$. Let $\tau = \tau_n^* + \mu$, then $\mu = 0$ is the Hopf bifurcation value of system (B.1). Define $U(t) = (u(x,t) - u_*, v(x,t) - v_*)^T$, system (B.1) can be written as the following form in the space $\mathscr{C} = C([-1,0],X)$,

$$\dot{U}(t) = \tilde{D}\Delta U(t) + L_\mu(U_t) + F(\mu,U_t), \tag{B.2}$$

where

$$\begin{aligned} &\tilde{D} = (\tau_n^* + \mu)D, \\ &L_\mu(\phi) = (\tau_n^* + \mu)(A\phi(0) + B\phi(-1)), \\ &F(\mu,\phi) = (\tau_n^* + \mu)\begin{pmatrix} a_{11}\phi_1^2(0) + a_{12}\phi_1(0)\phi_2(0) + a_{22}\phi_2^2(0) + a_{111}\phi_1^3(0) \\ + a_{112}\phi_1^2(0)\phi_2(0) + a_{122}\phi_1(0)\phi_2^2(0) + a_{222}\phi_2^3(0) \\ \\ b_{22}\phi_2^2(0) + b_{23}\phi_2(0)\phi_1(-1) + b_{33}\phi_1^2(-1) \\ + b_{233}\phi_2(0)\phi_1^2(-1) + b_{333}\phi_1^3(-1) \end{pmatrix} + \mathcal{O}(4), \end{aligned}$$

with

$$\begin{aligned} &a_{11} = -2r + \frac{2ab(3u_*^2 - a)v_*}{(a+u_*^2)^3}, \quad a_{12} = -\frac{r\beta}{(1+\beta v_*)^2} - \frac{2abu_*}{(a+u_*^2)^2}, \quad a_{22} = \frac{2r\beta^2u_*}{(1+\beta v_*)^3}, \\ &a_{111} = \frac{24abu_*(a-u_*^2)v_*}{(a+u_*^2)^4}, \quad a_{112} = \frac{2ab(3u_*^2-a)}{(a+u_*^2)^3}, \quad a_{122} = \frac{2r\beta^2}{(1+\beta v_*)^3}, \quad a_{222} = -\frac{6r\beta^3u_*}{(1+\beta v_*)^4}, \\ &b_{22} = -2, \quad b_{23} = \frac{2acu_*}{(a+u_*^2)^2}, \quad b_{33} = \frac{2ac(a-3u_*^2)v_*}{(a+u_*^2)^3}, \quad b_{233} = \frac{2ac(a-3u_*^2)}{(a+u_*^2)^3}, \quad b_{333} = \frac{24acu_*(u_*^2-a)v_*}{(a+u_*^2)^4}. \end{aligned}$$

Then the linear system of equation (B.2) at $(0,0)$ is

$$\dot{U}(t) = \tilde{D}\Delta U(t) + L_\mu(U_t). \tag{B.3}$$

By the Riesz representation theorem, there exists a bounded variation function

$$\eta(\theta,\mu) = \begin{cases} -(\tau_n^* + \mu)B, & \theta = -1, \\ 0, & -1 < \theta < 0, \\ -\dfrac{\tilde{D}n^2}{l^2} + (\tau_n^* + \mu)A, & \theta = 0, \end{cases}$$

such that

$$-\frac{\tilde{D}n^2}{l^2}\phi(0) + L_\mu(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), \quad \phi(\theta) \in \mathscr{C}.$$

For $\phi(\theta) \in \mathscr{C}$, define

$$A_\mu(\phi(\theta)) = \begin{cases} -\dot{\phi}(\theta), & \theta \in [-1,0], \\ \tilde{D}\Delta\phi(0) + L_\mu(\phi), & \theta = 0, \end{cases}$$

and for $\psi(s) \in \mathscr{C}^* = C([-1,0], X^*)$, define the adjoint operator A^* of A_0

$$A^*(\psi(s)) = \begin{cases} -\dot{\psi}(s), & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(s,0)\psi(-s), & s = 0. \end{cases}$$

Consider the bilinear form

$$(\psi(s), \phi(\theta)) = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{\psi}(\xi - \theta)\mathrm{d}\eta(\theta,0)\phi(\xi)\mathrm{d}\xi, \quad \phi(\theta) \in \mathscr{C}, \quad \psi(s) \in \mathscr{C}^*.$$

From Section 2, it is clear to know $\pm i\omega_n\tau_n^*$ are eigenvalues of A_0 and A^*, and $q(\theta) = (q_1, q_2)^T \mathrm{e}^{i\omega_n\tau_n^*\theta}$ and $q^*(s) = \bar{\kappa}_n(q_1^*, q_2^*)\mathrm{e}^{i\omega_n\tau_n^* s}$ are eigenvectors of A_0 and A^* corresponding to $i\omega_n\tau_n^*$ and $-i\omega_n\tau_n^*$, respectively, where

$$\begin{aligned}
(q_1, q_2)^T &= \left(i\omega_n + d_2\frac{n^2}{l^2} + v_*, \frac{2acu_*v_*}{(a+u_*^2)^2}\mathrm{e}^{-i\omega_n\tau_n^*}\right), \\
(q_1^*, q_2^*) &= \left(-i\omega_n + d_2\frac{n^2}{l^2} + v_*, \frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2}\right), \\
\kappa_n &= \left(\left(i\omega_n + d_2\frac{n^2}{l^2} + v_*\right)^2 + \left(1 + \tau_n^*\left(i\omega_n + d_2\frac{n^2}{l^2} + v_*\right)\right)P\mathrm{e}^{-i\omega_n\tau_n^*}\right)^{-1}.
\end{aligned}$$

Let $\Phi(\theta) = (q(\theta), \bar{q}(\theta))$ and $\Psi(s) = (q^*(s), \bar{q}^*(s))$, then

$$(\Phi, \Psi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Define $f_n = (f_n^1, f_n^2)^T$, where $f_n^1 = (\gamma_n, 0)^T$, $f_n^2 = (0, \gamma_n)^T$ and

$$\gamma_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}\cos\dfrac{nx}{l}, & n \in \mathbb{N}. \end{cases}$$

By applying the notation of Wu [34], we define

$$\begin{aligned}
&c \cdot f_n = c_1 f_n^1 + c_2 f_n^2, \quad c = (c_1, c_2)^T \in \mathbb{C}^2, \\
&(\psi \cdot f_n)\theta = \psi(\theta) \cdot f_n, \quad \theta \in [-1,0],
\end{aligned}$$

and

$$\langle u, v\rangle = \frac{1}{l\pi}\int_0^{l\pi} u_1v_1 + u_2v_2\mathrm{d}x, \qquad u = (u_1,u_2), v = (v_1,v_2) \in X,$$

thus $\langle \phi, f_n\rangle = \left(\langle \phi, f_n^1\rangle, \langle \phi, f_n^2\rangle\right)^T$, where $\phi \in \mathscr{C}$. Therefore, we can decompose the space $\mathscr{C}$ into

$$\mathscr{C} = P_{CN}\mathscr{C} \oplus P_s\mathscr{C}, \quad P_{CN}\phi = \Phi(\Psi, \langle \phi, f_n\rangle)\cdot f_n, \quad \phi \in \mathscr{C}.$$

The computation formula for the direction of local Hopf bifurcation and the stability of periodic solutions is only associated with $\mu = 0$, so we make $\mu = 0$ for equation (B.2) and in $P_s\mathscr{C}$ get a center manifold

$$W(z,\bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{B.4}$$

Then the flow of equation (B.2) can be expressed as

$$U_t = \Phi(z(t), \bar{z}(t))^T \cdot f_n + W(z,\bar{z}), \tag{B.5}$$

where

$$\begin{aligned}\dot{z}(t) =& i\omega_n\tau_n^* z(t) + \bar{q^*}(0)\left\langle F(0, \Phi(z(t),\bar{z}(t))^T\cdot f_n + W(z,\bar{z})), f_n\right\rangle \\ =& i\omega_n\tau_n^* z(t) + g(z,\bar{z}),\end{aligned} \tag{B.6}$$

with

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{B.7}$$

For convenience, denote

$$\begin{aligned}
G_1^n &= a_{11}q_1^2 + a_{12}q_1q_2 + a_{22}q_2^2;\\
G_2^n &= b_{22}q_2^2 + b_{23}q_1q_2\mathrm{e}^{-i\omega_n\tau_n^*} + b_{33}q_1^2\mathrm{e}^{-2i\omega_n\tau_n^*};\\
H_1^n &= 2a_{11}q_1\bar{q_1} + a_{12}(q_1\bar{q_2} + \bar{q_1}q_2) + 2a_{22}q_2\bar{q_2};\\
H_2^n &= 2b_{22}q_2\bar{q_2} + b_{23}(q_1\bar{q_2}\mathrm{e}^{-i\omega_n\tau_n^*} + \bar{q_1}q_2\mathrm{e}^{i\omega_n\tau_n^*}) + 2b_{33}q_1\bar{q_1}.
\end{aligned} \tag{B.8}$$

After some calculations, we have

$$\begin{aligned}
g_{20} &= \begin{cases} 2\kappa_0\tau_0\left(\bar{q_1^*}G_1^0 + \bar{q_2^*}G_2^0\right), & n=0,\\ 0, & n\in\mathbb{N},\end{cases}\\
g_{11} &= \begin{cases} \kappa_0\tau_0\left(\bar{q_1^*}H_1^0 + \bar{q_2^*}H_2^0\right), & n=0,\\ 0, & n\in\mathbb{N},\end{cases}\\
g_{02} &= \begin{cases} 2\kappa_0\tau_0\left(\bar{q_1^*}\bar{G}_1^0 + \bar{q_2^*}\bar{G}_2^0\right), & n=0,\\ 0, & n\in\mathbb{N},\end{cases}\\
g_{21} &= \frac{2\kappa_n\tau_n^*}{l\pi}\int_0^{l\pi}\left(\bar{q_1^*}\left(Q_1\gamma_n^2 + Q_2\gamma_n^4\right) + \bar{q_2^*}\left(Q_3\gamma_n^2 + Q_4\gamma_n^4\right)\right)\mathrm{d}x, \quad n\in\mathbb{N}_0,
\end{aligned} \tag{B.9}$$

where

$$
\begin{aligned}
Q_1 =& 2a_{11}\left(q_1 W_{11}^{(1)}(0) + \frac{\bar{q}_1}{2} W_{20}^{(1)}(0)\right) + a_{12}\Big(q_1 W_{11}^{(2)}(0) + \frac{\bar{q}_1}{2} W_{20}^{(2)}(0) + q_2 W_{11}^{(1)}(0) + \frac{\bar{q}_2}{2} W_{20}^{(1)}(0)\Big) \\
&+ 2a_{22}\left(q_2 W_{11}^{(2)}(0) + \frac{\bar{q}_2}{2} W_{20}^{(2)}(0)\right), \\
Q_2 =& 3a_{111} q_1^2 \bar{q}_1 + a_{112}(q_1^2 \bar{q}_2 + 2q_1 \bar{q}_1 q_2) + a_{122}(\bar{q}_1 q_2^2 + 2q_1 q_2 \bar{q}_2) + 3a_{222} q_2^2 \bar{q}_2, \\
Q_3 =& 2b_{22}\left(q_2 W_{11}^{(2)}(0) + \frac{\bar{q}_2}{2} W_{20}^{(2)}(0)\right) + b_{23}\Big(q_1 W_{11}^{(2)}(0) \mathrm{e}^{-i\omega_n \tau_n^*} + \frac{\bar{q}_1}{2} W_{20}^{(2)}(0) \mathrm{e}^{i\omega_n \tau_n^*} + q_2 W_{11}^{(1)}(-1) \\
&+ \frac{\bar{q}_2}{2} W_{20}^{(1)}(-1)\Big) + 2b_{33}\left(q_1 W_{11}^{(1)}(-1) \mathrm{e}^{-i\omega_n \tau_n^*} + \frac{\bar{q}_1}{2} W_{20}^{(1)}(-1) \mathrm{e}^{i\omega_n \tau_n^*}\right), \\
Q_4 =& b_{233}\left(q_1^2 \bar{q}_2 \mathrm{e}^{-2i\omega_n \tau_n^*} + 2q_1 \bar{q}_1 q_2\right) + 3b_{333} q_1^2 \bar{q}_1 \mathrm{e}^{-i\omega_n \tau_n^*}.
\end{aligned}
$$

Clearly, in order to calculate g_{21}, we also have to calculate $W_{11}(\theta)$ and $W_{20}(\theta)$.
From Wu [34], we get

$$\dot{W} = A_0 W + H(z, \bar{z}, \theta), \tag{B.10}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{B.11}$$

It follows that

$$(2i\omega_n \tau_n^* - A_0) W_{20} = H_{20}, \quad -A_0 W_{11} = H_{11}. \tag{B.12}$$

According to equation (B.11), one has

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \quad H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta), \quad -1 \le \theta < 0,$$

and

$$
\begin{aligned}
H_{20}(0) &= -g_{20} q(0) - \bar{g}_{02} \bar{q}(0) + \tau_n^* (G_1^n, G_2^n)^T \gamma_n^2, \\
H_{11}(0) &= -g_{11} q(0) - \bar{g}_{11} \bar{q}(0) + \tau_n^* (H_1^n, H_2^n)^T \gamma_n^2.
\end{aligned}
$$

From equation (B.12), it can be seen

$$
\begin{aligned}
W_{20}(\theta) &= \left(\frac{i g_{20}}{\omega_n \tau_n^*} q(\theta) + \frac{i\bar{g}_{20}}{3\omega_n \tau_n^*} \bar{q}(\theta)\right) \cdot f_n + E_1 \mathrm{e}^{2i\omega_n \tau_n^* \theta}, \\
W_{11}(\theta) &= \left(\frac{-i g_{11}}{\omega_n \tau_n^*} q(\theta) + \frac{i\bar{g}_{11}}{\omega_n \tau_n^*} \bar{q}(\theta)\right) \cdot f_n + E_2,
\end{aligned} \tag{B.13}
$$

where

$$E_1 = \frac{1}{2}\begin{pmatrix} 2i\omega_n + \frac{4d_1n^2}{l^2} + \alpha(u_*) & \frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2} \\ \frac{2acu_*v_*}{(a+u_*^2)^2}\mathrm{e}^{-2i\omega_n\tau_n^*} & 2i\omega_n + \frac{4d_2n^2}{l^2} + v_* \end{pmatrix}^{-1} \begin{pmatrix} G_1^n \\ G_2^n \end{pmatrix} \cos\frac{2nx}{l}$$
$$+ \frac{1}{2}\begin{pmatrix} 2i\omega_n + \alpha(u_*) & \frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2} \\ \frac{2acu_*v_*}{(a+u_*^2)^2}\mathrm{e}^{-2i\omega_n\tau_n^*} & 2i\omega_n + v_* \end{pmatrix}^{-1} \begin{pmatrix} G_1^n \\ G_2^n \end{pmatrix},$$

$$E_2 = \frac{1}{2}\begin{pmatrix} 2i\omega_n + \frac{4d_1n^2}{l^2} + \alpha(u_*) & \frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2} \\ \frac{2acu_*v_*}{(a+u_*^2)^2} & 2i\omega_n + \frac{4d_2n^2}{l^2} + v_* \end{pmatrix}^{-1} \begin{pmatrix} H_1^n \\ H_2^n \end{pmatrix} \cos\frac{2nx}{l}$$
$$+ \frac{1}{2}\begin{pmatrix} 2i\omega_n + \alpha(u_*) & \frac{r\beta u_*}{(1+\beta v_*)^2} + \frac{bu_*^2}{a+u_*^2} \\ \frac{2acu_*v_*}{(a+u_*^2)^2} & 2i\omega_n + v_* \end{pmatrix}^{-1} \begin{pmatrix} H_1^n \\ H_2^n \end{pmatrix}.$$

Summarizing the above analysis, it is clear that each g_{ij} can be expressed in terms of the parameters. Therefore, we can compute the following values related to the property of Hopf bifurcation:

$$C_1(0) = \frac{i}{2\omega_n\tau_n^*}(g_{20}g_{11} - 2|g_{11}|^2) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{\Re(C_1(0))}{\tau_n^*\text{Sign } \xi_1'(\tau_n^*)}, \quad \beta_2 = 2\Re(C_1(0)). \tag{B.14}$$

Theorem B.1. *Suppose that* (H_0)*–*(H_2) *hold, and* $\Re(C_1(0)) \neq 0$*. For system (1.4),*

(i) μ_2 *determines the direction of local Hopf bifurcation: if* $\mu_2 > 0$ $(\mu_2 < 0)$*, then the Hopf bifurcation is supercritical (subcritical);*

(ii) β_2 *determines the stability of bifurcating periodic solutions: if* $\beta_2 < 0$ $(\beta_2 > 0)$*, then the bifurcating periodic solutions are asymptotically stable (unstable).*

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References

[1] A.J. Lotka, Elements of Physical Biology. Williams and Wilkins, Princeton, NJ (1925).
[2] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. *Mem. Acad. Lincei.* **2** (1926) 31–113.
[3] G.F. Gause, The Struggle for Existence. Williams and Wilkins, Baltimore (1934).
[4] G.F. Gause, N.P. Smaragdova and A.A. Witt, Further studies of interaction between predators and prey. *J. Anim. Ecol.* **5** (1936) 1–18.
[5] H.I. Freedman, Deterministic Mathematical Models in Population Ecology. Marcel Dekker, New York (1980).
[6] R.E. Kooij and A. Zegeling, Predator–prey models with non-analytical functional response. *Chaos Soliton. Fract.* **123** (2019) 163–172.

[7] G.Y. Tang, S.Y. Tang and R.A. Cheke, Global analysis of a Holling type II predator–prey model with a constant prey refuge. *Nonlinear Dyn.* **76** (2014) 635–647.

[8] Y.Y. Li and J.F. Wang, Spatiotemporal patterns of a predator–prey system with an Allee effect and Holling type III functional response. *Int. J. Bifurcat. Chaos* **25** (2016) 1650088.

[9] Y. Lamontagne, C. Coutu and C. Rousseau, Bifurcation analysis of a predator–prey system with generalised Holling type III functional response. *J. Dyn. Differ. Equ.* **20** (2008) 535–571.

[10] S.G. Ruan and D.M. Xiao, Global analysis in a predator–prey system with nonmonotonic functional response. *SIAM J. Appl. Math.* **61** (2001) 1445–1472.

[11] T.W. Hwang, Uniqueness of limit cycles of the predator–prey system with Beddington–DeAngelis functional response. *J. Math. Anal. Appl.* **290** (2004) 113–122.

[12] Z.X. Li, L.S. Chen and J.M. Huang, Permanence and periodicity of a delayed ratio-dependent predator–prey model with Holling type functional response and stage structure. *J. Comput. Appl. Math.* **233** (2009) 173–187.

[13] J. Yang, S.Y. Tang and R.A. Cheke, Global stability and sliding bifurcations of a non-smooth Gause predator–prey system. *Appl. Math. Comput.* **224** (2013) 9–20.

[14] W. Ko and K. Ryu, A qualitative study on general Gause-type predator–prey models with constant diffusion rates. *J. Math. Anal. Appl.* **344** (2008) 217–230.

[15] X.X. Liu and L.H. Huang, Permanence and periodic solutions for a diffusive ratio-dependent predator–prey system. *Appl. Math. Model.* **33** (2009) 683–691.

[16] Z. Xu and Y.L. Song, Bifurcation analysis of a diffusive predator–prey system with a herd behavior and quadratic mortality. *Math. Method. Appl. Sci.* **38** (2015) 2994–3006.

[17] S.L. Yuan, C.Q. Xu and T.H. Zhang, Spatial dynamics in a predator–prey model with herd behavior. *Chaos* **23** (2013) 003102.

[18] S. Ghorai and S. Poria, Emergent impacts of quadratic mortality on pattern formation in a predator–prey system. *Nonlinear Dyn.* **87** (2017) 2715–2734.

[19] S. Rana, A.R. Bhowmick and T. Sardar, Invasive dynamics for a predator–prey system with Allee effect in both populations and a special emphasis on predator mortality. *Chaos* **31** (2021) 033150.

[20] E. Beretta and Y. Kuang, Convergence results in a well-known delayed predator–prey system. *J. Math. Anal. Appl.* **204** (1996) 840–853.

[21] S.G. Ruan, On nonlinear dynamics of predator–prey models with discrete delay. *Math. Model. Nat. Phenom.* **4** (2009) 140–188.

[22] J. Xia, Z.H. Liu, R. Yuan and S.G. Ruan, The effects of harvesting and time delay on predator–prey systems with holling type II functional response. *SIAM J. Appl. Math.* **70** (2009) 1178–1200.

[23] S.S. Chen, J.P. Shi and J.J. Wei, The effect of delay on a diffusive predator–prey system with Holling type-II predator functional response. *Commun. Pur. Appl. Anal.* **12** (2012) 481–501.

[24] F.R. Zhang and Y. Li, Stability and Hopf bifurcation of a delayed-diffusive predator–prey model with hyperbolic mortality and nonlinear prey harvesting. *Nonlinear Dyn.* **88** (2017) 1397–1412.

[25] W. Cresswell, Predation in bird populations. *J. Ornithol.* **152** (2011) 251–263.

[26] E.L. Preisser and D.I. Bolnick, The many faces of fear: comparing the pathways and impacts of nonconsumptive predator effects on prey populations. *PLoS One* **3** (2008) e2465.

[27] W.J. Ripple and R.L. Beschta, Wolves and the ecology of fear: can predation risk structure ecosystems. *Bioscience* **54** (2004) 755–766.

[28] L.Y. Zanette, A.F. White, M.C. Allen and M. Clinchy, Perceived predation risk reduces the number of offspring songbirds produce per year. *Science* **334** (2011) 1398–1401.

[29] F.Y. Hua, K.E. Sieving, R.J. Fletcher and C.A. Wright, Increased perception of predation risk to adults and offspring alters avian reproductive strategy and performance. *Behav. Ecol.* **25** (2014) 509–519.

[30] X.Y. Wang, L. Zanette and X.F. Zou, Modelling the fear effect in predator–prey interactions. *J. Math. Biol.* **73** (2016) 1179–1204.

[31] A. Kumar and B. Dubey, Modeling the effect of fear in a prey–predator system with prey refuge and gestation delay. *Int. J. Bifurcat. Chaos* **29** (2019) 1950195.

[32] S. Halder, J. Bhattacharyya and S. Pal, Comparative studies on a predator–prey model subjected to fear and Allee effect with type I and type II foraging. *J. Appl. Math. Comput.* **62** (2020) 93–118.

[33] Q. Liu and D.Q. Jiang, Influence of the fear factor on the dynamics of a stochastic predator–prey model. *Appl. Math. Lett.* **112** (2021) 106756.

[34] J.H. Wu, Theory and Applications of Partial Functional Differential Equations. Springer-Verlag, New York (1996).

[35] Y.L. Song and J.J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator–prey system. *J. Math. Anal. Appl.* **301** (2005) 1–21.

[36] C.J. Sun, M.A. Han, Y.P. Lin and Y.Y. Chen, Global qualitative analysis for a predator–prey system with delay. *Chaos Soliton. Fract.* **32** (2007) 1582–1596.

[37] X.P. Yan and W.T. Li, Bifurcation and global periodic solutions in a delayed facultative mutualism system. *Physica D* **227** (2007) 51–69.

[38] P.J. Pal, T. Saha, M. Sen and M. Banerjee, A delayed predator–prey model with strong Allee effect in prey population growth. *Nonlinear Dyn.* **68** (2012) 23–42.

[39] Y. Su, J.J. Wei and J.P. Shi, Hopf bifurcation in a diffusive Logistic equation with mixed delayed and instantaneous density dependence. *J. Dyn. Differ. Equ.* **24** (2012) 897–925.

[40] X.F. Xu and J.J. Wei, Bifurcation analysis of a spruce budworm model with diffusion and physiological structures. *J. Differ. Equ.* **262** (2017) 5206–5230.

[41] X.F. Xu and M. Liu, Global Hopf bifurcation of a general predator–prey system with diffusion and stage structures. *J. Differ. Equ.* **269** (2020) 8370–8386.

[42] M. Liu, J. Cao and X.F. Xu, Global Hopf bifurcation in a phytoplankton-zooplankton system with delay and diffusion. *Int. J. Bifurcat. Chaos* **31** (2021) 2150114.

[43] S.B. Hsu, A survey of constructing Lyapunov functions for mathmatical models in population biology. *Taiwan. J. Math.* **9** (2005) 151–173.

[44] T. Faria, Normal forms and Hopf bifurcation for partial differential equations with delays. *Trans. Am. Math. Soc.* **352** (2000) 2217–2238.