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A Nonlinear Parabolic Model in Processing of Medical Image

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Abstract. The image's restoration is an essential step in medical imaging. Several Filters are developped to remove noise, the most interesting are filters who permits to denoise the image preserving semantically important structures. One class of recent adaptive denoising methods is the nonlinear Partial Differential Equations who knows currently a significant success. This work deals with mathematical study for a proposed nonlinear evolution partial differential equation for image processing. The existence and the uniqueness of the solution are established. Using a finite differences method we experiment the validity of the proposed model and we illustrate the efficiently of the method using some medical images. The Signal to Noise Ration (SNR) number is used to estimate the quality of the restored images.

Key words: nonlinear parabolic model, image processing, Hilbert space
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1. Introduction

Restoration is a crucial step in image processing. Numerous algorithms have been proposed recently to tackle the problems of noise removal and image restoration in real images ([4, 7, 8, 9, 14, 17]).

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We are interested to restore the noisy image u_0 using the following nonlinear PDE problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \ \text{div}[\mu(\mid \nabla u \mid)\nabla u] = 0 \ in \ Q \\ u(x,t) = 0 \qquad \forall x \in \partial\Omega \qquad \forall t \in [0,T], \\ u(x,0) = u_0 \qquad \forall x \in \Omega. \end{cases} \tag{1.1}$$

Where Ω is an open bounded subset of $\mathbb{R}^n$, $n = 2$ or $n = 3$, with boundary $\partial\Omega$, $Q = \Omega \mathrm{x}[0,T]$ with some given $T > 0$.

Many algorithms are proposed for image processing [6, 10, 12, 15, 16]. The existence and the uniqueness of a solution of the used PDE problems are in general difficult to establish, an approach using the regularizing kernel is proposed in several works, see [3, 4]. In [1] the authors prove the existence and the uniqueness solution in Orlicz space. In this work we establishe the existence and uniqueness in $H^1(\Omega)$ space, under suitable hypothesis on μ.

The problem (1.1) is equivalent to the following variational problem:
Find $u \in V$ such that:

$$\int_\Omega \frac{\partial u(t)}{\partial t} v dx + \int_\Omega \mu[\mid \nabla u(t) \mid]\nabla u(t).\nabla v dx = 0. \tag{1.2}$$

Notice that $\mathcal{V} = \mathcal{D}(\Omega)$, $V = H_0^1(\Omega)$ is the adherence of $\mathcal{V}$ in $H^1(\Omega)$. $H_0^1(\Omega)$ will be provided with the scalar product $((u,v)) = \displaystyle\int_\Omega \nabla u.\nabla v \, dx$ of associated norm $\| \ . \ \|$. We indicate by H the adherence of $\mathcal{V}$ in $L^2(\Omega)$. The space H is provided with the scalar product of $L^2(\Omega)$ defined by: $(u,v) = \displaystyle\int_\Omega uv \, dx$. The associated norm is noted by $\mid \ . \ \mid$, see [13]. In the following we will establish the existence and the uniqueness of the weak solution of (1.1) under the following hypothesis, see [5, 11], on μ:

(1) $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$
(2) μ is continuous function
(3) $\lim\limits_{s \to +\infty} [\mu(s)] = \mu_0$ with $\mu_0 > 0$
(4) μ is differential continuous
(5) $s \mid \mu'(s) \mid \leq \mu(s) \ \forall s \in \mathbb{R}_+$

The hypothesis $((1) - (3))$ involve that μ is bounded.
Notice that $\sup\limits_{s \in \mathbb{R}_+} \mu(s) = a$ and $\inf\limits_{s \in \mathbb{R}_+} \mu(s) = b$, with $a \geq 0$ and $b \geq 0$.

We denote by A the operator defined by

$$(Av, w) = \int_\Omega \mu(\mid \nabla v \mid)\nabla v.\nabla w dx \quad for \, v, \, w \in V \tag{1.3}$$

According the hypothesis on μ we have $Av \in V'$ if $v \in V$, where $V' = H^{-1}(\Omega)$ is the dual space of V. We prove, see [5], that A is an operator monotone hemicontinuous, satisfying for all $u,\ v\ \in V$

$$(Au - Av, u - v) \geq b \parallel u - v \parallel^2 \tag{1.4}$$

2. Existence theorem

Theorem 1. *Let $u_0 \in H$ and μ satisfying (1)-(5). Then there exists at least one weak solution u of problem (1) such that $u \in L^2(0, T, V) \cap L^\infty(0, T, H)$.*
If $b = 0$ the existence takes place for any finish T.
If $b \neq 0$ the existence is global.

Proof. To show the existence, we use a Faedo-Galerkin method. We consider the spectral problem

$$((w, v)) = \lambda(w, v) \qquad \forall v \in V. \tag{2.1}$$

the injection of V in H is compact, the problem (2.1) admits a sequence of eigenvalues λ_j associated of eigenvectors w_j such that

$$((w_j, v)) = \lambda_j(w_j, v) \qquad \forall v \in V. \tag{2.2}$$

and $\{w_j\}_{j\in\mathbb{N}}$ is orthonormal in H and orthogonal in V. We denote $u_N(t)$ the approximate solution of (1.1) defined by

$$u_N(x, t) = u_N(t)(x) \in [w_1, \ldots, w_N] \qquad u_N(x, t) = \sum_{j=1}^{N} C_j^N(t) w_j(x) \tag{2.3}$$

We have then

$$\begin{cases} (u'_N(t), w_j) + (\mu(\mid \nabla u_N(t) \mid)\nabla u_N(t), \nabla w_j) = 0 \quad 1 \leq j \leq N,\ \ t \in [0, T] \\ \text{with } u_N(., 0) = u_{0N}(.) \in [w_1, \ldots, w_N],\ \text{and } u_{0N} \to u_0 \text{ in } V. \end{cases} \tag{2.4}$$

Each $C_j^N(t)$ verifies $\dfrac{dC_j^N(t)}{dt} = G_j(t, C_1^N(t), \ldots, C_N^N(t))$, where G_j is a continuous function, then by using the Cauchy-Péano theorem we deduce that there exists a local solution $u_N(t)$ of (2.4) on $[0, T_N]$.
By multiplying (2.4) by $C_j^N(t)$ and by adding, we deduct that:

$$\int_\Omega \frac{\partial u_N(t)}{\partial t} u_N(t) dx + \int_\Omega \mu[\mid \nabla u_N(t) \mid](\nabla u_N(t))^2 dx = 0$$

Then

$$\frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + \int_\Omega \mu[\mid \nabla u_N(t) \mid](\nabla u_N(t))^2 dx = 0 \tag{2.5}$$

case 1: $b > 0.$
As

$$\int_\Omega \mu[\mid \nabla u_N(t) \mid](\nabla u_N(t))^2 dx \geq b \int_\Omega (\nabla u_N(t))^2 dx, \tag{2.6}$$

From (2.5) we have

$$\frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + b \int_\Omega (\nabla u_N(t))^2 dx \leq 0, \tag{2.7}$$

Then

$$\frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + b \parallel u_N(t) \parallel^2 \leq 0, \tag{2.8}$$

There exists thus a constant $C_1 = \mid u_0 \mid > 0$ and a constant $C_2 = \dfrac{\mid u_0 \mid^2}{2b} > 0$ depending of b and u_0 such that

$$\mid u_N(t) \mid \leq C_1 \quad \text{and} \quad \int_0^t \parallel u_N(\tau) \parallel^2 \, d\tau \leq C_2 \qquad \forall\, t \in [0, T_N],\ \forall\, N \in \mathbb{N}. \tag{2.9}$$

case 2: $b = 0.$
In (2.5), we will rewrite the second term in the form

$$\int_\Omega \mu\{\nabla u_N(t)\}(\nabla\, u_N(t))^2 dx = \int_\Omega \{\mu[\nabla u_N(t)] - \mu_0\}(\nabla[u_N(t)])^2 dx + \mu_0 \int_\Omega (\nabla\, u_N(t))^2 dx$$

Where μ_0 is given by (3). From (2.5) we have

$$\left\{ \begin{array}{lcl} \frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + \mu_0 \parallel u_N(t) \parallel^2 & = & -\int_\Omega \{\mu[\nabla u_N(t)] - \mu_0\}(\nabla\, u_N(t))^2 dx \\ & \leq & \int_\Omega \mid \mu[\nabla u_N(t)] - \mu_0 \mid (\nabla\, u_N(t))^2 dx. \end{array} \right. \tag{2.10}$$

Notice that $\mu_1(s) = \mu(s) - \mu_0$, and by (3), it was $\lim\limits_{s \to +\infty} \mu_1(s) = 0$ then for all $\varepsilon > 0$, there exists $A > 0$ so that for all $(x, t) \in \Omega \times [0, T_N]$

$$\mid \mu_1\{\nabla u_N(x, t)\} \mid < \varepsilon \qquad \text{if} \quad \mid \nabla u_N(x, t) \mid > A$$

Let $t \in [0, T_N]$ fixed, we consider the following sets

$$\Omega_1^t = \{x \in \Omega / \mid \nabla u_N(x, t) \mid \leq A\}$$
$$\Omega_2^t = \{x \in \Omega / \mid \nabla u_N(x, t) \mid > A\}$$
$$\Omega_1^t \cap \Omega_2^t = \emptyset \qquad \Omega_1^t \cup \Omega_2^t = \Omega$$

We have then:

$$\begin{cases} \int_\Omega \mid \mu_1\{\nabla u_N(x,t)\} \mid (\nabla\, u_N(x,t))^2 dx = \int_{\Omega_1^t} \mid \mu_1\{\nabla u_N(x,t)\} \mid (\nabla\, u_N(x,t))^2 dx \\ \qquad + \int_{\Omega_2^t} \mid \mu_1\{\nabla u_N(x,t)\} \mid (\nabla\, u_N(x,t))^2 dx \\ \qquad \leq A^2 \int_{\Omega_1^t} \mid \mu_1\{\nabla u_N(x,t)\} \mid dx + \varepsilon \int_{\Omega_2^t} (\nabla\, u_N(x,t))^2 dx \\ \qquad \leq (\mu_0 + a) A^2 mes(\Omega) + \varepsilon \parallel u_N(t) \parallel^2 \end{cases}$$

From (2.10) we have

$$\frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + \mu_0 \parallel u_N(t) \parallel^2 \leq (\mu_0 + a) A^2 mes(\Omega) + \varepsilon \parallel u_N(t) \parallel^2$$

we take $\varepsilon = \frac{\mu_0}{2}$ and we set $(\mu_0 + a)A^2 mes(\Omega) = C(\Omega, \mu_0, a)$, then

$$\frac{1}{2}\frac{d}{dt} \mid u_N(t) \mid^2 + \frac{\mu_0}{2} \parallel u_N(t) \parallel^2 \leq C(\Omega, \mu_0, a)$$

hence, by the same reasoning as the first case, we deduced that there exists a constant $C_1 = \sqrt{2C(\Omega, \mu_0, a) + \mid u_0 \mid^2} > 0$ and a constant $C_2 = \frac{1}{\mu_0}(2C(\Omega, \mu_0, a)T + \mid u_0 \mid^2) > 0$ depending of $\Omega,\ \mu_0,\ a,\ u_0$ and T such that

$$\mid u_N(t) \mid \leq C_1 \qquad \text{and} \quad \int_0^t \|u_N(t)\|^2\, dt \leq C_2 \qquad \forall t \in [0, T_N],\ \forall N \in \mathbb{N}$$

Thus in both cases there exists two constants $C_1 > 0$ and $C_2 > 0$ independent of N such that

$$\begin{cases} \mid u_N(t) \mid \leq C_1 & \forall t \in [0, T_N],\ \forall N \in \mathbb{N} \\ \int_0^t \parallel u_N(\tau) \parallel^2 d\tau \leq C_2 \end{cases} \tag{2.11}$$

Moreover $T_N = T$ for all $N \in \mathbb{N}$ and the problem (2.4) admits a unique solution

$u_N \in L^2(0, T, V) \cap L^\infty(0, T, H)$ on $[0, T]$ and according to the monotony and the hemicontinuity of the operator A we deduce that the approximate solution u_N of the problem (2.1) converges towards a weak solution u of the problem (1.1) (see [2, 5]).

□

Theorem 2. *Under hypothesis of the theorem 1. with* $b \neq 0$ *we have*

$$u \in L^2(0, +\infty, V) \cap L^\infty(0, +\infty, H)$$

and $\lim_{t \to +\infty} \mid u(t) \mid = 0$.

Proof. From the theorem 1., there exists $u \in L^2(0, +\infty, V) \cap L^\infty(0, +\infty, H)$.
We have :

$$\frac{1}{2}\frac{d}{dt} \mid u(t) \mid^2 + \int_\Omega \mu[\| \nabla u(t) \|](\nabla u(t))^2 dx = 0, \tag{2.12}$$

we deduce :

$$\frac{1}{2}\frac{d}{dt} \mid u(t) \mid^2 + b \| u(t) \|^2 \leq 0. \tag{2.13}$$

Then

$$\frac{1}{2}\frac{d}{dt} \mid u(t) \mid^2 + \frac{b}{C(\Omega)^2} \mid u(t) \mid^2 \leq 0 \tag{2.14}$$

where $C(\Omega)$ is a Poincare constant.
From (2.14) we deduce that $\mid u(t) \mid$ converge to 0 exponentially when $t \to +\infty$.

□

3. Uniqueness Theorem

Theorem 3. *Under Hypothesis of the existence theorem, the weak solution u of the problem* (1.1) *is unique and* $u' = \dfrac{du}{dt} \in L^2(0, T, V')$.

Proof. Let u_1 and u_2 two solutions of the problem (1.1), we have then for all $v \in V$

$$(\frac{\partial u_1}{\partial t}(t) - \frac{\partial u_2}{\partial t}(t), v) + (Au_1(t) - Au_2(t), v) = 0. \tag{3.1}$$

We take $u_1 - u_2 = w$ and $v = w(t)$ in (3.1), we have then from (1.4):

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 = -(Au_1(t) - Au_2(t), w(t)) \leq 0$$

Then

$$|w(t)|^2 \leq |w(0)|^2 = 0$$

Thus $w(t) = u_1(t) - u_2(t) = 0$. And we obtain the uniqueness solution.
Besides $u'(t) = -Au(t)$ in V', from (1.2) and (1.3), and for all $v \in V$, we have

$$(Au(t), v) \leq a \| u(t) \| \| v \|$$

Then $\| Au(t) \|_{V'} \leq a \| u(t) \| \qquad \forall t \in [0, T]$.
So $Au \in L^2(0, T, V')$ and $u' \in L^2(0, T, V')$ □

4. Regularity and Stability Theorems

For the regularity proof in two-dimensional case, we will use the following lemma [18]:

Lemma 4. *For any open* Ω *in* $\mathbb{R}^2$ *and* $v \in H_0^1(\Omega)$ *we have*

$$\| v \|_{L^4(\Omega)}^2 \leq \sqrt{2} \mid v \mid_{L^2(\Omega)} \| v \|_{H_0^1(\Omega)} .$$

Theorem 5. *Under Hypothesis of the existence theorem 1., the solution* u *of the problem* (1.1) *exists in* $L^4(Q)$*:*

$$u \in L^4(Q)$$

Proof. We know according to the theorem 1. that:

$$u \in L^\infty(0, T, H) \cap L^2(0, T, V).$$

For all $t \in [0, T]$, we have from the lemma 4.

$$\|u(t)\|_{L^4(\Omega)}^2 \leq 2^{1/2} |u(t)| \, \|u(t)\|,$$

therefore:

$$\|u(t)\|_{L^4(\Omega)}^2 \leq 2^{1/2} |u|_{L^\infty(0,T,H)} \, \|u(t)\|,$$

$$\left(\int_0^T \| u(t) \|_{L^4(\Omega)}^4 \, dt\right)^{1/4} \leq 2^{1/4} \mid u \mid_{L^\infty(0,T,H)}^{1/2} \left(\int_0^T \| u(t) \|^2 \, dt\right)^{1/4}$$

hence

$$\left(\int_0^T \|u(t)\|_{L^4(\Omega)}^4 dt\right)^{1/4} \leq 2^{1/4} |u|_{L^\infty(0,T,H)}^{1/2} \, \|u\|_{L^2(0,T,V)}^{1/2}.$$

□

For the regularity proof in three dimensional case, it sufficient to check that the following lemma [18], is satisfied:

Lemma 6. *For any open* Ω *in* $\mathbb{R}^3$ *we have*

$$\| u \|_{L^4(\Omega)} \leq \sqrt{2} \mid u \mid_{L^2(\Omega)}^{1/4} \| u \|_{H_0^1(\Omega)}^{3/4} \qquad \forall u \in H_0^1(\Omega).$$

Theorem 7. *Under Hypothesis of the existence of the theorem 1., the solution* u *of the problem* (1.1) *verifies:*

$$u \in L^{8/3}[0, T, L^4(\Omega)]$$

Proof. For all $t \in [0, T]$, we have from the lemma 6.

$$\|u(t)\|_{L^4(\Omega)} \leq \sqrt{2}|u(t)|^{1/4}\,\|u(t)\|^{3/4}.$$

From the theorem 1., $u \in L^\infty(0,T,H) \cap L^2(0,T,V)$ and we have then:

$$\|u(t)\|^{8/3}_{L^4(\Omega)} \leqq 2^{4/3}|u(t)|^{2/3}_{L^\infty(0,T,H)}\,\|u(t)\|^2,$$

and $u \in L^{8/3}[0,T,L^4(\Omega)]$.

□

Theorem 8 (Stability theorem)**.** *Let* μ_1 *and* μ_2 *verifie* $(1)-(5)$*. It is assumed more that*

$$\inf_{s\in\mathbb{R}_+} \mu_1(s) = b_1 \neq 0 \;\; or \;\; \inf_{s\in\mathbb{R}_+} \mu_2(s) = b_2 \neq 0$$

Then there exists a constant $K > 0$ *such that*

$$|\, u_1 - u_2 \,|_{L^\infty(0,T,H)} \leq K \sup_{s\in\mathbb{R}_+} |\, \mu_1(s) - \mu_2(s) \,|$$

with $K = \dfrac{1}{\sqrt{b_1}}\|u_1\|_{L^2(0,T,V)}$ *if* $b_1 \neq 0$ *or* $K = \dfrac{1}{\sqrt{b_2}}\|u_2\|_{L^2(0,T,V)}$ *if* $b_2 \neq 0$.

Proof. From theorem 1., there exists u_1 and u_2 two solutions of the problem (1.1) with μ_1 and μ_2 (respectively) such that:

$$u_1 \in L^2(0,T,V) \cap L^\infty(0,T,H)$$
$$u_2 \in L^2(0,T,V) \cap L^\infty(0,T,H)$$

We take $w = u_1 - u_2$ and $v = w(t)$. We obtain from the equation (1.2):

$$\frac{1}{2}\frac{d}{dt} \mid w(t) \mid^2 + \int_\Omega \Big\{\mu_1[\mid \nabla u_1(t) \mid]\nabla u_1(t) - \mu_2[\mid \nabla u_2(t) \mid]\nabla u_2(t)\Big\}\nabla w(t)dx = 0. \tag{4.1}$$

In the case where $\inf_{s\in\mathbb{R}^+} \mu_2(s) \neq 0$ we add and subtract the term

$$\int_\Omega \mu_2[\mid \nabla u_1(t) \mid]\nabla u_1(t)\nabla w(t)dx,$$

we obtain

$$\left\{\begin{array}{rl} & \frac{1}{2}\frac{d}{dt} \mid w(t) \mid^2 + \int_\Omega \mu_2[\mid \nabla u_1(t) \mid]\nabla u_1(t)\nabla w(t)dx - \int_\Omega \mu_2[\mid \nabla u_2(t) \mid]\nabla u_2(t)\nabla w(t)dx \\ = & -\int_\Omega \mu_1[\mid \nabla u_1(t) \mid]\nabla u_1(t)\nabla w(t)dx + \int_\Omega \mu_2[\mid \nabla u_1(t) \mid]\nabla u_1(t)\nabla w(t)dx. \end{array}\right. \tag{4.2}$$

Then
$$\left\{\begin{array}{rl} \dfrac{1}{2}\dfrac{d}{dt} \mid w(t) \mid^2 +(Au_1(t)-Au_2(t),w(t)) & = \\ \displaystyle\int_\Omega \Big\{\mu_2[\mid \nabla u_1(t) \mid] - \mu_1[\mid \nabla u_1(t) \mid]\Big\} \nabla u_1(t) \nabla w(t) dx, & \end{array}\right. \tag{4.3}$$

where A is an operator defined as in (1.3) with μ_2.
By using (1.4)
$$(Au_1(t)-Au_2(t),w(t)) \geq b_2 \parallel w(t) \parallel^2,$$

we have then:
$$\frac{1}{2}\frac{d}{dt} \mid w(t) \mid^2 + b_2 \parallel w(t) \parallel^2 \leq \sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \parallel u_1(t) \parallel \parallel w(t) \parallel . \tag{4.4}$$

It's known, that:
$$\Big(\frac{1}{\sqrt{2b_2}} \sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \parallel u_1(t) \parallel - \frac{\sqrt{b_2}}{\sqrt{2}} \parallel w(t) \parallel \Big)^2 \geq 0,$$

hence it follows:
$$\left\{\begin{array}{rcl} \displaystyle\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \parallel u_1(t) \parallel \parallel w(t) \parallel & \leq & \dfrac{1}{2b_2}\Big(\displaystyle\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \Big)^2 \Big(\parallel u_1(t) \parallel \Big)^2 \\ & + & \dfrac{b_2}{2}\Big(\parallel w(t) \parallel \Big)^2 \end{array}\right. \tag{4.5}$$

we get then:
$$\frac{1}{2}\frac{d}{dt} \mid w(t) \mid^2 + \frac{b_2}{2}\Big(\parallel w(t) \parallel \Big)^2 \leq \frac{1}{2b_2}\Big(\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \Big)^2 \Big(\parallel u_1(t) \parallel \Big)^2. \tag{4.6}$$

As
$$\frac{b_2}{2}\Big(\parallel w(t) \parallel \Big)^2 > 0,$$

it results then:
$$\frac{1}{2}\frac{d}{dt} \mid w(t) \mid^2 \leq \frac{1}{2b_2}\Big(\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid \Big)^2 \Big(\parallel u_1(t) \parallel \Big)^2. \tag{4.7}$$

Since $w(0)=0$, we deduce:
$$\mid w(t) \mid^2 \leq \frac{1}{b_2}(\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid)^2 \int_0^T \parallel u_1(\tau) \parallel^2 d\tau \tag{4.8}$$
$$\mid w(t) \mid^2 \leq \frac{1}{b_2} \parallel u_1 \parallel^2_{L^2(0,T,V)} (\sup_{s\in \mathbb{R}_+} \mid \mu_1(s)-\mu_2(s) \mid)^2 \tag{4.9}$$

We have then:

$$\sup_{t\in[0,T]} \mid u_1(t) - u_2(t) \mid \leq K \sup_{s\in\mathbb{R}_+} \mid \mu_1(s) - \mu_2(s) \mid . \tag{4.10}$$

With $K = \dfrac{1}{\sqrt{b_2}} \parallel u_1 \parallel_{L^2(0,T,V)}$.

Therefore

$$\mid u_1 - u_2 \mid_{L^\infty(0,T,H)} \leq K \sup_{s\in\mathbb{R}_+} \mid \mu_1(s) - \mu_2(s) \mid . \tag{4.11}$$

Notice that we can obtain the same result, in the case where $\inf_{s\in\mathbb{R}^+} \mu_1(s) \neq 0$, if we add and subtract the term

$$\int_\Omega \mu_1[\mid \nabla u_2(t) \mid] \nabla u_2(t) \nabla w(t) dx.$$

This completes the proof of the theorem.

□

Note that all results presented give us sufficient conditions of existence, uniqueness and regularity of the solution. These results also allow us to make a suitable choice data problem considered with a view to approaching numerically.

In this paper we propose a numerical approach by the finite difference methods for the following model:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\Big((\dfrac{1}{\sqrt{1+(\mid \nabla u \mid)^2}} + \alpha) \nabla u \Big) = 0 \quad in\ Q = \Omega \text{x}]0,T[\\ u(x,0) = u_0 \quad in\ \Omega, \qquad u(x,t) = 0 \ \ \forall\ x \in \partial\Omega \text{x} [0,T]. \end{cases} \tag{4.12}$$

where u represents the processed image.
We consider a bounded domain Ω in $\mathbb{R}^2$, containing strictly the image, in this case we can consider the homogeneous Dirichlet condition on $\partial\,\Omega$.

4.1. Discrete PDE's model

Notice that the problem (4.12) is discretized using the finite differences method. We use an explicit scheme, we denote respectively by h and k the spatial and time steps sizes. In this sequel, we take $h = 1$.
For the temporal discretization of the time derivative we have:

$$\frac{\partial u}{\partial t}(i,j) = \frac{u^{n+1}(i,j) - u^n(i,j)}{k},$$

the superscript n and $n+1$ denote the time levels t_n and t_{n+1}, respectively.

For the approximation of the spatial derivatives, a first-order explicit scheme is employed:

$$\nabla u^n(i,j) = \begin{cases} \dfrac{\partial u^n}{\partial x}(i,j) &= \dfrac{u^n(i+1,j)-u^n(i,j)}{h}, \\ \dfrac{\partial u^n}{\partial y}(i,j) &= \dfrac{u^n(i,j+1)-u^n(i,j)}{h}. \end{cases} \tag{4.13}$$

Then

$$|\nabla u^n(i,j)| = \sqrt{(\frac{u^n(i+1,j)-u^n(i,j)}{h})^2 + (\frac{u^n(i,j+1)-u^n(i,j)}{h})^2}.$$

The equations (4.12) can be used as a diffusion problem involving in image processing. We define for every field $p = (p_1, p_2) \in \mathbb{R}^2$, the discrete divergence approximation:

$$(\text{div}(p))_{i,j} = \begin{cases} p_1^n(i,j) - p_1^n(i-1,j) & \text{if } 1 < i < N_1 \\ p_1^n(i,j) & \text{if } i = 1 \\ -p_1^n(i-1,j) & \text{if } i = N_1 \end{cases}$$

$$+ \begin{cases} p_2^n(i,j) - p_2^n(i,j-1) & \text{if } 1 < j < N_1 \\ p_2^n(i,j) & \text{if } j = 1 \\ -p_2^n(i,j-1) & \text{if } j = N_1 \end{cases}$$

where N_1 is an integer greater than 2, and

$$p^n(i,j) = \begin{cases} p_1^n(i,j) = (\dfrac{1}{\sqrt{1 + (\mid \nabla u^n(i,j) \mid)^2}} + \alpha) \dfrac{\partial u^n}{\partial x}(i,j), \\ p_2^n(i,j) = (\dfrac{1}{\sqrt{1 + (\mid \nabla u^n(i,j) \mid)^2}} + \alpha) \dfrac{\partial u^n}{\partial y}(i,j), \end{cases} \tag{4.14}$$

Then, an iterative method based on the explicit scheme is used to solve the discrete model:

$$u^{n+1}(i,j) = u^n(i,j) + k\,\text{div}(p^n(i,j)),\ 1 \le n \le M,$$

where

$$p^n(i,j) = (p_1^n(i,j), p_2^n(i,j))$$

$u^n(i,j) = u(x_i, y_j, t_n), x_i = ih,\ y_j = jh,\ t_n = nk$ and $k = \dfrac{T}{M} = 0.1$ (where M is the number of iterations).

5. Numerical Results

In this section we will present the numerical results obtained by the proposed model. We recall that the SNR is the Signal-to-noise ratio used to estimate the quality of an image I_2 with respect to a reference image I_1, defined by the expression:

$$SNR(I_1/I_2) = 10\log_{10}\left[\frac{\sigma^2(I_1)}{\sigma^2(I_1 - I_2)}\right]$$

where σ is the variance.

We begin by presenting a few 2D processed images, obtained by the nonlinear parabolic model with ($h = 1$, $k = 0.1$ and M varies from 50 to 500) for processing different medical images corrupted by a Gaussian and Speckle noises, see Figure 1 and 2. We remark that the model gives a better solution for a small values of α.

Table (1) is a complete description of the parameter SNR number of experiments that yielded best results for the nonlinear model on images deteriorated by a Gaussian and Speckle noises for $\alpha = 10^{-6}$.

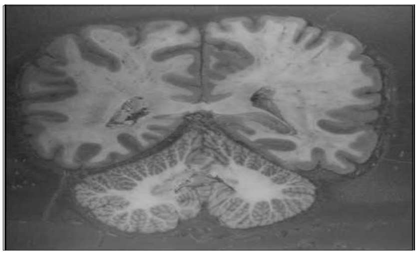
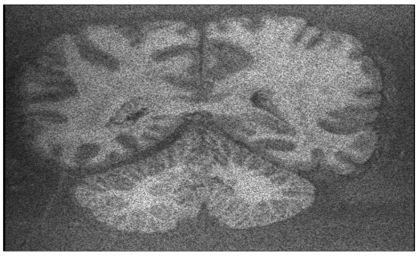
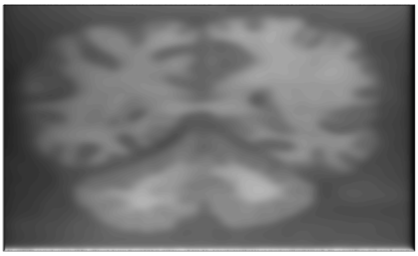


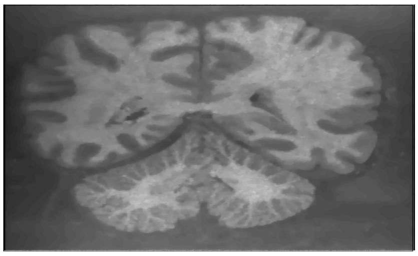
Image original (a)



Noisy Image (Speckle)



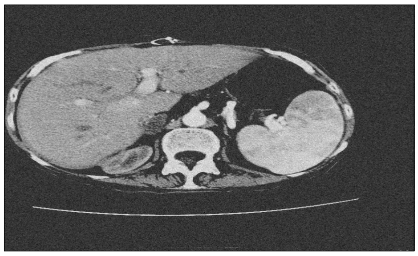
Restored Image with $\alpha = 1$



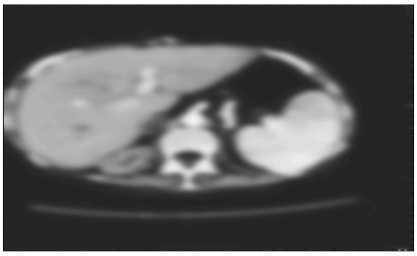
Restored Image with $\alpha = 10^{-6}$

Figure 1. Influence of the parametr α on the Restored d'image (Speckle noise)

Image originale (b)



Noisy Image (Gaussian 15%)



Restored Image with $\alpha = 1$

Restored Image with $\alpha = 10^{-6}$

Figure 2. Influence of the parametr α on the Restored d'image (Gaussian noise)

Images	SNR
image (a), Speckle Noise	17.45
image (b), Gaussian Noise 15%	26.35
image (b), Gaussian Noise 20%	25.93
image (b), Gaussian Noise 30%	24.51

Table 1: Evaluation of the noise suppression by the nonlinear restoration method

6. Conclusion

We proposed in this work a nonlinear parabolic PDE model for the medical image processing. After the mathematical analysis of the PDE problem, we tested the model numerically. The obtained results show the effectiveness of the approach.

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