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# Global Existence and Boundedness of Solutions to a Model of Chemotaxis

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**Abstract.** A model of chemotaxis is analyzed that prevents blow-up of solutions. The model consists of a system of nonlinear partial differential equations for the spatial population density of a species and the spatial concentration of a chemoattractant in $n$-dimensional space. We prove the existence of solutions, which exist globally, and are $L^{\infty}$-bounded on finite time intervals. The hypotheses require nonlocal conditions on the species-induced production of the chemoattractant.

**Key words:** chemotaxis, global solution, boundedness, nonlocal conditions, diffusion, analytic semigroup, fractional power
**AMS subject classification:** 92C17, 92B05, 92D25, 47D03, 47H20, 35M10

## 1. Introduction

Chemotaxis is the directed movement of bacteria, eukaryotic cells, or multicellular organisms toward concentrations of environmental chemoattractants. Bacterial chemotaxis is based on biased alteration of nondirectional tumbling and unidirectional swimming motions in response to chemical gradients [2]. Eukaryotic chemotaxis is based on detection of molecular signaling gradients through intracellular pathways [6]. Mathematical models of chemotaxis have been used extensively to model biological processes such as migratory behavior, pattern formation, and aggregation phenomena. The origin of these models began with the investigations of C.S. Patlak [30] and E.F. Keller and L.A. Segal [20], and a comprehensive

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review is given in [18]. A classic example of a chemotaxis model describes the formation of the spore-bearing stalk sporangiophore in the life cycle of the slime mold *Dictyostelium discoideum* ([10], [12], [24], [26], [31]).

Many models of chemotaxis exhibit blow-up behavior of solutions. In 1-spatial dimension it has been shown relatively recently that blow-up does not occur ([13], [28]). In spatial dimensions $\geq 2$, blow-up is frequently present, depending on model parameters and initial data ([4], [5], [15], [16], [19], [25]). Blow-up phenomena in chemotaxis models result from the extreme concentration of the species or the chemoattractant in a spatial region whose effective size may be much smaller than the size of a physically meaningful realization. Even in 1-dimension models for which it is proven that blow-up does not occur, "numerical blow-up" is frequently observed due to the inability of discretization approximations to capture this extreme behavior.

Our objective is to develop a model of chemotaxis in which blow-up is prevented. Other researchers have developed chemotaxis models that avoid blow-up, including [3], [14], [17], [21], [22], [27], [29], [33], [39]. One approach is to allow the processes that drive chemotactic behavior to depend nonlocally on the constituents, rather than point-wise in the spatial region. In [14] a chemotaxis model is developed which avoids blow-up by using a nonlocal gradient sensing term to model the effective sampling radius of the species. Our approach here is to use a nonlocal term to model the species-induced production of the chemoattractant. We argue that such a nonlocal assumption is biologically realistic, because biological quantities, such as cells and molecules, are not dimensionless points, but geometric objects occupying irreducible volumes.

In Section 2 we describe the chemotaxis model, in Section 3 we present preliminaries needed for the analysis, in Section 4 we prove local existence of solutions, and in Section 5 we establish regularity of these solutions, in Section 6 we prove global existence of solutions and show that solutions are $L^\infty$-bounded on finite time intervals, in Section 7 we establish positivity of solutions, in Section 8 we give $L^1$ growth bounds for $p$, in Section 9 we provide an illustrative example, and in Section 10 we give some concluding remarks.

# 2. The Model of Chemotaxis

The model describes the spatial movement of a species in the presence of a chemoattractant produced by the species. Both species and chemoattractant are diffusing in a spatial region $\Omega$, and the species is attracted up-gradient the concentration of the chemoattractant. The variables of the model are

$$c(x,t) = \text{concentration of chemoattractant at position } x \in \Omega \text{ at time } t,$$

$$p(x,t) = \text{population density of the species at position } x \in \Omega \text{ at time } t.$$

Throughout we take $\Omega \subset \mathbb{R}^n$, $n \geq 1$, to be a non-empty open bounded set with smooth boundary $\partial\Omega$. The equations of the model are

$$\frac{\partial}{\partial t}c(x,t) = \underbrace{\eta\Delta c(x,t)}_{diffusion} - \underbrace{\omega_c c(x,t)}_{degradation} + \underbrace{\gamma(x,c(\cdot,t),p(\cdot,t))}_{species\ induced\ production} \tag{2.1}$$

$$\frac{\partial}{\partial t}p(x,t) = \underbrace{\xi\Delta p(x,t)}_{motility} - \underbrace{\Big(\omega_p p(x,t) + \mu(x,c(\cdot,t),p(\cdot,t))p(x,t)\Big)}_{loss} - \underbrace{\nabla\cdot\Big(\chi(x,c(\cdot,t),p(\cdot,t))p(x,t)\,\nabla c(x,t)\Big)}_{chemotaxis} \tag{2.2}$$

We take Neumann boundary conditions

$$\left.\frac{\partial c(x,t)}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial p(x,t)}{\partial n}\right|_{\partial\Omega} = 0 \tag{2.3}$$

and initial conditions,

$$c(\cdot,0) = c_0, \quad p(\cdot,0) = p_0. \tag{2.4}$$

We assume the constants $\eta > 0$, $\xi > 0$, $\omega_c > 0$ and $\omega_p > 0$. The nonlocal character of the production term $\gamma$ in (2.1) incorporates a distributed sensing of the chemoattractant $c(\cdot,t)$ and the species density $p(\cdot,t)$, rather than a strict point-wise sensing. This form of $\gamma$ enables an analysis of the equations, in which the spatial regularity required in (2.2) is subsumed by the spatial regularity hypotheses imposed on $\gamma$. It is lack of regularity, under point-wise hypotheses, that produces blow-up in chemotaxis models in $\mathbb{R}^n$, $n \geq 2$.

# 3. Preliminaries

We denote by $||\cdot||_r$ the norm in $L^r(\Omega)$, by $||\cdot||_\infty$ the norm in $C(\bar{\Omega})$ and by $||\cdot||_{s,r}$ the norm in the Sobolev(-Slobodeckii) space $W^{s,r}(\Omega)$. Let $W^{2,r}(\Omega;B) = \{u \in W^{2,r}(\Omega) : Bu|_{\partial\Omega} = 0\}$, where $Bu = \frac{\partial u}{\partial n}$, the normal derivative. Define the operator $A_p : D(A_p) \subset L^1(\Omega) \to L^1(\Omega)$ as the closure in $L^1(\Omega)$ of the operator $A_{p,r}$ where, for any $1 < r < \infty$, $A_{p,r} : D(A_{p,r}) \subset L^r(\Omega) \to L^r(\Omega)$ is defined by

$$\begin{aligned} A_{p,r}u &= (-\xi\Delta + \omega_p)u,\ u \in D(A_{p,r}) \quad \text{with} \\ D(A_{p,r}) &= W^{2,r}(\Omega;B). \end{aligned} \tag{3.1}$$

It is well known (see [1]) that $-A_{p,r}$ generates a positive analytic semigroup $\{T_{p,r}(t)\}_{t\geq 0}$, $\|T_{p,r}(t)\| \leq e^{-\omega_p t}$ and $-A_p$ generates a positive analytic semigroup $\{T_p(t)\}_{t\geq 0}$, with $\|T_p(t)\| \leq e^{-\omega_p t}$ (where $\|\ \|$ is the operator norm). We define $A_c : D(A_c) \subset C(\bar{\Omega}) \to C(\bar{\Omega})$, by

$$\begin{aligned} A_c u &= (-\eta\Delta + \omega_c)u \quad \text{with} \\ D(A_c) &= \{u \in \bigcap_{1\leq q<\infty} W^{2,q}(\Omega) : \Delta u \in C(\bar{\Omega}) \text{ and } Bu|_{\partial\Omega} = 0\}. \end{aligned} \tag{3.2}$$

Then $-A_c$ generates a positive analytic semigroup $\{T_c(t)\}_{t\geq 0}$, with $||T_c(t)|| \leq Me^{-\omega_c t}$ (see [1], [23], [34], [35]). Note 0 is in the resolvent set of $A_p$ and $A_c$ as $\omega_p > 0$ and $\omega_c > 0$.

We denote by $B_i$, $i = 1, \dots, n$, the closed linear operator $B_i : D(B_i) \subset L^1(\Omega) \to L^1(\Omega)$,

$$B_i u = \frac{\partial u}{\partial x_i}, \quad \text{with} \quad D(B_i) = W^{1,1}(\Omega). \tag{3.3}$$

Thus the mild form of (2.1) – (2.4) is the following:

$$c(t) = T_c(t)c_0 + \int_0^t T_c(t-s)\gamma(c(s), p(s))\mathrm{d}s \tag{3.4}$$

$$\begin{aligned} p(t) = T_p(t)p_0 + \int_0^t T_p(t-s)\Bigg( &- \sum_{i=1}^n \Bigg( \chi(c(s), p(s))B_i p(s) \frac{\partial}{\partial x_i} c(s) \\ &+ \chi(c(s), p(s))p(s) \frac{\partial^2}{\partial x_i^2} c(s) + p(s) \frac{\partial}{\partial x_i} \chi(c(s), p(s)) \frac{\partial}{\partial x_i} c(s) \Bigg) \\ &- \mu(c(s), p(s))p(s) \Bigg) \mathrm{d}s, \end{aligned} \tag{3.5}$$

where here and in the sequel, we suppress the variable $x$ in $\gamma(x, c, p)$ etc.

To study the solutions of this problem we use the fractional powers, $A_p^\alpha$, of $A_p$ (see [32]). Take $\alpha$ such that $\frac{1}{2} < \alpha < 1$. Set $v(t) = A_p^\alpha p(t)$, and equation (3.5) becomes

$$\begin{aligned} v(t) = T_p(t)A_p^\alpha p_0 + \int_0^t A_p^\alpha T_p(t-s)\Bigg( &- \sum_{i=1}^n \Bigg( \chi(c(s), A_p^{-\alpha}v(s))B_i A_p^{-\alpha}v(s) \frac{\partial}{\partial x_i} c(s) \\ &+ \chi(c(s), A_p^{-\alpha}v(s))A_p^{-\alpha}v(s) \frac{\partial^2}{\partial x_i^2} c(s) + A_p^{-\alpha}v(s) \frac{\partial}{\partial x_i} \chi(c(s), A_p^{-\alpha}v(s)) \frac{\partial}{\partial x_i} c(s) \Bigg) \\ &- \mu(c(s), A_p^{-\alpha}v(s))A_p^{-\alpha}v(s) \Bigg) \mathrm{d}s. \end{aligned} \tag{3.6}$$

This equation is easier to study because, as we will see from Lemma 3.1, the operator $B_i A_p^{-\alpha}$, unlike $B_i$, is continuous.

We have, from [32] Theorem 2.6.13 and Lemma 2.6.3:

i if $\alpha > 0$, for every $u \in D(A_p^\alpha)$, $T_p(t)A_p^\alpha u = A_p^\alpha T_p(t)u$;

ii if $\alpha > 0$, then for every $t > 0$ the operator $A_p^\alpha T_p(t)$ is a bounded linear operator in $L^1(\Omega)$ and there exists $C_\alpha > 0$ such that

$$||A_p^\alpha T_p(t)|| \leq C_\alpha t^{-\alpha} e^{-\omega_p t}; \tag{3.7}$$

iii if $0 \leq \alpha \leq 1$, then $A_p^{-\alpha}$ is a bounded linear operator in $L^1(\Omega)$ and there exists a constant $E \geq 1$ such that for all $u \in L^1(\Omega)$,

$$||A_p^{-\alpha}u||_1 \leq E||u||_1.$$

Throughout $D(A_p^\alpha)$ is a Banach space with the graph norm $||u||_{D(A_p^\alpha)} = ||A_p^\alpha u||_1$ and similarly for $D(A_{p,r}^\alpha)$. The following two lemmas come from [7].

**Lemma 3.1.** *(i) If $0 < \beta < \alpha < 1$, then $D(A_p^\alpha) \hookrightarrow W^{2\beta,1}(\Omega)$. Thus for $\frac{1}{2} < \alpha < 1$, $D(A_p^\alpha) \hookrightarrow D(B_i)$ and $B_i A_p^{-\alpha}$ is a bounded linear operator on $L^1(\Omega)$.*

*(ii) If $1 < r < \infty$, $0 < \beta < \alpha < 1$, then $D(A_{p,r}^\alpha) \hookrightarrow W^{2\beta,r}(\Omega)$. Thus for $\frac{1}{2} < \alpha < 1$, $D(A_{p,r}^\alpha) \hookrightarrow D(B_i)$ and $B_i A_{p,r}^{-\alpha}$ is a bounded linear operator on $L^r(\Omega)$.*

**Lemma 3.2.** *There is a constant $L > 0$ such that for all $c_1 \in D(A_c)$, $||c_1||_\infty \le L||A_c c_1||_\infty$, $||\frac{\partial}{\partial x_i} c_1||_\infty \le L||A_c c_1||_\infty$ and $||\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} c_1||_\infty \le L||A_c c_1||_\infty$.*

**Proof.** Note first that $\frac{\partial}{\partial x_i} A_c^{-1}$ is bounded (see [32], Corollary 2.6.11). Thus

$$\left\| \frac{\partial}{\partial x_i} c_1 \right\|_\infty = \left\| \frac{\partial}{\partial x_i} A_c^{-1} A_c c_1 \right\|_\infty \le L||A_c c_1||_\infty.$$

But $A_c^{-1}$ is a bounded operator, say $||A_c^{-1}|| = N$. So $||c_1||_\infty = ||A_c^{-1} A_c c_1||_\infty \le N||A_c c_1||_\infty$. The result now follows.

Our approach to the chemotaxis problem is similar to that developed in [7] – [9] where we study mathematical models of tumor invasion with haptotaxis and some of the proofs here are natural adaptations of those given in these papers.

# 4. Local existence of the mild solution

Take

$$y(t) = \begin{bmatrix} c(t) \\ v(t) \end{bmatrix}$$

and work in the space $Y = C([0, t_0]; Z)$, for suitable $t_0 > 0$, where $Z = D(A_c) \times L^1(\Omega)$, with norm $||[c, v]^t||_Z = ||A_c c||_\infty + ||v||_1$ and $||y||_Y = \max_{0 \le t \le t_0} ||y(t)||_Z$.

Take $c_0 \in D(A_c)$ and $p_0 \in D(A_p^\alpha)$ and set

$$y_0 := y_0(t) = \begin{bmatrix} c_0 \\ v_0 \end{bmatrix} \in Y,$$

where $v_0 = A_p^\alpha p_0$. For $y \in Y$, define $\mathcal{F} : Y \to Y$ by

$$\mathcal{F}y(t) = \begin{bmatrix} T_c(t)c_0 + \int_0^t T_c(t-s)\gamma(c(s), A_p^{-\alpha}v(s))\mathrm{d}s \\ T_p(t)v_0 + \int_0^t A_p^\alpha T_p(t-s)\Bigg( -\sum_{i=1}^n \Big(\chi(c(s), A_p^{-\alpha}v(s))B_i A_p^{-\alpha}v(s)\frac{\partial}{\partial x_i}c(s) + \\ \chi(c(s), A_p^{-\alpha}v(s))A_p^{-\alpha}v(s)\frac{\partial^2}{\partial x_i^2}c(s) + A_p^{-\alpha}v(s)\frac{\partial}{\partial x_i}\chi(c(s), A_p^{-\alpha}v(s))\frac{\partial}{\partial x_i}c(s)\Big) \\ -\mu(c(s), A_p^{-\alpha}v(s))A_p^{-\alpha}v(s)\Bigg)\mathrm{d}s \end{bmatrix} \tag{4.1}$$

and for convenience we will write $\mathcal{F}y(t) = [\mathcal{F}_1 y(t), \mathcal{F}_2 y(t)]^t$.

Finally, if $||y_0||_Y = R$, set

$$\mathcal{N}_R = \{y \in Y : ||y||_Y \le (M+3)R\}.$$

We will show that for $t_0 > 0$, but small enough, $\mathcal{F}$ has a unique fixed point $y \in \mathcal{N}_R$ and this will yield a mild solution of our model.

We make the hypotheses

H(4.1) There exists $\frac{1}{2} < \alpha < 1$ such that $\gamma : \Omega \times D(A_c) \times D(A_p^\alpha) \to \mathbb{R}$. Also, for all $c \in D(A_c)$ and $p \in D(A_p^\alpha)$, $\gamma(\cdot, c, p) \in D(A_c)$ and, for any $R > 0$, there is a constant $H_R > 0$, independent of $x$, such that if $||A_c c_j||_\infty \le R$ and $||A_p^\alpha p_j||_1 \le R$, $j = 1, 2$, then for all $x \in \Omega$,

$$||A_c\gamma(x, c_1, p_1) - A_c\gamma(x, c_2, p_2)||_\infty \le H_R \left\{||A_c c_1 - A_c c_2||_\infty + ||A_p^\alpha p_1 - A_p^\alpha p_2||_1\right\}.$$

H(4.2) $\mu : \Omega \times D(A_c) \times D(A_p^\alpha) \to \mathbb{R}$ is measurable in $x$, $\chi : \Omega \times D(A_c) \times D(A_p^\alpha) \to \mathbb{R}$, and for all $c \in D(A_c)$ and $p \in D(A_p^\alpha)$, $\chi(\cdot, c, p) \in C^1(\bar{\Omega})$ and for any $R > 0$, there is a constant $J_R > 0$, independent of $x$, such that whenever $\max\{||A_c c_j||_\infty, ||A_p^\alpha p_j||_1\} \le R$, $j = 1, 2$, and $x \in \Omega$,

$$|\mu(x, c_1, p_1) - \mu(x, c_2, p_2)| \le J_R\{||A_c c_1 - A_c c_2||_\infty + ||A_p^\alpha p_1 - A_p^\alpha p_2||_1\}, \tag{4.2}$$

$$\max_{1\le i\le n}\left\{\left\|\frac{\partial\chi(c_1, p_1)}{\partial x_i} - \frac{\partial\chi(c_2, p_2)}{\partial x_i}\right\|_\infty, ||\chi(c_1, p_1) - \chi(c_2, p_2)||_\infty\right\} \le J_R\{||A_c c_1 - A_c c_2||_\infty + ||A_p^\alpha p_1 - A_p^\alpha p_2||_1\} \tag{4.3}$$

Also $\mu(\cdot, 0, 0) \in L^\infty(\Omega)$.

**Remark 4.1.** The key idea here is to use a nonlocal dependence on $p$ of the nonlinearity in the $c(x,t)$ equation to obtain smoothness and boundedness information about spatial derivatives of $c(x,t)$ for use in the $p(x,t)$ equation. For example, we could use

$$\gamma(x, c, p) = \beta(x, p)\tau_c(c(x)),\ x \in \Omega,\ c \in D(A_c),\ p \in D(A_p^\alpha),$$

where $\tau_c : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, $\tau_c''$ is Lipschitz continuous on bounded sets, and $\beta$ satisfies

H(4.1)$'$ There exists $\frac{1}{2} < \alpha < 1$ such that $\beta : \Omega \times D(A_p^\alpha) \to \mathbb{R}$, $\beta(\cdot, p) \in C^2(\bar{\Omega})$ with $B\beta(\cdot, p)|_{\partial\Omega} = 0$ for all $p \in D(A_p^\alpha)$. In addition, for any $R > 0$, there is a constant $H_R' > 0$, independent of $x$, such that if $||A_p^\alpha p_j||_1 \le R$, $j = 1, 2$, then for all $x \in \Omega$,

$$\max\Big\{|\beta(x, p_1) - \beta(x, p_2)|, \sum_{i=1}^n \left|\tfrac{\partial}{\partial x_i}\beta(x, p_1) - \tfrac{\partial}{\partial x_i}\beta(x, p_2)\right|, \sum_{i=1}^n \left|\tfrac{\partial^2}{\partial x_i^2}\beta(x, p_1) - \tfrac{\partial^2}{\partial x_i^2}\beta(x, p_2)\right|\Big\} \le H_R'||A_p^\alpha p_1 - A_p^\alpha p_2||_1.$$

Adapting Lemma 3 of [7] it can be seen that this $\gamma$ satisfies H(4.1).

Another possibility is

$$\gamma(c,p)(x) = \int_\Omega \kappa(x,\hat{x}) f(\hat{x}, c(\hat{x}), \nabla c(\hat{x}), \Delta c(\hat{x})) g(\hat{x}, p(\hat{x}), \nabla p(\hat{x})) \mathrm{d}\hat{x},$$

where $\kappa(x,\hat{x}) \in D(A_c)$ for all $\hat{x}$ and $(A_c\kappa)(x,\hat{x}) \in C(\bar{\Omega} \times \bar{\Omega})$; and $f : \Omega \times \mathbb{R}^{n+2} \to \mathbb{R}$ is locally Lipschitz continuous, $g : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ is globally Lipschitz continuous, both uniformly with respect to $x$, and also $f(x,0,0,0) \in L^\infty(\Omega)$ and $g(x,0,0) \in L^1(\Omega)$. To satisfy H(6.1) (see Section 6) we would also require $f$ is globally Lipschitz continuous, uniformly with respect to $x$, $f(x,0,0,0) = 0$ for all $x$ and $g$ is globally bounded. For example to satisfy all the hypotheses we could take $f(\hat{x}, y) = \sum_1^{n+2} a_i(\hat{x}) y_i$ where $a(\hat{x}) \in L^\infty(\Omega)$ and $g(\hat{x}, y) = \cos(\sum_1^{n+1} b_i(\hat{x}) y_i)$ where $b_i(\hat{x}) \in L^\infty(\Omega)$.

Note that in the first example the dependence of $\gamma(x,c,p)$ on $c$ is local. Similarly condition H(4.2) allows the dependence on $c$ of the coefficients in equation (2.2) to be local. For example the case $\mu(x,c,p) = \hat{\mu}(c(x))$ is included, where $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ and is locally Lipschitz continuous.

It follows from H(4.1) and H(4.2) that there exists a constant $k_R$, which depends only on $R$, such that, for all $x \in \Omega$, $y = [c,v]^t \in \mathcal{N}_R$, and $0 \le t \le t_0$

$$\max_{1\le i\le n} \left\{ |\mu(x, c(t), A_p^{-\alpha} v(t)))|, ||\chi(c(t), A_p^{-\alpha} v(t))||_\infty, \left\| \frac{\partial \chi(c(t), A_p^{-\alpha} v(t)))}{\partial x_i} \right\|_\infty, \right. \\ \left. ||A_c \gamma(c(t), A_p^{-\alpha} v(t))||_\infty \right\} \le k_R. \tag{4.4}$$

The proof proceeds through a series of lemmas.

**Lemma 4.2.** *(i) For $c_1 \in D(A_c)$ and $v_1 \in L^1(\Omega)$, $\frac{1}{2} < \alpha < 1$, define*

$$G(c_1, v_1) = \nabla \cdot (\chi(c_1, A_p^{-\alpha} v_1) A_p^{-\alpha} v_1 \nabla c_1) \tag{4.5}$$

$$:= \sum_{i=1}^n \Big( \chi(c_1, A_p^{-\alpha} v_1) B_i A_p^{-\alpha} v_1 \frac{\partial}{\partial x_i} c_1 + \chi(c_1, A_p^{-\alpha} v_1) A_p^{-\alpha} v_1 \frac{\partial^2}{\partial x_i^2} c_1 + A_p^{-\alpha} v_1 \frac{\partial}{\partial x_i} \chi(c_1, A_p^{-\alpha} v_1) \frac{\partial}{\partial x_i} c_1 \Big). \tag{4.6}$$

*Then there exists a constant $L_\alpha > 0$ such that*

$$||G(c_1, v_1)||_1 \le L_\alpha ||v_1||_1 ||A_c c_1||_\infty \max_{1\le i\le n} \left\{ ||\chi(c_1, A_p^{-\alpha} v_1)||_\infty, \left\| \frac{\partial \chi(c_1, A_p^{-\alpha} v_1)}{\partial x_i} \right\|_\infty \right\}.$$

*(ii) If $c_j, v_j$, $j = 1,2$ are such that $||A_c c_j||_\infty \le R$ and $||v_j||_1 \le R$ then there exists a constant $F_R > 0$ such that*

$$||G(c_1, v_1) - G(c_2, v_2)||_1 \le F_R \{ ||A_c c_1 - A_c c_2||_\infty + ||v_1 - v_2||_1 \},$$

*and for* $y = [c, v]^t \in Y$*,* $G(c(\cdot), v(\cdot)) \in C([0, t_0]; L^1(\Omega))$*.*
**Proof.** Part (i) follows immediately from Lemma 3.1(i) and Lemma 3.2. Part (ii) is similar.

It now follows that

**Lemma 4.3.** $\mathcal{F} : Y \to Y$ *is well defined and there exists* $t_0 > 0$ *such that* $\mathcal{F}(\mathcal{N}_R) \subset \mathcal{N}_R$ *.*
**Proof.** Let $y = [c, v]^t \in Y$. By H(4 .1), $A_c\gamma(x, c(t), A_p^{-\alpha}v(t)) \in C([0, t_0]; C(\bar{\Omega}))$, so, $\mathcal{F}_1 y(t) \in D(A_c)$,

$$A_c\mathcal{F}_1 y(t) = T_c(t)A_c c_0 + \int_0^t T_c(t-s)A_c\left(\gamma(c(s), A_p^{-\alpha}v(s))\right) \mathrm{d}s, \tag{4.7}$$

the integral exists and if $y \in \mathcal{N}_R$,

$$||A_c\mathcal{F}_1 y(t)||_\infty \le (M + 1)R,$$

for $t_0$ small enough.

Also, from (3.7), for $y \in Y$,

$$\int_0^t ||A_p^\alpha T_p(t-s)(-G(c(s), v(s)) - \mu(c(s), A_p^{-\alpha}v(s))A_p^{-\alpha}v(s))||_1 \mathrm{d}s$$
$$\le \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\omega_p(t-s)}(||G(c(s), v(s))||_1 + |\mu(c(s), A_p^{-\alpha}v(s))|||v(s)||_1)\mathrm{d}s.$$

Thus the integral exists and if $y \in \mathcal{N}_R$ then for $t_0$ small enough

$$||\mathcal{F}_2 y(t)||_1 \le R + K_1 \int_0^t (t-s)^{-\alpha} e^{-\omega_p(t-s)}\mathrm{d}s \le 2R,$$

where $K_1 > 0$ is a constant depending on $R$.

**Lemma 4.4.** *For* $t_0$ *sufficiently small* $\mathcal{F}$ *is a contraction on* $\mathcal{N}_R$*.*
**Proof.** We prove that if $t_0$ is small enough, then given $y_1, y_2 \in \mathcal{N}_R$,

$$\|\mathcal{F}y_1 - \mathcal{F}y_2\|_Y \le \frac{2}{3}\|y_1 - y_2\|_Y.$$

We first show that for $0 \le t \le t_0$, $t_0$ small enough,

$$\|\mathcal{F}_2 y_1(t) - \mathcal{F}_2 y_2(t)\|_1 \le \frac{1}{3}\|y_1 - y_2\|_Y.$$

Using H(4 .2), (3.7), (4.4) and Lemma 4.2 we have for a suitable constant $K_2$, depending on $R$,

$$\begin{aligned}\|\mathcal{F}_2 y_1(t) - \mathcal{F}_2 y_2(t)\|_1 \le& \int_0^t C_\alpha (t-s)^{-\alpha} e^{-\omega_p(t-s)}\Big(\|G(c_1(s), v_1(s)) - G(c_2(s), v_2(s))\|_1 \\ &+ \|\mu(c_1(s), A_p^{-\alpha}v_1(s))A_p^{-\alpha}v_1(s) - \mu(c_2(s), A_p^{-\alpha}v_2(s))A_p^{-\alpha}v_2(s)\|_1\Big)\mathrm{d}s \\ \le& K_2\|y_1 - y_2\|_Y \int_0^t (t-s)^{-\alpha} e^{-\omega_p(t-s)}\mathrm{d}s \\ \le& \frac{1}{3}\|y_1 - y_2\|_Y,\end{aligned}$$

for $0 \leq t \leq t_0$ and $t_0$ small enough. Further, using (4.7) and H(4.1) it can be seen that $\|A_c\mathcal{F}_1 y_1(t) - A_c\mathcal{F}_1 y_2(t)\|_\infty \leq \frac{1}{3}\|y_1 - y_2\|_Y$.

The following local existence and uniqueness result is a consequence of Lemmas 4.3 and 4.4.

**Theorem 4.5.** *Suppose that hypotheses H(4.1) and H(4.2) hold and that* $\frac{1}{2} < \alpha < 1$. *Take* $[c_0, p_0]^t \in D(A_c) \times D(A_p^\alpha)$. *The problem (3.4) – (3.5) has a unique mild solution* $[c, p]^t \in C([0, t_0]; D(A_c) \times D(A_p^\alpha))$, *for* $t_0 > 0$ *small enough.*

# 5. Regularity

We prove that the mild solution of the problem (2.1) – (2.4), $[c, p]^t \in C([0, t_0]; D(A_c) \times D(A_p^\alpha))$, given in Theorem 4.5 is in fact classical.

**Theorem 5.1.** *Suppose that hypotheses H(4.1) and H(4.2) hold and take* $[c_0, p_0]^t \in D(A_c) \times D(A_p^\alpha)$ *where* $\frac{1}{2} < \alpha < 1$. *Suppose that* $[c, p]^t \in C([0, t_0]; D(A_c) \times D(A_p^\alpha))$ *is the solution of (3.4) – (3.5) on* $[0, t_0]$.

*Then* $[c, p]^t$ *is the classical solution on* $[0, t_0]$ *of the problem (2.1) – (2.4) in the sense that*

$$
\begin{aligned}
& c \in C([0, t_0]; C(\bar{\Omega})) \cap C^1((0, t_0]; C(\bar{\Omega})), \ \textit{and for } 0 \leq t \leq t_0, \ c(t) \in D(A_c), \\
& p \in C([0, t_0]; L^1(\Omega)) \cap C^1((0, t_0); L^1(\Omega)), \ \textit{and for } 0 < t \leq t_0, \ p(t) \in D(A_p),
\end{aligned} \tag{5.1}
$$

*and*

$$
\frac{\mathrm{d}}{\mathrm{d}t} c(t) = \eta \Delta c(t) - \omega_c c(t) + \gamma(c(t), p(t)), \quad 0 \leq t \leq t_0, \ c(0) = c_0 \tag{5.2}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p(t) = {} & \xi \Delta p(t) - \omega_p p(t) - \mu(c(t), p(t)) p(t) \\
& - \nabla \cdot (\chi(c(t), p(t)) p(t) \, \nabla c(t)), \quad 0 < t \leq t_0, \ p(0) = p_0.
\end{aligned} \tag{5.3}
$$

**Proof.** The proof is very like that in [8] but is given here for completeness. We can regard $\gamma(c(t), p(t))$, $\mu(c(t), p(t))$, $\chi(c(t), p(t))$ as known functions of $x$ and $t$. Note first that $A_c(\gamma(c(\cdot)), A_p^{-\alpha} v(\cdot)) \in C([0, t_0]; C(\bar{\Omega}))$. Thus by [32] Theorem 4.3.1, $A_c c(t)$ is locally Hölder continuous.

Further, clearly $H(c(t), v(t)) := G(c(t), v(t)) - \mu(c(t), A_p^{-\alpha} v(t)) A_p^{-\alpha} v(t) \in C([0, t_0]; L^1(\Omega))$, so that, using the same argument as in [32] Theorem 6.3.1, $v(t) = A^\alpha p(t)$ is locally Hölder continuous. Thus $H(c(t), v(t))$ and $\gamma(c(t), p(t))$ are also locally Hölder continuous and the result follows using [32] Corollary 4.3.3.

# 6. Global existence of solutions and $L^\infty$- boundedness

We now prove that the mild solution of (3.4) – (3.5) is a global solution. We make the additional assumption

H(6.1) $\mu$, $\chi$, $\frac{\partial}{\partial x_i}\chi$, $(i = 1, \ldots, n)$, are all globally bounded and for all $[c, v]^t \in D(A_c) \times L^1(\Omega)$, there exist constants $P > 0$ and $Q > 0$ such that

$$\|A_c\gamma(c, A_p^{-\alpha}v)\|_\infty \leq P + Q\|A_c c\|_\infty. \tag{6.1}$$

For example, (6.1) is satisfied if H(4.1) is satisfied with $H_R$ independent of $R$ and also there exists a constant $P' > 0$ such that $\|A_c(\gamma(0, p))\|_\infty \leq P'$ for all $p \in D(A_p^\alpha)$.

First we show that we have continuous dependence on the initial data. Set $U_\alpha = C([0, t_0]; D(A_c) \times D(A_p^\alpha))$, with norm

$$||u||_{U_\alpha} = \max_{0\leq t\leq t_0} \left\{||A_c c(t)||_\infty + ||p(t)||_{D(A_p^\alpha)}\right\}.$$

Let $u = [c, p]^t \in U_\alpha$ and $\hat{u} = [\hat{c}, \hat{p}]^t \in U_\alpha$ be the solutions of (3.4) – (3.5) with associated initial data $u_0 = [c_0, p_0]^t \in D(A_c) \times D(A_p^\alpha)$ and $\hat{u}_0 = [\hat{c}_0, \hat{p}_0]^t \in D(A_c) \times D(A_p^\alpha)$ respectively.

**Proposition 6.1.** *Suppose that conditions H(4.1), H(4.2) and H(6.1) hold and that $\frac{1}{2} < \alpha < 1$. Then solutions of (3.4) – (3.5) depend continuously on the initial data for $u \in U_\alpha$. That is, given any $\hat{u}_0 \in D(A_c) \times D(A_p^\alpha)$ and $\epsilon > 0$, there exists $\delta > 0$ such that $||u_0 - \hat{u}_0||_{U_\alpha} < \delta$ implies $||u - \hat{u}||_{U_\alpha} < \epsilon$.*

*Further, for any $t_0 > 0$, there is at most one solution of (3.4) – (3.5) in $U_\alpha$.*

**Proof.** We work in Y. So set $v = A_p^\alpha p$, $\hat{v} = A_p^\alpha \hat{p}$, $v_0 = A_p^\alpha p_0$, $\hat{v}_0 = A_p^\alpha \hat{p}_0$ and write $y = [c, v]^t$, $\hat{y} = [\hat{c}, \hat{v}]^t$, $y_0 = [c_0, v_0]^t$, $\hat{y}_0 = [\hat{c}_0, \hat{v}_0]^t$. Note that $\|T_c(t)\| \leq M$. Assume without loss of generality that $M \geq 1$. All the constants $C_j$ used below are independent of $y_0$. It follows easily, from (4.7) and H(6.1) that, for $t \leq t_0$,

$$\|A_c c(t)\|_\infty \leq M\|A_c c_0\|_\infty + M\int_0^t (P + Q\|A_c c(s)\|_\infty)\, \mathrm{d}s \tag{6.2}$$

so that, by Gronwall's inequality,

$$\|A_c c(t)\|_\infty \leq (M\|A_c c_0\|_\infty + MPt)e^{MQt}.$$

Also

$$\|v(t)\|_1 \leq \|v_0\|_1 + C_1\int_0^t (t-s)^{-\alpha}(\|v(s)\|_1||A_c c(s)||_\infty + ||v(s)||_1)\, \mathrm{d}s. \tag{6.3}$$

Then

$$\|y(t)\|_Z \leq M\|y_0\|_Z + C_2(t_0)(||A_c c_0||_\infty + 1)\int_0^t (t-s)^{-\alpha}\|y(s)\|_Z\, \mathrm{d}s. \tag{6.4}$$

So if $\lambda > 0$ then

$$e^{-t\lambda}\|y(t)\|_Z \leq M\|y_0\|_Z + C_2(t_0)(||A_c c_0||_\infty + 1)\int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}e^{-\lambda s}\|y(s)\|_Z \,\mathrm{d}s. \quad (6.5)$$

So, if we choose $\lambda$ such that $C_2(t_0)(2+||\hat{y}_0||_Z)\Gamma(1-\alpha)\lambda^{\alpha-1} < 1$, where $\Gamma(\beta) = \int_0^\infty e^{-x}x^{\beta-1}\,\mathrm{d}x$ is the Euler Gamma function, then for any $y_0$ such that $||y_0 - \hat{y}_0||_Z < 1$

$$\|y\|_Y \leq \frac{M(\|\hat{y}_0\|_Z + 1)}{1 - C_2(t_0)(2 + ||\hat{y}_0||_Z)\Gamma(1-\alpha)\lambda^{\alpha-1}}e^{\lambda t_0} = R_0 = R(t_0, ||\hat{y}_0||_Z) \quad (6.6)$$

(see Lemma 1.1 in [37]).

Thus $||y||_Y \leq R_0$ for all solutions $y \in Y$ such that $||y_0 - \hat{y}_0||_Z < 1$. Now we can use very similar arguments to those used above and in the proof of Lemma 4.4 to show that there is a constant $C_3 = C_3(t_0, ||\hat{y}_0||_Z)$ such that if we take $\lambda$ large enough

$$\|y - \hat{y}\|_Y \leq \frac{M\|y_0 - \hat{y}_0\|_Z}{1 - C_3\Gamma(1-\alpha)\lambda^{\alpha-1}}e^{\lambda t_0},$$

and continuous dependence and uniqueness follows.

**Theorem 6.2.** *Suppose that the conditions of Theorem 4.5 and H(6.1) hold, then, for* $[c_0, p_0]^t \in D(A_c) \times D(A_p^\alpha)$ *the problem (3.4) – (3.5) has a unique global solution.*

**Proof.** Let $t_{\max}$ be the sup of $t_0 > 0$ such that the problem (3.4) – (3.5) has a unique mild solution on $[0, t_0]$, so that the problem has a unique mild solution on $[0, t_{\max})$. Assume $t_{\max} < +\infty$. We claim that the solution is bounded on bounded intervals of time. It is easily seen that, given $R > 0$, $t_0$ found in Lemmas 4.3 and 4.4 can be chosen independently of the initial data $y_0$ when $||y_0||_Y \leq R$.

Let $T > t_{\max}$. Set $y_\epsilon = \max_{0\leq t\leq t_{\max}-\epsilon} e^{-\lambda t}\|y(t)\|_Z$. Then (6.5) holds for $t < t_{\max}$ so

$$y_\epsilon \leq M\|y_0\|_Z + C_2(T)(||A_c c_0||_\infty + 1)\Gamma(1-\alpha)\lambda^{\alpha-1}y_\epsilon.$$

Choose $\lambda$ big enough so that $C_2(T)\Gamma(1-\alpha)\lambda^{\alpha-1}(||A_c c_0||_\infty + 1) < 1$. Then

$$y_\epsilon \leq \frac{M\|y_0\|_Z}{1 - C_2(T)(||A_c c_0||_\infty + 1)\Gamma(1-\alpha)\lambda^{\alpha-1}}.$$

It follows that

$$\sup_{0\leq t<t_{\max}} \|y(t)\|_Z \leq \frac{M\|y_0\|_Z}{1 - C_2(T)(||A_c c_0||_\infty + 1)\Gamma(1-\alpha)\lambda^{\alpha-1}}e^{\lambda T} \quad (6.7)$$

and so the mild solution is bounded in $[0, t_{\max})$ and global existence follows as in [38] Theorem 2.3. The solution is classical from Theorem 5.1.

We can now see that, for suitable initial data, for any $T > 0$, solutions $c(x,t)$, $p(x,t)$ are bounded on $\bar{\Omega} \times [0, T]$.

**Theorem 6.3.** *Suppose that hypotheses H(4.1), H(4.2) and H(6.1) hold and that* $\frac{1}{2} < \alpha < 1$. *If we now take* $[c_0, p_0]^t \in D(A_c) \times D(A_{p,r}^\alpha)$*, for some* $r > n$*, then the unique solution of*

*the problem (3.4) – (3.5),* $[c(t), p(t)]^t$*, lies in* $C([0,\infty) \times \bar{\Omega}) \times C([0,\infty) \times \bar{\Omega})$*, and hence* $||c(t)||_\infty + ||p(t)||_\infty$ *is bounded on bounded subsets of* $[0,\infty)$*.*
**Proof.** Note first that we can carry out the same computations as in Sections 4 and 6 in

$$Y^r = C([0,\ t_0];\ D(A_c) \times L^r(\Omega)),$$

for $1 < r < \infty$ and suitable $t_0 > 0$, endowed with the norm

$$||y||_{Y^r} = \max_{0 \le t \le t_0} \{||A_c c(t)||_\infty + ||v(t)||_r\}.$$

Using Lemma 3.1(ii) we can show that under the same conditions as Theorem 6.2 for $[c_0, p_0]^t \in D(A_c) \times D(A^\alpha_{p,r})$, (3.4) – (3.5) has a unique global solution

$$[c,p]^t \in C([0,\infty); D(A_c) \times D(A^\alpha_{p,r})).$$

Also $T_p(t) : L^r(\Omega) \to L^r(\Omega)$, and if restricted to $L^r(\Omega)$ coincides with $T_{p,r}(t)$. Thus these solutions coincide on their common domain. So in particular $[c_0, p_0]^t \in D(A_c) \times D(A^\alpha_{p,r})$ implies $[c(t), p(t)]^t \in D(A_c) \times D(A^\alpha_{p,r})$.

However, from Lemma 3.1(ii), if $1 < r < \infty$, $0 < \beta < \alpha < 1$, then $D(A^\alpha_{p,r}) \hookrightarrow W^{2\beta,r}(\Omega)$. So $p \in C([0,\infty); W^{2\beta,r}(\Omega))$ and if we take $r > n$ and $\frac{1}{2} < \beta < \alpha < 1$, then $W^{2\beta,r}(\Omega) \hookrightarrow W^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})$ (see [11] Theorem 11.1, Part 1), so for any $T > 0$, $p \in C([0,T] \times \bar{\Omega})$.

# 7. Positivity

We make the additional hypotheses

H(7.1) $c_0 \ge 0$ and $p_0 \ge 0$.

H(7.2) $\gamma(x, c, p) \ge 0$ for $c \in D(A_c)$ with $c \ge 0$ and $p \in D(A^\alpha_p)$ with $p \ge 0$.

**Theorem 7.1.** *Suppose that hypotheses H(4.1), H(4.2), H(7.1), and H(7.2) are satisfied, that* $\frac{1}{2} < \alpha < 1$ *and* $[c_0, p_0]^t \in D(A_c) \times D(A^\alpha_{p,r})$ *for some* $r > n$*. Then if* $[c,p]^t \in C([0,t_0]; D(A_c) \times D(A^\alpha_p))$ *is the solution of (3.4) – (3.5) on* $[0, t_0]$*, it is positive. If also H(6.1) is satisfied then solutions are positive for all* $[c_0, p_0]^t \in D(A_c) \times D(A^\alpha_p)$*.*
**Proof.** The proof is essentially the same as [36] Section 4 and [8] Theorem 9. Take first $p_0 \in D(A^\alpha_{p,r})$, for some $r > n$. According to [28] p.45 there exists a function $H \in C^2(\mathbb{R})$ and a constant $c_0$ such that $H(z) = 0$ for $z \ge 0$ and $H(z) > 0$ for $z < 0$ and such that

$$0 \le H''(z)z^2 \le c_0 H(z), \quad z \in \mathbb{R}, \tag{7.1}$$
$$0 \le H'(z)z \le c_0 H(z), \quad z \in \mathbb{R}. \tag{7.2}$$

From Theorem 6.3, $\int_\Omega H(p(x,t))\mathrm{d}x$ is well defined. For $t \in [0, t_0]$, define $M(t) \in C([0, t_0]) \cap$

$C^1((0,t_0])$ by

$$
\begin{aligned}
M(t) &= \int_\Omega H(p(x,t))\,\mathrm{d}x, \qquad \text{so} \\
M'(t) &= \int_\Omega H'(p)p'\,\mathrm{d}x \\
&= \int_\Omega H'(p)(\xi\Delta p - \omega_p p - \mu(c,p)p - \nabla\cdot(\chi(c,p)p\nabla c))\,\mathrm{d}x \\
&= -\xi\int_\Omega H''(p)|\nabla p|^2\,\mathrm{d}x + \int_\Omega H''(p)p\chi(c,p)\nabla p\cdot\nabla c\,\mathrm{d}x - \int_\Omega H'(p)p(\omega_p+\mu(c,p))\,\mathrm{d}x.
\end{aligned}
$$

But

$$
|p\chi(c,p)\nabla p\cdot\nabla c| \le \frac{\xi}{2}|\nabla p|^2 + \frac{1}{2\xi}p^2\chi(c,p)^2|\nabla c|^2,
$$

and by Theorem 4.5, $\nabla c \in C([0,t_0]\times\bar{\Omega})$, so

$$
M'(t) \le \frac{1}{2\xi}\int_\Omega H''(p)p^2\chi(c,p)^2|\nabla c|^2\,\mathrm{d}x + \int_\Omega H'(p)p||\omega_p+\mu(c,p)||_\infty\,\mathrm{d}x \le C(t_0)M(t), \quad (7.3)
$$

for some constant $C(t_0)$.

So $M(t)=0$ as $M(0)=0$. Thus $p(t)\ge 0$. The result for $p_0\in D(A_p^\alpha)$ follows from the density of $D(A_{p,r}^\alpha)$ in $D(A_p^\alpha)$ and Proposition 6.1. That $c(t)\ge 0$ is now straightforward.

# 8. An $L^1$ Growth Bound for $p$

In this section we establish an $L^1$ growth bound for $p(x,t)$.

**Theorem 8.1.** *Suppose that hypotheses H(4.1), H(4.2), H(6.1), H(7.1) and H(7.2) are satisfied, $\frac{1}{2}<\alpha<1$, $[c_0,p_0]^t\in D(A_c)\times D(A_p^\alpha)$. Then for all $t\ge 0$,*

$$
||p(t)||_1 \le e^{-(\omega_p+\inf\mu)t}||p_0||_1. \quad (8.1)
$$

**Proof.** Note first that $-A_p-\inf\mu I$ generates a positive analytic semigroup $\{\tilde{T}_p(t)\}_{t\ge 0}$ , say, such that $||\tilde{T}_p(t)||\le e^{-(\omega_p+\inf\mu)t}$ and we can work with this operator in place of $-A_p$. Also $\int_\Omega \tilde{T}_p(t)z\mathrm{d}x = e^{-(\omega_p+\inf\mu)t}\int_\Omega z\mathrm{d}x$, for all $z\in L^1(\Omega)$. We have

$$
\begin{aligned}
\int_\Omega p(t)\mathrm{d}x &= \int_\Omega \tilde{T}_p(t)p_0\mathrm{d}x + \int_0^t\int_\Omega \tilde{T}_p(t-s)(-\nabla\cdot(\chi(c(s),p(s))p(s)\nabla c(s)) \\
&\qquad -(\mu(c(s),p(s))-\inf\mu)p(s))\,\mathrm{d}x\,\mathrm{d}s \\
&= e^{-(\omega_p+\inf\mu)t}\int_\Omega p_0\,\mathrm{d}x + \int_0^t e^{-(\omega_p+\inf\mu)(t-s)}\int_\Omega(-\nabla\cdot(\chi(c(s),p(s))p(s)\nabla c(s)) \\
&\qquad -(\mu(c(s),p(s))-\inf\mu)p(s))\,\mathrm{d}x\,\mathrm{d}s \\
&\le e^{-(\omega_p+\inf\mu)t}\int_\Omega p_0\,\mathrm{d}x.
\end{aligned}
$$

So (8.1) holds as required.

## 9. An Example

We illustrate the theoretical results for the chemotaxis model (2.1) – (2.4) with an example. For simplicity we consider the 1-dimensional spatial region $\Omega = (0, 10)$. We take $\eta = 0.1$, $\omega_c = 1.0$, $\xi = 0.1$, $\omega_p = 0.001$, $\mu = 0.0$, and $\chi = 1.0$. We assume the production term $\gamma$ in (2.1) has the form

$$\gamma(x, c, p) = \beta_\epsilon(x) \int_0^{10} \kappa_\sigma(x, \hat{x})\tau(p(\hat{x}))\,\mathrm{d}\hat{x}$$

with $\epsilon = 1.0$ and $\sigma = 1.5$. Here $\beta_\epsilon(x)$ is the regularization with respect to Neumann boundary conditions satisfying $\beta_\epsilon(x) \equiv 1$ on $[\epsilon, 10-\epsilon]$, $\beta_\epsilon(x)$ is the $4^{th}$ degree polynomial on $[0, \epsilon]$, which is 0 at 0, 1 at $\epsilon$, has $1^{st}$ derivative 0 at 0, and has $1^{st}$ and $2^{nd}$ derivative 0 at $\epsilon$, and $\beta_\epsilon(x)$ is the $4^{th}$ degree polynomial on $[10-\epsilon, 10]$, which is 0 at 10, 1 at $10-\epsilon$, has $1^{st}$ derivative 0 at 10, and has $1^{st}$ and $2^{nd}$ derivative 0 at $10-\epsilon$. Also, $\tau(z) = z/(1.0 + .01z)$ and $\kappa_\sigma(x, \hat{x})$ is the regularization with respect to spatial derivatives given by the normal probability distribution

$$\kappa_\sigma(x, \hat{x}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\hat{x})^2}{2\sigma^2}}.$$

The function $\beta(x, p) = \beta_\epsilon(x) \int_0^{10} \kappa_\sigma(x, \hat{x})\tau(p(\hat{x}))\,\mathrm{d}\hat{x}$ satisfies H(4.1)$'$. In fact for $p_1, p_2 \in D(A_p^\alpha)$,

$$\begin{aligned} |\beta(x, p_1) - \beta(x, p_2)| &\leq \text{const.} \times \int_0^{10} |p_1(\hat{x}) - p_2(\hat{x})|\,\mathrm{d}\hat{x} \\ &= \text{const.} \times \|A_p^{-\alpha} A_p^\alpha p_1 - A_p^{-\alpha} A_p^\alpha p_2\|_1 \\ &\leq \text{const.} \times \|A_p^\alpha p_1 - A_p^\alpha p_2\|_1, \end{aligned}$$

and similarly for $|\frac{\partial}{\partial x_i}\beta(x, p_1) - \frac{\partial}{\partial x_i}\beta(x, p_2)|$ and $|\frac{\partial^2}{\partial x_i^2}\beta(x, p_1) - \frac{\partial^2}{\partial x_i^2}\beta(x, p_2)|$. So $\gamma(x, c, p) = \beta(x, p)$ satisfies H(4.1). Also, as $\tau$ is bounded, we have $\|A_c\gamma(p, c)\|_\infty \leq P$ for all $p \in D(A_p^\alpha)$ and so also H(6.1) is satisfied.

This form of $\gamma$ incorporates an influence of the species density on production of chemoattractant at a point $x$ through a weighted sensing of the species near $x$. The initial conditions are taken as $c(x, 0) = 2.0 - \cos(0.2\pi x)$ and $p(x, 0) \equiv 1.0$. The solution is graphed in Fig.1. The species and chemoattractant both avoid blow-up. If the production term is given by the point-wise form $\gamma\,\tau(p(x))$ with $\gamma \equiv 1.0$ and with $\tau$ and all other parameters and initial data as above, then the solutions exhibit "numerical blow-up", as illustrated in Fig.2. Although the solutions in this case are known to exist globally and remain $L^\infty$ bounded in finite time intervals, the numerical simulation shows an apparent blow-up, because the chemotactic driven concentration is so extreme.

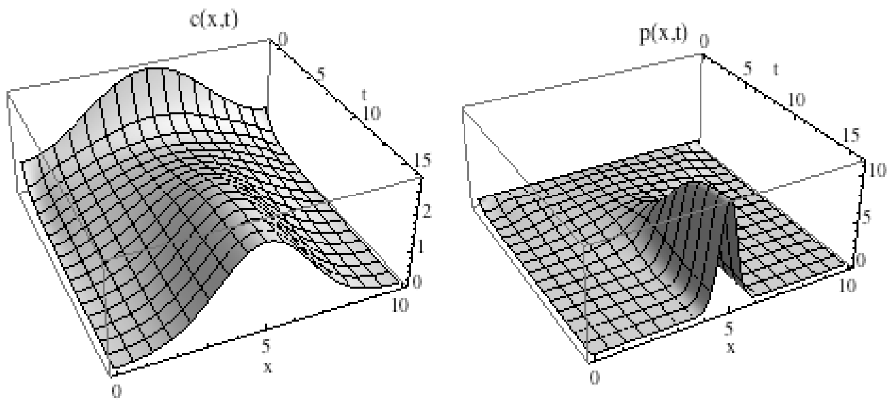


Figure 1: The chemoattractant concentration $c(x,t)$ and the species population density $p(x,t)$ in the case of a nonlocal production term. Both $c(x,t)$ and $p(x,t)$ avoid blow-up, because the chemical production is sufficiently dispersed away from the highest density of the species.

# 10. Concluding Remarks

We have analyzed the chemotaxis model (2.1) – (2.4) and proven the existence of global solutions which remain bounded in $L^\infty$ on finite intervals of time. Our analysis has three key elements. The first is the choice of state spaces for the chemoattractant concentration $c$ and the species population density $p$. The form of equations (2.1) – (2.4) is consistent with higher regularity for the state space for $c$ than for $p$. We take the state space for $c$ as $D(A_c) \subset C(\bar{\Omega})$ with norm $\|A_c c\|_\infty$, $c \in D(A_c)$, where $A_c$ is the diffusion operator, defined as in (3.2). We take the state space for $p$ as $D(A_p^\alpha) \subset L^1(\Omega)$, where $A_p^\alpha$ is the fractional power of the diffusion operator $A_p$ in $L^1(\Omega)$. The second key element of our analysis is the use of analytic semigroups and fractional powers of the generators to reformulate (2.1) – (2.4) in the mild form (3.4), (3.6) using the semigroups $T_c(t), t \geq 0$ and $T_p(t), t \geq 0$ generated by $-A_c$ and $-A_p$, respectively. The third key element is the nonlocal form of the chemical production term $\gamma$ in (2.1), which regularizes the spatial dependence of $\gamma$ on $c$ and $p$. Together these three elements provide the regularity and estimates needed to prove the local existence of solutions to (3.4), (3.6). In particular, classical solutions of the model exist in the sense of (5.2), (5.3) for initial data $[c_0, p_0]^t$ in $D(A_c) \times D(A_p^\alpha)$, where $\frac{1}{2} < \alpha < 1$. The continuability of the solutions follows using the Gronwall-type inequality (6.7) for fractional powers and the boundedness assumptions in H(6.1). The positivity and boundedness properties of solutions are further established in this framework.

The distinction between blow-up and non-blow-up in models with both diffusion and taxis is subtle, and intimately tied to model formulation, as well as analytic techniques. Models of haptotaxis, in which directed movement is mediated by an intermediate constituent between

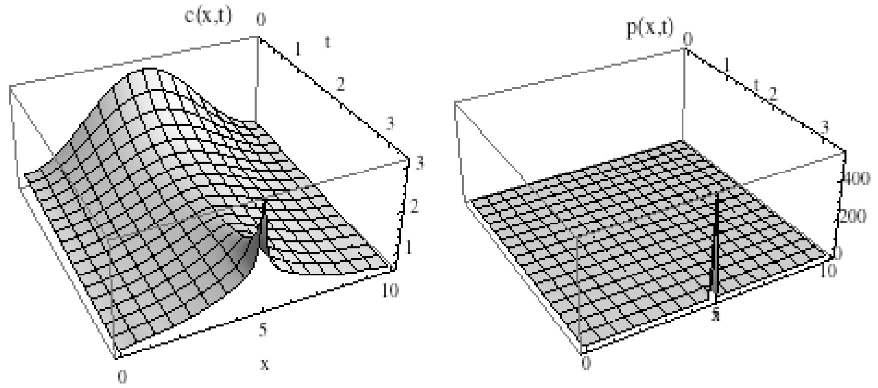


Figure 2: The chemoattractant concentration $c(x,t)$ and the species population density $p(x,t)$ in the case of a local point-wise production term. The solutions exhibit "numerical blow-up".

species and a bound chemoattractant, do not exhibit blow-up behavior ([7], [8], [9], [36]). In chemotaxis models these two spatial movements - diffusion away from concentrations and chemotactic attraction toward concentrations - are directly confronted. In many models of chemotaxis, with point-wise dependence on the species and the chemical, chemotaxis uncontrollably dominates this confrontation, resulting in blow-up of solutions. The nonlocal dependence we use here mitigates this imbalance, and prevents blow-up independently of spatial dimension, parametric input, and initial data. We argue that this nonlocal form is valid for biological applications in which the species density $p$ describes a population of individuals occupying intrinsically limited minimal space.

The model (2.1) – (2.4) incorporates key features of chemotaxis phenomena, but lacks many elements important in biological applications. In particular, the model does not consider proliferation of the species, which requires incorporation of the cell cycle and cell division. In future work the authors will investigate models of chemotaxis such as (2.1) – (2.4) which include proliferative processes.

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