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An Age and Spatially Structured Population Model for *Proteus Mirabilis* Swarm-Colony Development

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Abstract. *Proteus mirabilis* are bacteria that make strikingly regular spatial-temporal patterns on agar surfaces. In this paper we investigate a mathematical model that has been shown to display these structures when solved numerically. The model consists of an ordinary differential equation coupled with a partial differential equation involving a first-order hyperbolic aging term together with nonlinear degenerate diffusion. The system is shown to admit global weak solutions.

Key words: population models, age structure, degenerate diffusion
AMS subject classification: 92C17, 35G25, 35M20, 35K65, 47N20

1. Introduction

Bacteria of the species *Proteus mirabilis* are ubiquitous throughout nature. In human beings, *Proteus mirabilis* is found as part of the normal flora of the gut. Its main pathological role is in infections of the urinary tract, but it can also cause wound infections and *septicaemia*. Even though most of the human urinary tract infections are due to the bacterium *Escherichia coli*, urinary tract infections due to *Proteus mirabilis* are also well-documented. It commonly invades the urinary tract when the normal function of the tract is disturbed by instrumentation such as catheterization. Once attached to urinary tract, *Proteus mirabilis* infects the kidney more commonly than *Escherichia coli* and characteristically leads to urinary stones.

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Proteus mirabilis can exist in two distinct morphological and physiological forms known as "swimmer" cells and "swarmer" cells, respectively. Broth cultures of *Proteus mirabilis* consist virtually exclusively of mononuclear cells (swimmers) approximately 1 μm wide with short flagella. Swimmer cells go through a prototypical bacterial cell growth and division cycle. However, when inoculated onto agar surfaces, some cells cease division but continue to grow and produce many lateral flagella to form elongated multi-nucleoid hyperflagellated swarmer cells up to 100 μm in length which aggregate in parallel arrays to form motile multicellular "rafts". The process in which dividing cells become swarmers is called "differentiation" and occurs only above a critical dividing-cell density. Rafts of swarmer cells are capable of translocation while swimmer cells are immobile. The movement of *Proteus mirabilis* through raft building requires two things, namely sufficient maturity in swarmer cells to contribute to raft building and sufficient biomass of mature cells to form the rafts. After some time migrating, when the multinuclear swarmer cells approach a maximal size, they cease movement and rapidly "dedifferentiate" again into single nucleus swimmer cells. This coordinated burst of swarming activity interspersed with a consolidation to the swimmer state results in characteristic concentric rings of growth. It is due to these strikingly regular spatial and temporal patterns that *Proteus mirabilis* has attracted attention in the mathematical biology literature [6, 9, 12].

The ability to form swarmer cells seems to allow rapid colonization of solid surfaces and the establishment of extensive *Proteus mirabilis* biofilms. Apparently, *Proteus mirabilis* can also swarm over the surfaces of all the major catheter. Swarming may thus play roles in both the initiation of catheter associated infections and the subsequent spread of the biofilm over the catheter surface.

The processes involved in the evolution of *Proteus mirabilis* and the formation of regular patterns are rather complex. A key ingredient of the mathematical representation is the age dependence of swarmer cell behavior. An age and spatially structured model for *Proteus mirabilis* swarm colony development was presented in [9], and - in slightly modified form - in [12] and [6, 7]. If $v = v(t,x)$ denotes the swimmer cell biomass density in dependence of time $t \geq 0$ and spatial position $x \in \Omega$ (with a spatial region $\Omega \subset \mathbb{R}^n$) and $u = u(t,a,x)$ denotes the swarmer cell density which additionally depends on age $a \geq 0$, the models in [6, 7, 9, 12] can be re-cast in the form

$$\partial_t u + \partial_a u = \mathrm{div}_x\big(D(\Lambda(t,x))\,\nabla_x u\big) - \mu(a)\,u\ , \quad (t,a,x) \in (0,\infty)\times(0,\infty)\times\Omega\ , \tag{1.1}$$

$$\partial_t v = \frac{1}{\tau}\big(1 - \xi(v)\big)\,v + \int_0^\infty e^{a/\tau}\,\mu(a)\,u(t,a,x)\,\mathrm{d}a\ , \quad (t,x)\in(0,\infty)\times\Omega\ , \tag{1.2}$$

$$u(t,0,x) = \frac{1}{\tau}\,\xi\big(v(t,x)\big)\,v(t,x)\ , \quad (t,x)\in(0,\infty)\times\Omega\ , \tag{1.3}$$

$$u(0,a,x) = u^0(a,x)\ , \qquad v(0,x) = v^0(x)\ , \qquad (a,x)\in(0,\infty)\times\Omega\ , \tag{1.4}$$

$$\partial_\nu u = 0\ , \quad (t,a,x)\in(0,\infty)\times(0,\infty)\times\partial\Omega\ , \tag{1.5}$$

where

$$\Lambda(t,x) := \int_{a_0}^\infty e^{a/\tau}\,u(t,a,x)\,\mathrm{d}a\ , \quad (t,x)\in(0,\infty)\times\Omega\ . \tag{1.6}$$

The major differences of the models [6, 9, 12] and their philosophies are in different choices of the functions D, μ, and ξ. The meaning of the various terms are as follows: $\Lambda = \Lambda(t,x)$ represents the total motile swarmer cell biomass, where $a_0 \geq 0$ is the minimal age of swarmer cells required to stimulate the collective movement of the cells and thus participate actively in group migration. The exponential comes in since biomass increase during swarm development is assumed to occur at the same rate as during the swimmer cell cycle. The parameter τ is the time it takes a cell to subdivide.

Equation (1.1) expresses the change in time of swarmer cells of a given age a. Movement of *Proteus mirabilis* occurs if sufficiently many swarmers above the critical age a_0 group together to build a mass above a certain threshold $\Lambda_{min} \geq 0$. Thus, the diffusivity D depends on Λ and is small (or zero) for Λ small. For instance, D may be of the form

$$D(\Lambda) \;=\; D_0 \left(\Lambda - \Lambda_{min}\right)_+^{m-1} \tag{1.7}$$

as in [6] with $D_0 \in (0,\infty)$ and $m = 2$. Here and below, $r_+ := \max\{0, r\}$ denotes the positive part of the real number r. In [9] also a dependence of D on v and on a memory term is included, something we will refrain of taking into account. Note that the exponential weighting in Λ means in (1.7) that older cells contribute more to swarming than younger cells. The age dependent function μ in (1.1) is the dedifferentiation modulus, which is higher for older swarmers than for younger ones. A typical shape for μ is a narrow hump located around a maximal age a_{max} and zero elsewhere. The limit choice $\mu(a) = \mu_0\, \delta_{a=a_{max}}$ has also been considered in [9].

The change in time of the swimmer population is given by equation (1.2). The population grows exponentially with rate $1/\tau$. Some of the swimmer cells cease division and differentiate with rate $\xi(v)/\tau$ into swarmers of age 0. This increase in swarmer cells is reflected by equation (1.3). As pointed out in [6] the function ξ should be zero for v small. Indeed, the incorporation of a lag phase in swarmer cell production triggers the development of a consolidation phase after a swarm phase and thus prevents a self-sustaining soliton caused by swarmers that dedifferentiate into swimmers immediately differentiating into new swarmers. This lag in the onset of differentiation was observed in [18] and included in the models in [6, 7, 12]. The integral term in (1.2) represents dedifferentiation of swarmer cells into swimmer cells.

The basis of equations (1.1)-(1.5) was presented in [9] and extensions and modifications of these equations were proposed in [6, 7, 12]. In [9] and [6, 7] the main focus - besides the modeling aspect - were computational results displaying the spatial and temporal patterns of concentric rings with equal width. In [7] numerical results were presented examining the necessity of a sharp age of dedifferentiation from swarmer to swimmer cells. All papers [6, 7, 9] use explicit age dependence in the evolution of the swarmers. As pointed out in [6], explicit age structure provides a mechanism for controlling - at least numerically - the ratio of time spent swarming to time spent in consolidation without changing the total cycle time. In [12] a reaction-diffusion model for *Proteus mirabilis* swarm-colony development based on averaging over the age variable was used and results on the long time distribution Λ/v were derived. A model for the periodic swarming of *Proteus* ignoring the age structure from the outset was introduced in [8].

For further reading concerning morphology and pathogenicity of *Proteus mirabilis* and for numerical results for models of this bacteria we refer to [6, 7, 8, 9, 12, 13, 14, 17, 18] and the references therein. Moreover, as population models structured by age and spatial position have a long history we merely refer the interested reader to [19] for an extensive list of literature.

Fewer analytical than numerical results seem to be available for models of *Proteus mirabilis*, and [10] contains the only result regarding the mathematical well-posedness of models for *Proteus mirabilis* we are aware of. Existence and uniqueness of weak solutions to (1.1)-(1.5) are shown in [10] for the case of non-degenerate diffusion including memory.

The purpose of this paper is to prove an existence result for diffusion coefficients $D(\Lambda)$ that may degenerate for $\Lambda = 0$ in the sense that $D(0) = 0$ and thus to get closer to the biological reality. Still we cannot handle the case where D is given by (1.7) but expect that the outcome of the model with diffusivity

$$D(\Lambda) \;=\; D_0 \left(\Lambda - \Lambda_{min}\right)_+^{m-1} + e^{-1/(\varepsilon\,\Lambda)}$$

(to which our result applies if $m \geq 3$) for small $\varepsilon > 0$ resembles that for $\varepsilon = 0$ from a numerical viewpoint. However, proving the formation of regular spatio-temporal patterns is beyond the scope of this paper.

The outline of the paper is as follows: In the next section, we first establish an existence and uniqueness result for the non-degenerate case by a fixed point argument; that is, when D is bounded below by a positive constant. Our method for proving this result is completely different from that in [10] in which a compactness argument is used. Section 3 then shows how to handle certain degenerate diffusivity.

Throughout the paper we assume that the minimal age a_0 required for swarmer cells to participate actively in the collective motion is positive. The case $a_0 = 0$ turns out to be easier and could also be handled with minor modifications.

2. The Non-Degenerate Case

Throughout this section we suppose that the diffusivity D satisfies

$$D \in C^{2-}(\mathbb{R}) \qquad \text{and} \qquad D(z) \geq d_0 > 0 \quad \text{for} \quad z \in \mathbb{R}\ , \tag{2.1}$$

where C^{k-} (resp. C_b^{k-}) for $k \in \mathbb{N} \setminus \{0\}$ denotes the set of C^{k-1}-smooth functions with a Lipschitz continuous (resp. uniformly Lipschitz continuous on bounded subsets) $(k-1)$-th derivative. As for the differentiation rate ξ we assume that

$$\xi \in C^{3-}(\mathbb{R}) \qquad \text{and} \qquad 0 \leq \xi(z) \leq 1 \quad \text{for} \quad z \in \mathbb{R}\ , \tag{2.2}$$

while the dedifferentiation modulus

$$\mu \in BC(\mathbb{R}) := C(\mathbb{R}) \cap L_\infty(\mathbb{R}) \quad \text{is non-negative} . \tag{2.3}$$

Note that only the values of D, ξ, and μ for non-negative real numbers are relevant from the biological point of view and the previous assumptions are made for simplicity.

Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. For $p > n$ fixed, $W^{2\sigma}_{p,\mathcal{B}}$ denotes either the space $W^{2\sigma}_p := W^{2\sigma}_p(\Omega)$ if $2\sigma \le 1+1/p$ or the subspace of $W^{2\sigma}_p(\Omega)$ consisting of those elements satisfying homogeneous Neumann boundary conditions if $2\sigma > 1+1/p$. For abbreviation we put

$$E_\sigma := L_1(\mathbb{R}^+, W^{2\sigma}_{p,\mathcal{B}}, e^{a/\tau}\mathrm{d}a) \quad \text{and} \quad \|u\|_{E_\sigma} := \int_0^\infty \|u(.,a)\|_{W^{2\sigma}_{p,\mathcal{B}}}\, e^{a/\tau}\mathrm{d}a \tag{2.4}$$

for $u \in E_\sigma$ and $\sigma \in [0,1]$.

In the following, let $c(T)$, $c(R)$, and $c(T,R)$ denote constants that depend on their respective arguments in a monotone increasing way; these constants may differ from occurrence to occurrence.

We first prove an auxiliary result regarding the solvability of (1.2).

Lemma 2.1. *Let $T > 0$, $2\sigma \in (1+n/p, 2)$, and assume that $v^0 \in W^2_{p,\mathcal{B}}$ and $u \in C([0,T], E_\sigma)$ are given non-negative functions. Then there is a unique solution $v = v_u \in C^1([0,T], W^{2\sigma}_{p,\mathcal{B}})$ to (1.2) subject to the initial condition $v(0) = v^0$. This solution is non-negative and belongs to $C([0,T], W^2_{p,\mathcal{B}})$ if $u \in L_1([0,T], E_1)$. Moreover, if u and $\bar{u}$ both belong to $C([0,T], E_\sigma)$ and satisfy* $\max\{\|u(t)\|_{E_\sigma}, \|\bar{u}(t)\|_{E_\sigma}\} \le R$ *for $t \in [0,T]$ and some $R > 0$, then*

$$\|v_u(t) - v_u(s)\|_{W^{2\sigma}_p} \le c(T,R)\,|t-s| \ , \quad 0 \le t,s \le T \tag{2.5}$$

and

$$\|v_u(t) - v_{\bar{u}}(t)\|_{W^{2\sigma}_p} \le c(T,R)\,\|u-\bar{u}\|_{C([0,T],E_\sigma)}, \quad 0 \le t \le T \tag{2.6}$$

for some constant $c(T,R) > 0$.

Proof. First note that the regularity of ξ and [3, Theorem 4.2] imply

$$[v \mapsto \xi(v)v] \in C^{1-}_b(W^{2\sigma}_{p,\mathcal{B}}, W^{2\sigma}_{p,\mathcal{B}}) \tag{2.7}$$

since $2\sigma > 1+n/p$. In addition, (2.3) ensures that the integral term in (1.2) belongs to $C([0,T], W^{2\sigma}_{p,\mathcal{B}})$. The existence of a unique non-negative solution $v_u \in C^1(J, W^{2\sigma}_{p,\mathcal{B}})$ is now obvious, where either $J = [0,T]$ or $J = [0,\tilde{T})$ with $\tilde{T} < T$ and $\|v_u(t)\|_{W^{2\sigma}_p} \to \infty$ as $t \nearrow \tilde{T}$. Next, (2.2) and the embedding $W^{2\sigma}_p \hookrightarrow L_\infty$ ensure $v_u \in L_\infty(J, L_\infty)$. Then, taking the gradient with respect to x on both sides of (1.2) we similarly obtain $v_u \in L_\infty(J, W^1_\infty)$. Recalling that pointwise multiplication satisfies

$$W^{2\sigma-1}_p \times W^1_\infty \hookrightarrow W^{2\sigma-1}_p \times W^{2\sigma-1}_p \hookrightarrow W^{2\sigma-1}_p$$

according to [3, Theorem 4.2] since $2\sigma - 1 > n/p$ we deduce that

$$\|\xi(v_u(t))\, v_u(t)\|_{W^{2\sigma}_p} \le c\big(1 + \|v_u(t)\|_{W^{2\sigma}_p}\big) \ , \quad t \in J \ .$$

From this we first conclude that $v_u \in L_\infty(J, W_p^{2\sigma})$, whence $J = [0,T]$, and then $\xi(v)v \in L_\infty(J, W_p^{2\sigma})$ so that (2.5) follows by (1.2). Property (2.6) is implied by (2.3) and (2.7). Finally, if $u \in L_1([0,T], E_1)$, then the integral term in (1.2) belongs to $L_1([0,T], W^2_{p,\mathcal{B}})$ due to (2.3), and we readily infer that v_u belongs to $C([0,T], W^2_{p,\mathcal{B}})$.

The solvability of (1.1) is based on the following formal observation: Suppose that the function u is sufficiently smooth so that the function $\Lambda = \Lambda_u$, given by (1.6), leads to a well-defined evolution system $U_{A_u}(t,s)$ on L_p corresponding to the differential operator

$$A_u(t)\, w := -\mathrm{div}_x\big(D(\Lambda_u(t))\nabla_x w\big)\ , \quad w \in W^2_{p,\mathcal{B}}\ . \tag{2.8}$$

Then (1.1), (1.3)-(1.5) can be re-written as a problem in L_p of the form

$$\begin{aligned} \partial_t u + \partial_a u + \mu(a)u &= -A_u(t)u\ , && a>0\ ,\ 0<t\le T\ ,\\ u(t,0,\cdot) &= \frac{1}{\tau}\,\xi\big(v_u(t)\big)\,v_u(t)\ , && 0<t\le T\ ,\\ u(0,a,\cdot) &= u^0(a,\cdot)\ , && a>0\ , \end{aligned}$$

where v_u is the corresponding solution to (1.2). Applying the method of characteristics we derive that u is a fixed point of the map Φ, given by

$$\Phi(u)(t,a) := \begin{cases} \dfrac{1}{\tau}\, e^{-\int_0^a \mu(r)\mathrm{d}r}\, U_{A_u}(t,t-a)\,\xi(v_u(t-a))\,v_u(t-a)\ , & 0\le a\le t\ ,\\[2ex] e^{-\int_{a-t}^a \mu(r)\mathrm{d}r}\, U_{A_u}(t,0)\,u^0(a-t)\ , & 0\le t<a\ . \end{cases} \tag{2.9}$$

We now show that this map Φ indeed has a fixed point in a suitable space involving E_σ defined in (2.4) and thus (1.1)-(1.5) admits a unique solution. More precisely, we have:

Theorem 2.2. *Suppose* (2.1)-(2.3) *and fix* $p>n$, $q>1$ *and* $2\omega \in (1+n/p, 2)$. *Consider non-negative initial values* $v^0 \in W^2_{p,\mathcal{B}}$ *and*

$$u^0 \in E_1 \cap C^1([0,\infty), L_p) \cap L_q\big((0,a_0), W_p^{2\omega}\big) \cap C([0,a_0], W_p^{2\omega-1})$$

satisfying the compatibility condition $\xi(v^0)\,v^0 = \tau\, u^0(0,\cdot)$ *in* Ω. *Then problem* (1.1)-(1.5) *possesses a unique non-negative solution* (v,u) *with*

$$v \in C^1([0,\infty), W^{2\eta}_{p,\mathcal{B}}) \cap C([0,\infty), W^2_{p,\mathcal{B}})\ , \quad u \in C([0,\infty), E_\eta) \cap L_{\infty,loc}([0,\infty), E_1)$$

for any $\eta \in (0,1)$ *and such that* u *satisfies*

$$\begin{aligned} &\partial_t u(t,\cdot)\ ,\ \partial_a u(t,\cdot) \in C([0,t], L_p) \cap C((t,\infty), L_p)\ ,\\ &\partial_t u(\cdot,a)\ ,\ \partial_a u(\cdot,a) \in C([0,a), L_p) \cap C([a,\infty), L_p) \end{aligned}$$

for all $t,a>0$ *and solves* (1.1) *in* L_p *for* $t \neq a$.

Proof. Given $\eta \in (0,1)$ we fix numbers ϑ, σ, and ϱ such that

$$1 + n/p < 2\vartheta < 2\omega < 2\sigma < 2\varrho < 2$$

and $\eta < \sigma$ and choose $\kappa \in (0, \min\{\sigma - \vartheta, 1/q'\})$, where q' is the dual exponent of q. Note that we may assume without loss of generality that $q\omega < 1$ by making q smaller if necessary. Let c_0 be the norm of the natural injection $W^{2\varrho}_{p,\mathcal{B}} \hookrightarrow W^{2\sigma}_{p,\mathcal{B}}$ and let $R > 0$ be such that

$$c_0\, e^{1/\tau}\, \|u^0\|_{E_\varrho} + \|u^0\|_{L_q((0,a_0),W^{2\omega}_p)} \;\leq\; R \;. \tag{2.10}$$

For $T \in (0,1)$, $\mathcal{V}_T$ denotes the space consisting of all non-negative $u \in C([0,T], E_\sigma)$ such that $\|u(t)\|_{E_\sigma} \leq R+1$ and $\|\Lambda_u(t) - \Lambda_u(s)\|_{W^{2\vartheta}_p} \leq |t-s|^\kappa$ for $0 \leq t, s \leq T$, where Λ_u is given by (1.6). Then, given any $u \in \mathcal{V}_T$, it follows that the operator $-A_u(t)$ defined in (2.8) is for each $t \in [0,T]$ the generator of a positive analytic semigroup on L_p (e.g. [1, 16]). Moreover, due to the embedding $W^{2\sigma}_p \hookrightarrow W^{2\vartheta}_p \hookrightarrow W^1_\infty$ we have

$$\|A_u(t) - A_u(s)\|_{\mathcal{L}(W^2_{p,\mathcal{B}},L_p)} \;\leq\; c(R)\,|t-s|^\kappa \;, \qquad 0 \leq t,s \leq T \;,$$

and

$$\|A_u(t) - A_{\bar{u}}(t)\|_{\mathcal{L}(W^2_{p,\mathcal{B}},L_p)} \;\leq\; c(R)\,\|u - \bar{u}\|_{\mathcal{V}_T} \;, \qquad 0 \leq t \leq T \;, \quad u, \bar{u} \in \mathcal{V}_T \;, \tag{2.11}$$

with the notation

$$\|u\|_{\mathcal{V}_T} := \sup_{t\in[0,T]} \|u(t)\|_{E_\sigma} \quad \text{for} \quad u \in \mathcal{V}_T \;,$$

the space E_σ being defined in (2.4). Therefore, invoking Corollary II.4.4.2, Lemma II.5.1.3, Lemma II.5.1.4, Equation (II.5.3.8), and Section II.6.4 in [4] and using standard interpolation results on Sobolev spaces with boundary conditions we derive that, for any $u \in \mathcal{V}_T$, there exists a unique positive evolution system $U_{A_u}(t,s)$, $0 \leq s \leq t \leq T$ on L_p such that

$$\|U_{A_u}(t,s)\|_{\mathcal{L}(W^{2\alpha}_{p,\mathcal{B}})} \;+\; (t-s)^{\gamma-\alpha}\|U_{A_u}(t,s)\|_{\mathcal{L}(W^{2\beta}_{p,\mathcal{B}},W^{2\gamma}_{p,\mathcal{B}})} \;\leq\; c(R) \tag{2.12}$$

for $0 \leq s < t \leq T$, $0 \leq \alpha < \beta \leq \gamma \leq 1$ with $2\beta, 2\gamma \neq 1 + 1/p$, and

$$\|U_{A_u}(t,r) - U_{A_u}(s,r)\|_{\mathcal{L}(W^{2\gamma}_{p,\mathcal{B}},W^{2\beta}_{p,\mathcal{B}})} \;\leq\; c(R)\,(t-s)^{\gamma-\beta} \tag{2.13}$$

for $0 \leq r < s < t \leq T$, $0 < \beta \leq \gamma < 1$ with $2\beta, 2\gamma \neq 1 + 1/p$. In addition, if $\bar{u}$ is another function in $\mathcal{V}_T$, we have

$$\|U_{A_u}(t,s) - U_{A_{\bar{u}}}(t,s)\|_{\mathcal{L}(W^{2\alpha}_{p,\mathcal{B}},W^{2\beta}_{p,\mathcal{B}})} \;\leq\; c(R)\,(t-s)^{\alpha-\beta}\,\|u - \bar{u}\|_{\mathcal{V}_T} \tag{2.14}$$

for $0 \leq s < t \leq T$, $0 \leq \alpha, \beta \leq 1$ with $\alpha \neq 0$, $\beta \neq 1$, $2\alpha, 2\beta \neq 1 + 1/p$.

Since $-\Delta_x$ subject to homogeneous Neumann conditions on the boundary generates a contraction semigroup on $W^{2\sigma}_{p,\mathcal{B}}$ according to [4, Corollary V.2.1.4] it follows from (2.14) that

$$\|U_{A_u}(t,s)\|_{\mathcal{L}(W^{2\varrho}_{p,\mathcal{B}},W^{2\sigma}_{p,\mathcal{B}})} \;\leq\; c(R)\,(t-s)^{\varrho-\sigma} + c_0 \tag{2.15}$$

for $0 \le s < t \le T$. Also note that

$$\|U_{A_u}(t,s)\|_{\mathcal{L}(L_r)} \le 1 \ , \qquad r \in (1,\infty] \ , \quad 0 \le s \le t \le T \ . \tag{2.16}$$

Defining Φ by (2.9) we now claim that $\Phi : \mathcal{V}_T \to \mathcal{V}_T$ is a contraction provided $T = T(R) \in (0,1)$ is chosen sufficiently small. To prove this we fix $u \in \mathcal{V}_T$ and observe that $v_u \in C^1([0,T], W^{2\sigma}_{p,\mathcal{B}})$ is well-defined due to Lemma 2.1. Furthermore, Lemma 2.1 and (2.7) entail

$$\|\xi(v_u(t))v_u(t)\|_{W_p^{2\sigma}} \le c(R) \ , \quad 0 \le t \le T \ . \tag{2.17}$$

We put $\lambda(a) := \mathbf{1}_{(a_0,\infty)}(a)\, e^{a/\tau}$ so that

$$\Lambda_u(t,x) = \int_0^\infty \lambda(a)\, u(t,a,x)\,\mathrm{d}a \ , \quad (t,x) \in [0,T] \times \Omega \ .$$

Then we deduce from (2.3), (2.10), (2.12), (2.13), and (2.17) that, for $0 \le s \le t \le T \le 1$,

$$\begin{aligned}
&\|\Lambda_{\Phi(u)}(t) - \Lambda_{\Phi(u)}(s)\|_{W_p^{2\vartheta}} \\
&\le \int_s^t \|U_{A_u}(t,a)\|_{\mathcal{L}(W^{2\vartheta}_{p,\mathcal{B}})} \|\xi(v_u(a))v_u(a)\|_{W_p^{2\vartheta}}\, e^{(t-a)/\tau}\,\mathrm{d}a \\
&\quad + \int_0^s \left| e^{-\int_0^{t-a}\mu(r)\mathrm{d}r} - e^{-\int_0^{s-a}\mu(r)\mathrm{d}r} \right| \|U_{A_u}(t,a)\|_{\mathcal{L}(W^{2\vartheta}_{p,\mathcal{B}})} \|\xi(v_u(a))v_u(a)\|_{W_p^{2\vartheta}}\, e^{(t-a)/\tau}\,\mathrm{d}a \\
&\quad + \int_0^s \|U_{A_u}(t,a) - U_{A_u}(s,a)\|_{\mathcal{L}(W^{2\sigma}_{p,\mathcal{B}}, W^{2\vartheta}_{p,\mathcal{B}})} \|\xi(v_u(a))v_u(a)\|_{W_p^{2\sigma}}\, e^{(t-a)/\tau}\,\mathrm{d}a \\
&\quad + \int_0^s \|U_{A_u}(s,a)\|_{\mathcal{L}(W^{2\vartheta}_{p,\mathcal{B}})} \|\xi(v_u(a))v_u(a)\|_{W_p^{2\vartheta}} \left|\lambda(t-a) - \lambda(s-a)\right|\mathrm{d}a \\
&\quad + \|U_{A_u}(t,0) - U_{A_u}(s,0)\|_{\mathcal{L}(W^{2\sigma}_{p,\mathcal{B}}, W^{2\vartheta}_{p,\mathcal{B}})} \int_0^\infty \|u^0(a)\|_{W_p^{2\sigma}}\, e^{(a+t)/\tau}\,\mathrm{d}a \\
&\quad + \|U_{A_u}(s,0)\|_{\mathcal{L}(W^{2\vartheta}_{p,\mathcal{B}})} \int_0^\infty \left| e^{-\int_a^{t+a}\mu(r)\mathrm{d}r} - e^{-\int_a^{s+a}\mu(r)\mathrm{d}r} \right| \|u^0(a)\|_{W_p^{2\vartheta}}\, e^{(a+t)/\tau}\,\mathrm{d}a \\
&\quad + \|U_{A_u}(s,0)\|_{\mathcal{L}(W^{2\vartheta}_{p,\mathcal{B}})} \int_0^\infty \|u^0(a)\|_{W_p^{2\vartheta}} \left|\lambda(a+t) - \lambda(a+s)\right|\mathrm{d}a \\
&\le c(R)\,(t-s) + c(R)\,(t-s)^{\sigma-\vartheta} + c(R) \int_0^s \left|\lambda(t-a) - \lambda(s-a)\right|\mathrm{d}a \\
&\quad + c(R) \int_0^\infty \|u^0(a)\|_{W_p^{2\vartheta}} \left|\lambda(a+t) - \lambda(a+s)\right|\mathrm{d}a \ .
\end{aligned}$$

Next note that

$$\begin{aligned}
&\int_0^\infty \|u^0(a)\|_{W_p^{2\vartheta}} \left|\lambda(a+t) - \lambda(a+s)\right|\mathrm{d}a \\
&\qquad \le \int_{(a_0-t)_+}^{(a_0-s)_+} \|u^0(a)\|_{W_p^{2\vartheta}}\, e^{(t+a)/\tau}\,\mathrm{d}a + \int_{(a_0-s)_+}^\infty \|u^0(a)\|_{W_p^{2\vartheta}} \left| e^{(a+t)/\tau} - e^{(a+s)/\tau} \right| \mathrm{d}a \\
&\qquad \le e^{(a_0+1)/\tau}\, \|u^0\|_{L_q((0,a_0),W_p^{2\vartheta})} \left|(a_0-s)_+ - (a_0-t)_+\right|^{1/q'} + \tau^{-2}\, e^{1/\tau}\,(t-s)\,\|u^0\|_{E_\vartheta} \\
&\qquad \le c(R)\left((t-s)^{1/q'} + (t-s)\right)
\end{aligned}$$

owing to (2.10) while

$$\begin{aligned}\int_0^s \big|\lambda(t-a)-\lambda(s-a)\big|\,\mathrm{d}a \;&\le\; \int_0^{(s-a_0)_+} \big(e^{(t-a)/\tau}-e^{(s-a)/\tau}\big)\,\mathrm{d}a \,+\, \int_{(s-a_0)_+}^{(t-a_0)_+} e^{(t-a)/\tau}\,\mathrm{d}a\\ &\le\; \tau^{-2}\,e^{1/\tau}\,(t-s) \,+\, e^{1/\tau}\,\big((t-a_0)_+-(s-a_0)_+\big)\\ &\le\; \big(1+\tau^{-2}\big)\,e^{1/\tau}\,(t-s)\ .\end{aligned}$$

Therefore,

$$\|\Lambda_{\Phi(u)}(t)-\Lambda_{\Phi(u)}(s)\|_{W_p^{2\vartheta}} \;\le\; |t-s|^\kappa\ ,\quad 0\le t,s\le T\ ,$$

due to the choice of κ provided that $T=T(R)\in(0,1)$ is chosen sufficiently small. Furthermore, using (2.12), (2.15), and (2.17) we obtain for $0\le t\le T$

$$\begin{aligned}\|\Phi(u)(t)\|_{E_\sigma} \;&\le\; \int_0^t \|U_{A_u}(t,t-a)\|_{\mathcal{L}(W^{2\sigma}_{p,\mathcal{B}})}\,\|\xi(v_u(t-a))\,v_u(t-a)\|_{W_p^{2\sigma}}\,e^{a/\tau}\,\mathrm{d}a\\ &\quad+\int_t^\infty \|U_{A_u}(t,0)\|_{\mathcal{L}(W^{2\varrho}_{p,\mathcal{B}},W^{2\sigma}_{p,\mathcal{B}})}\,\|u^0(a-t)\|_{W_p^{2\varrho}}\,e^{a/\tau}\,\mathrm{d}a\\ &\le\; c(R)\,T \,+\, \big(c(R)\,t^{\varrho-\sigma}+c_0\big)\,e^{1/\tau}\,\|u^0\|_{E_\varrho}\\ &\le\; 1\,+\,R\end{aligned}$$

provided that $T=T(R)\in(0,1)$ is chosen sufficiently small. Since $\Phi(u)$ is obviously nonnegative and $\Phi(u)\in C([0,T],E_\sigma)$ holds by similar arguments as used to prove the Hölder continuity of $\Lambda_{\Phi(u)}$, we conclude that Φ maps $\mathcal{V}_T$ into itself. That it is a contraction follows from the observation that if $u,\bar u\in\mathcal{V}_T$ and $0\le t\le T$, then

$$\begin{aligned}&\|\Phi(u)(t)-\Phi(\bar u)(t)\|_{E_\sigma}\\ &\le \int_0^t \|U_{A_u}(t,t-a)-U_{A_{\bar u}}(t,t-a)\|_{\mathcal{L}(W^{2\sigma}_{p,\mathcal{B}})}\,\|\xi(v_u(t-a))\,v_u(t-a)\|_{W_p^{2\sigma}}\,e^{a/\tau}\,\mathrm{d}a\\ &\quad+\int_0^t \|U_{A_{\bar u}}(t,t-a)\|_{\mathcal{L}(W^{2\sigma}_{p,\mathcal{B}})}\,\|\xi(v_u(t-a))\,v_u(t-a)-\xi(v_{\bar u}(t-a))\,v_{\bar u}(t-a)\|_{W_p^{2\sigma}}\,e^{a/\tau}\,\mathrm{d}a\\ &\quad+\int_t^\infty \|U_{A_u}(t,0)-U_{A_{\bar u}}(t,0)\|_{\mathcal{L}(W^{2\varrho}_{p,\mathcal{B}},W^{2\sigma}_{p,\mathcal{B}})}\,\|u^0(a-t)\|_{W^{2\varrho}_{p,\mathcal{B}}}\,e^{a/\tau}\,\mathrm{d}a\end{aligned}$$

and hence, using (2.6), (2.7), (2.10), (2.12), (2.14), and (2.17),

$$\|\Phi(u)(t)-\Phi(\bar u)(t)\|_{E_\sigma} \;\le\; c(R)\,\|u-\bar u\|_{\mathcal{V}_T}\int_0^t e^{a/\tau}\,\mathrm{d}a \,+\, c(R)\,t^{\varrho-\sigma}\,\|u-\bar u\|_{\mathcal{V}_T} \le \frac12\,\|u-\bar u\|_{\mathcal{V}_T}$$

provided that $T=T(R)\in(0,1)$ is chosen sufficiently small. Therefore, by Banach's fixed point theorem there exists a unique $u\in\mathcal{V}_T$ such that $\Phi(u)=u$. Note that (2.12) and (2.17) imply $u=\Phi(u)\in L_\infty([0,T],E_1)$ since $u^0\in E_1$, the space E_1 being defined in (2.4), whence $v_u\in C([0,T],W^2_{p,\mathcal{B}})$ by Lemma 2.1. Due to $u(T)\in E_1$, $v_u(T)\in W^2_{p,\mathcal{B}}$, and the fact that T was chosen depending only on R satisfying (2.10), we can iterate this argument and extend

u and v_u uniquely to functions $u \in C(J, E_\sigma) \cap L_{\infty,loc}(J, E_1)$ and $v \in C^1(J, W^{2\sigma}_{p,\mathcal{B}}) \cap C(J, W^2_{p,\mathcal{B}})$, where $t^+ := \sup J = \infty$ if

$$\sup_{0<t<\min\{t^+,T\}} \left\{ \|u(t)\|_{E_\varrho} + \|u(t)\|_{L_q((0,a_0),W^{2\omega}_p)} \right\} < \infty \quad \text{for all} \quad T > 0 \ . \tag{2.18}$$

Clearly, this so extended function u still satisfies

$$u(t,a) = \begin{cases} e^{-\int_0^a \mu(r)\mathrm{d}r}\, U_A(t, t-a)\, \xi(v(t-a))\, v(t-a) \ , & a < t \ , \\ e^{-\int_{a-t}^a \mu(r)\mathrm{d}r}\, U_A(t,0)\, u^0(a-t) \ , & a > t \ , \end{cases} \tag{2.19}$$

for $a > 0$ and $0 \le t < t^+$, where we simply write $A = A_u$ and $v = v_u$. Next recall that $u^0 \in E_1 \cap C^1(\mathbb{R}^+, L_p)$ and so, for $t \in (0, t^+)$ and $a > 0$ with $a \ne t$,

$$\begin{aligned} \partial_t u(t,a) = & \ \mathbf{1}_{[a<t]}(t,a)\, e^{-\int_0^a \mu(r)\mathrm{d}r} \Big\{ -A(t)\, U_A(t,t-a)\, \xi(v(t-a))\, v(t-a) \\ & \qquad + U_A(t,t-a)\big(\partial_t + A(t-a)\big)\big(\xi(v(t-a))\, v(t-a)\big) \Big\} \\ & - \mathbf{1}_{[a>t]}(t,a) \Big\{ \mu(a-t)\, u(t,a) + e^{-\int_{a-t}^a \mu(r)\mathrm{d}r} A(t)\, U_A(t,0)\, u^0(a-t) \\ & \qquad + e^{-\int_{a-t}^a \mu(r)\mathrm{d}r}\, U_A(t,0)\, \partial_a u^0(a-t) \Big\} \end{aligned}$$

and

$$\begin{aligned} \partial_a u(t,a) = & -\mu(a)\, u(t,a) \\ & - \mathbf{1}_{[a<t]}(t,a)\, e^{-\int_0^a \mu(r)\mathrm{d}r}\, U_A(t,t-a)\big(A(t-a) + \partial_t\big)\big(\xi(v(t-a))\, v(t-a)\big) \\ & + \mathbf{1}_{[a>t]}(t,a) \Big\{ e^{-\int_{a-t}^a \mu(r)\mathrm{d}r}\, U_A(t,0)\, \partial_a u^0(a-t) + \mu(a-t)\, u(t,a) \Big\} \ . \end{aligned}$$

Thus (v, u) is a solution to (1.1)-(1.5) with the regularity properties as stated in the assertion of the theorem.

It remains to prove that $t^+ = \infty$. We fix $T > 0$ arbitrarily and put $J_T := J \cap [0, T]$. Defining $\mathcal{M}_1(t,x) := \int_0^\infty e^{a/\tau}\, u(t,a,x)\, \mathrm{d}a$ we observe that (1.2) and (2.3) ensure

$$\partial_t v(t) \le v(t) + \|\mu\|_\infty\, \mathcal{M}_1(t) \ , \quad t \in J \ ,$$

whence

$$\|v(t)\|_\infty \le c(T) \left(\int_0^t \|\mathcal{M}_1(s)\|_\infty\, \mathrm{d}s + 1 \right) \ , \quad t \in J_T \ .$$

But since

$$\mathcal{M}_1(t) \le \|\xi\|_\infty\, e^{T/\tau} \int_0^t \|v(t-a)\|_\infty\, \mathrm{d}a + c(T)\, \|u^0\|_{E_1} \ , \quad t \in J_T \ ,$$

by (2.16), we conclude

$$\|v(t)\|_\infty + \|\mathcal{M}_1(t)\|_\infty \le c(T) \ , \quad t \in J_T \ . \tag{2.20}$$

Owing to $u^0 \in C([0,a_0], W_p^{2\omega-1})$ and $v \in C(J, W^2_{p,\mathcal{B}})$ it follows from

$$\|U_A(t,s) - U_A(t,r)\|_{\mathcal{L}(W^2_{p,\mathcal{B}}, W^{2\omega-1}_{p,\mathcal{B}})} \le c(t_0)\,(s-r)^{3/2-\omega}\ , \quad 0 \le r \le s \le t \le t_0 < t^+\ , \tag{2.21}$$

and (2.19) that $u(\cdot,a_0) \in C(J \setminus \{a_0\}, W_p^{2\omega-1})$. Property (2.21) is shown analogously to [4, Equation (II.5.3.8)]. Provided $a_0 < t^+$, (2.13) warrants that $\lim_{t\nearrow a_0} u(t,a_0) = u^0(0)$ in $W_p^{2\nu}$ with $n/p < 2\nu < 2\omega - 1$, while (2.21) warrants that $\lim_{t\searrow a_0} u(t,a_0) = \xi(v^0)v^0$ in $W_p^{2\omega-1}$. Thus, the imposed compatibility condition on u^0 and v^0 entails

$$u(\cdot,a_0) \in C(J, W_p^{2\nu}) \hookrightarrow C(J, C(\bar\Omega))\ .$$

Recalling that Λ_u is given by (1.6) we set

$$f(t,x) := e^{a_0/\tau}\, u(t,a_0,x) + \frac{1}{\tau}\,\Lambda_u(t,x) - \int_{a_0}^\infty e^{a/\tau}\,\mu(a)\,u(t,a,x)\,\mathrm{d}a\ , \quad (t,x) \in J_T \times \bar\Omega\ ,$$

and deduce $f \in C(J_T \times \bar\Omega)$ with

$$|f(t,x)| \le c(T)\ , \quad (t,x) \in J_T \times \bar\Omega \tag{2.22}$$

due to (2.20) and

$$\|u(t,a_0)\|_\infty \le \left\{ \begin{array}{ll} \|\xi(v(t-a_0))\,v(t-a_0)\|_\infty\ , & t > a_0 \\ \|u^0(a_0-t)\|_\infty\ , & t < a_0 \end{array} \right\} \le c(T)\ .$$

We then observe that $\Lambda = \Lambda_u$ solves the quasilinear parabolic problem

$$\partial_t\Lambda - \mathrm{div}_x\big(D(\Lambda)\nabla_x\Lambda\big) = f(t,x)\ , \quad (t,x) \in J_T \times \bar\Omega$$

subject to $\partial_\nu\Lambda = 0$ and $\Lambda(0) \in W^2_{p,\mathcal{B}}$. We refer to (2.1), (2.20), and (2.22) when using [2, Lemma 5.1(ii)] to obtain that $\Lambda \in BUC^\delta(J_T, C^\delta(\bar\Omega))$ for some $\delta > 0$, and hence $\Lambda \in BUC^\varepsilon(J_T, C^1(\bar\Omega))$ for some $\varepsilon > 0$ by [2, Lemma 4.2, Remark 4.3], where BUC^ε stands for 'bounded and uniformly ε-Hölder continuous'. But then $A = A_u$ is uniformly Hölder continuous, that is,

$$\|A_u(t) - A_u(s)\|_{\mathcal{L}(W^2_{p,\mathcal{B}}, L_p)} \le c(T)\,|t-s|^\varepsilon\ , \quad t,s \in J_T\ ,$$

so that [4, Lemma II.5.1.3] implies

$$\|U_{A_u}(t,s)\|_{\mathcal{L}(W^{2\varrho}_{p,\mathcal{B}})} + (t-s)^\varrho\,\|U_{A_u}(t,s)\|_{\mathcal{L}(L_p, W^{2\varrho}_{p,\mathcal{B}})} \le c(T) \tag{2.23}$$

for $t,s \in J_T$ with $s < t$. Note that $c(T)$ depends here on T only (but not on some norm of u). Combining (2.19), (2.20), and (2.23) we have

$$\begin{aligned} \|u(t)\|^q_{L_q((0,a_0),W_p^{2\omega})} \le\ & \int_0^{\min\{t,a_0\}} \|U_A(t,t-a)\|^q_{\mathcal{L}(L_p, W^{2\omega}_{p,\mathcal{B}})}\ \|\xi(v(t-a))\,v(t-a)\|^q_{L_p}\,\mathrm{d}a \\ & + \int_{\min\{t,a_0\}}^{a_0} \|U_A(t,0)\|^q_{\mathcal{L}(W^{2\omega}_{p,\mathcal{B}})}\ \|u^0(a-t)\|^q_{W_p^{2\omega}}\,\mathrm{d}a \\ \le\ & c(T)\int_0^{\min\{t,a_0\}} a^{-q\omega}\,\mathrm{d}a + c(T)\|u^0\|_{L_q((0,a_0),W_p^{2\omega})} \\ \le\ & c(T) \end{aligned}$$

for $t \in J_T$ thanks to $q\omega < 1$. Finally, from (2.19), (2.20), and (2.23) it follows analogously that $u \in L_\infty(J_T, E_\varrho)$. From this and (2.18) we deduce $t^+ = \infty$. This proves the theorem.

3. The Degenerate Case

We now turn to the "degenerate" case where D is allowed to vanish but only for $\Lambda = 0$. More precisely, we assume that $D \in C^{2-}(\mathbb{R})$ is such that $D(0) = 0$, $D(z) > 0$ if $z > 0$, and

$$I_D := \int_0^1 \frac{z\ D'(z)^2}{D(z)}\,\mathrm{d}z < \infty \quad \text{and} \quad \lim_{z\to 0} \frac{z\,|D'(z)|}{D(z)^{1/2}} = 0\ . \tag{3.1}$$

The function

$$\Phi_D(z) := z\ \int_1^z \frac{D'(y)^2}{D(y)}\,\mathrm{d}y - \int_0^z \frac{y\ D'(y)^2}{D(y)}\,\mathrm{d}y\ , \quad z \in [0,\infty)\ , \tag{3.2}$$

is then a well-defined smooth convex function satisfying $\Phi_D(z) \geq \Phi_D(1) = -I_D$ for $z \in [0,\infty)$. We also put

$$\widehat{D}(z) \ := \int_0^z D(y)\,\mathrm{d}y \quad \text{and} \quad \widehat{D}_1(z) \ := \int_0^z \widehat{D}(y)\,\mathrm{d}y\ , \quad z \in [0,\infty)\ .$$

We note that both $D(z) = z^{m-1}$, $m > 1$, and $D(z) = e^{-1/z}$ fulfil (3.1).

As for the differentiation and dedifferentiation rates ξ and μ, we assume that ξ fulfils (2.2) while μ satisfies

$$\mu \in W^1_\infty(\mathbb{R}) \quad \text{is non-negative and} \quad \mu(a) = 0 \quad \text{for} \quad a < a_0\ , \tag{3.3}$$

the latter assumption being stronger than (2.3).

Finally, the initial data are required to satisfy the following properties:

$$0 \leq u^0 \in L_1\big((0,\infty)\times\Omega, e^{a/\tau}\,\mathrm{d}a\mathrm{d}x\big) \cap L_\infty((0,\infty)\times\Omega) \quad \text{and} \quad 0 \leq v^0 \in L_\infty(\Omega) \tag{3.4}$$

and

$$\Lambda^0 \in L_\infty(\Omega) \quad \text{and} \quad \widehat{D}\left(\Lambda^0\right) \in W^1_2(\Omega)\ , \tag{3.5}$$

where

$$\Lambda^0(x) \ := \int_{a_0}^\infty e^{a/\tau}\, u^0(a,x)\,\mathrm{d}a\ , \quad x \in \Omega\ . \tag{3.6}$$

Theorem 3.1. *Let $T > 0$ and put $U := (0,T) \times (0,\infty) \times \Omega$. There are two non-negative functions $u \in L_\infty(U)$ and $v \in L_\infty((0,T)\times\Omega) \cap C([0,T], L_2(\Omega))$ satisfying*

$$\partial_t v(t,x) = \frac{1}{\tau}\big(1 - \xi(v(t,x))\big)\, v(t,x) + \int_0^\infty e^{a/\tau}\, \mu(a)\, u(t,a,x)\, \mathrm{d}a \quad a.e.\ in \quad (0,T)\times\Omega\ ,$$

and

$$\begin{aligned} 0 \;=\; & \int_0^T \int_\Omega \int_0^\infty u(t,a,x)\, \big(\partial_t\psi + \partial_a\psi\big)(t,a,x)\, \mathrm{d}a\mathrm{d}x\mathrm{d}t + \int_\Omega \int_0^\infty u^0(a,x)\, \psi(0,a,x)\, \mathrm{d}a\mathrm{d}x \\ & + \frac{1}{\tau}\int_0^T \int_\Omega \xi(v(t,x))\, v(t,x)\, \psi(t,0,x)\, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \int_0^\infty \mu(a)\, u(t,a,x)\, \psi(t,a,x)\, \mathrm{d}a\mathrm{d}x\mathrm{d}t \\ & - \int_0^T \int_\Omega \int_0^\infty \mathbf{J}(t,a,x)\, \nabla_x\psi(t,a,x)\, \mathrm{d}a\mathrm{d}x\mathrm{d}t \end{aligned}$$

for $\psi \in C_c^1([0,T) \times [0,\infty) \times \bar{\Omega})$ with $\partial_\nu\psi(t,a,x) = 0$ for $(t,a,x) \in (0,T) \times (0,\infty) \times \partial\Omega$, where the functions Λ and $\mathbf{J}$ are given by

$$\Lambda(t,x) \;:=\; \int_{a_0}^\infty e^{a/\tau}\, u(t,a,x)\, \mathrm{d}a \quad for\ a.e. \quad (t,x) \in (0,T)\times\Omega\ , \tag{3.7}$$

$$\mathbf{J} \;:=\; \nabla_x\big(u\, D(\Lambda)\big) - u\, \nabla_x D(\Lambda) \ \ in\ \ \mathcal{D}'(U,\mathbb{R}^n)\ , \tag{3.8}$$

and satisfy $\Lambda \in C([0,T], L_2(\Omega))$, $D(\Lambda) \in L_2((0,T), W_2^1(\Omega))$, and $\mathbf{J} \in L_2(U,\mathbb{R}^n)$.

The proof of Theorem 3.1 is performed by a compactness argument, approximating the diffusivity D by non-degenerate diffusivities $(D_\alpha)_{\alpha>0}$ for which we can apply Theorem 2.2 and obtain a sequence of solutions $(v_\alpha, u_\alpha)_{\alpha>0}$. The next step is to pass to the limit as $\alpha \to 0$ and we now point out the difficulties to be overcome: first, as $D(\Lambda)$ vanishes when $\Lambda = 0$, the equation (1.1) is no longer uniformly parabolic with respect to the space variable and $\nabla_x u$ is unlikely to be a function. Furthermore, as $a_0 > 0$, we may have $\Lambda(t,x) = 0$ but $u(t,a,x) \neq 0$ for $a \in (0,a_0)$ and (1.1) gives no information on u in that case. We therefore cannot expect to have strong convergence on the sequence (u_α). There are, however, nonlinear terms in (1.1) and (1.2) for which strong convergence is necessary to identify the limit. In particular, the strong compactness of $(v_\alpha)_\alpha$ is needed to pass to the limit in the term $\xi(v_\alpha)\, v_\alpha$. As v_α solves an ordinary differential equation, such a compactness can only be obtained as a consequence of the compactness of $\big((t,x) \longmapsto \int_0^\infty e^{a/\tau}\, \mu(a)\, u_\alpha(t,a,x)\, \mathrm{d}a\big)_\alpha$. One step in the proof is thus to show that certain integrals of u_α with respect to age enjoy some compactness properties with respect to the time and space variables. The strong compactness of $\Lambda_\alpha(t,x) = \int_{a_0}^\infty e^{a/\tau}\, u_\alpha(t,a,x)\, \mathrm{d}a$ will also follow from this step. Next, in order to identify $\mathbf{J}$, strong compactness is needed on $(\nabla_x D(\Lambda_\alpha))_\alpha$ to pass to the limit in the term $u_\alpha \nabla_x D(\Lambda_\alpha)$. This is proved by a suitable adaptation of an argument from [5].

We now begin the proof of Theorem 3.1. We fix $T > 0$ and consider a sequence $(D_\alpha)_{\alpha\in(0,1)}$ of functions in $C^{2-}(\mathbb{R})$ with the following properties: for every $\alpha \in (0,1)$, there is $d_\alpha > 0$ such that $D_\alpha(z) \geq d_\alpha$ for all $z \in \mathbb{R}$, and

$$D_\alpha(z) = D(z) \quad \text{for} \quad z \in \big[\alpha\, e^{-T\,\|\mu\|_\infty}, \infty\big)\ . \tag{3.9}$$

Next, let $(v^0_\alpha, u^0_\alpha)_{\alpha\in(0,1)}$ be a sequence of non-negative initial data fulfilling all the requirements of Theorem 2.2 together with the following properties:

$$\lim_{\alpha\to 0}\left\{\int_\Omega\int_0^\infty e^{a/\tau}\,\left|u^0_\alpha(a,x)-u^0(a,x)\right|\,\mathrm{d}a\mathrm{d}x+\left\|v^0_\alpha-v^0\right\|_1\right\}=0\ , \tag{3.10}$$

and there is $c_0>0$ such that

$$\int_\Omega\int_0^\infty e^{a/\tau}\,u^0_\alpha(a,x)\,\mathrm{d}a\mathrm{d}x+\left\|u^0_\alpha\right\|_\infty+\left\|v^0_\alpha\right\|_\infty+\left\|\Lambda^0_\alpha\right\|_\infty+\left\|\widehat{D}\left(\Lambda^0_\alpha\right)\right\|_{W^1_2(\Omega)}\le c_0 \tag{3.11}$$

with

$$\Lambda^0_\alpha(x)\ :=\int_{a_0}^\infty e^{a/\tau}\,u^0_\alpha(a,x)\,\mathrm{d}a\ge\alpha \tag{3.12}$$

for all $x\in\Omega$ and $\alpha\in(0,1)$.

Let (v_α,u_α) denote the solution to

$$\partial_t u_\alpha+\partial_a u_\alpha\ =\ \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x u_\alpha\big)\ -\ \mu(a)\,u_\alpha\ ,\quad (t,a,x)\in(0,\infty)\times(0,\infty)\times\Omega\ , \tag{3.13}$$

$$\partial_t v_\alpha\ =\ \frac{1}{\tau}\big(1\ -\ \xi(v_\alpha)\big)\,v_\alpha\ +\int_0^\infty e^{a/\tau}\,\mu(a)\,u_\alpha(t,a,x)\,\mathrm{d}a\ ,\ (t,x)\in(0,\infty)\times\Omega\ , \tag{3.14}$$

where

$$\Lambda_\alpha(t,x)\ :=\int_{a_0}^\infty e^{a/\tau}\,u_\alpha(t,a,x)\,\mathrm{d}a\ ,\quad (t,x)\in(0,\infty)\times\Omega\ , \tag{3.15}$$

subject to the boundary conditions

$$u_\alpha(t,0,x)\ =\ \frac{1}{\tau}\,\xi(v_\alpha(t,x))\,v_\alpha(t,x)\ ,\qquad (t,x)\in(0,\infty)\times\Omega\ , \tag{3.16}$$

$$D_\alpha(\Lambda_\alpha)\,\partial_\nu\,u_\alpha\ =\ 0\ ,\qquad (t,x)\in(0,\infty)\times\partial\Omega\ , \tag{3.17}$$

and the initial conditions

$$u_\alpha(0,a,x)=u^0_\alpha(a,x)\ ,\quad v_\alpha(0,x)=v^0_\alpha(x)\ ,\qquad (a,x)\in(0,\infty)\times\Omega\ . \tag{3.18}$$

We note that, thanks to (3.13), Λ_α solves

$$\partial_t\Lambda_\alpha\ =\ \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x\Lambda_\alpha\big)\ +g^1_\alpha-g^2_\alpha\quad\text{in}\quad(0,T)\times\Omega \tag{3.19}$$

with homogeneous Neumann boundary conditions and

$$g^1_\alpha(t,x)\ :=\ e^{a_0/\tau}\,u_\alpha(t,a_0,x)+\frac{\Lambda_\alpha(t,x)}{\tau}\ge 0\ ,\quad g^2_\alpha(t,x)\ :=\int_{a_0}^\infty\mu(a)\,e^{a/\tau}\,u_\alpha(t,a,x)\ \mathrm{d}a\ge 0\ .$$

For further use, we introduce the following functions:

$$\mathcal{M}_{1,\alpha}(t,x):=\int_0^\infty e^{a/\tau}\,u_\alpha(t,a,x)\,\mathrm{d}a\quad\text{and}\quad\mathcal{M}_{2,\alpha}(t,x):=\int_0^\infty e^{a/\tau}\,\mu(a)\,u_\alpha(t,a,x)\,\mathrm{d}a$$

for $(t,x) \in (0,T)\times\Omega$ and $\alpha \in (0,1)$. Using again (3.13) together with (3.15), we realize that $\mathcal{M}_{1,\alpha}$ and $\mathcal{M}_{2,\alpha}$ are solutions to

$$\partial_t \mathcal{M}_{1,\alpha} = \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x \mathcal{M}_{1,\alpha}\big) + \frac{1}{\tau}\,\xi(v_\alpha)\,v_\alpha + \frac{\mathcal{M}_{1,\alpha}}{\tau} - \int_0^\infty \mu(a)\,e^{a/\tau}\,u_\alpha(.,a,.)\,\mathrm{d}a \quad (3.20)$$

and

$$\partial_t \mathcal{M}_{2,\alpha} = \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x \mathcal{M}_{2,\alpha}\big) + \frac{\mathcal{M}_{2,\alpha}}{\tau} + \int_0^\infty \big(\mu'(a) - \mu(a)^2\big)\,e^{a/\tau}\,u_\alpha(.,a,.)\,\mathrm{d}a \quad (3.21)$$

in $(0,T)\times\Omega$, respectively, with homogeneous Neumann boundary conditions.

In the following, c and c_i, $i \ge 1$, denote positive constants depending on D, μ, ξ, a_0, τ, and c_0 in (3.11), but not on α. The dependence upon additional variables (such as T) will be indicated explicitly. As in the non-degenerate case, we establish L_∞-bounds for $\mathcal{M}_{1,\alpha}$, v_α, and u_α.

Lemma 3.2. *For $\alpha \in (0,1)$ and $t \in [0,T]$, we have*

$$\|\mathcal{M}_{1,\alpha}(t)\|_\infty + \|v_\alpha(t)\|_\infty + \|u_\alpha(t)\|_\infty \le c_1(T)\ , \quad (3.22)$$

$$\int_0^T\int_\Omega\int_0^\infty D_\alpha(\Lambda_\alpha)(t,x)\,|\nabla_x u_\alpha(t,a,x)|^2\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \le c_1(T)\ . \quad (3.23)$$

Proof. On the one hand, since μ belongs to $L_\infty(0,\infty)$, we have

$$\partial_t v_\alpha \le \frac{1}{\tau}\,v_\alpha + \|\mu\|_\infty\,\mathcal{M}_{1,\alpha}$$

by (3.14), from which we deduce that, for $t \in [0,T]$,

$$v_\alpha(t,x) \le v_\alpha^0(x)\ e^{t/\tau} + \|\mu\|_\infty \int_0^t \mathcal{M}_{1,\alpha}(s,x)\,e^{(t-s)/\tau}\,\mathrm{d}s\ .$$

Using (3.11) gives

$$\|v_\alpha(t)\|_\infty \le c(T)\ \left(1 + \int_0^t \|\mathcal{M}_{1,\alpha}(s)\|_\infty\,\mathrm{d}s\right)\ . \quad (3.24)$$

On the other hand, it follows from (3.20) and the boundedness of ξ that

$$\partial_t \mathcal{M}_{1,\alpha} \le \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x \mathcal{M}_{1,\alpha}\big) + \frac{\|v_\alpha\|_\infty}{\tau} + \frac{\mathcal{M}_{1,\alpha}}{\tau}\ .$$

The comparison principle then ensures that

$$\mathcal{M}_{1,\alpha}(t,x) \le \|\mathcal{M}_{1,\alpha}(0)\|_\infty\ e^{t/\tau} + \frac{1}{\tau}\int_0^t \|v_\alpha(s)\|_\infty\,e^{(t-s)/\tau}\,\mathrm{d}s \le c(T)\ \left(1 + \int_0^t \|v_\alpha(s)\|_\infty\,\mathrm{d}s\right)$$

for $t \in [0,T]$. We now combine this estimate with (3.24) and end up with

$$\|\mathcal{M}_{1,\alpha}(t)\|_\infty \le c(T)\left(1+\int_0^t\int_0^s \|\mathcal{M}_{1,\alpha}(\sigma)\|_\infty\,\mathrm{d}\sigma\mathrm{d}s\right) \le c(T)\left(1+\int_0^t \|\mathcal{M}_{1,\alpha}(s)\|_\infty\,\mathrm{d}s\right)$$

for $t \in [0,T]$. The Gronwall lemma then gives the claimed bound on $\|\mathcal{M}_{1,\alpha}\|_\infty$, which in turn gives that for $\|v_\alpha\|_\infty$ by (3.24).

Finally, by (3.13), (3.16), and (3.17), u_α satisfies

$$\partial_t u_\alpha + \partial_a u_\alpha \le \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x u_\alpha\big)$$

with $u_\alpha(t,0,x) = \xi(v_\alpha(t,x))\,v_\alpha(t,x)\,/\,\tau$ and subject to homogeneous Neumann boundary conditions for $t \in [0,T]$. On the one hand, the comparison principle readily implies that

$$u_\alpha(t,a,x) \le \|u_\alpha^0\|_\infty + \frac{1}{\tau}\sup_{s\in[0,T]}\|v_\alpha(s)\|_\infty\ , \quad (t,a,x)\in(0,T)\times(0,\infty)\times\Omega\ ,$$

which, together with (3.11) and the already established bound on $\|v_\alpha\|_\infty$, allows us to complete the proof of (3.22). On the other hand, since u_α is non-negative, we also have

$$\begin{aligned}
&\int_\Omega\int_0^\infty u_\alpha(T)^2\,\mathrm{d}a\mathrm{d}x + 2\int_0^T\int_\Omega\int_0^\infty D_\alpha(\Lambda_\alpha(t))\,|\nabla_x u_\alpha(t)|^2\,\mathrm{d}a\mathrm{d}x\mathrm{d}t\\
\le\ & \int_\Omega\int_0^\infty \left(u_\alpha^0\right)^2\,\mathrm{d}a\mathrm{d}x + \int_0^T\int_\Omega\int_0^\infty u_\alpha(t)^2\,\mathrm{d}a\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \left(\xi(v_\alpha(t))\,v_\alpha(t)\right)^2\,\mathrm{d}x\mathrm{d}t\\
\le\ & \|u_\alpha^0\|_\infty\,\|u_\alpha^0\|_1 + \int_0^T \|u_\alpha(t)\|_\infty\,\|u_\alpha(t)\|_1\,\mathrm{d}t + \int_0^T \|v_\alpha(t)\|_\infty^2\,\mathrm{d}t\\
\le\ & c(T)\ ,
\end{aligned}$$

the last inequality being a consequence of (3.11), (3.22), and the fact $\|u_\alpha(t)\|_1 \le \|\mathcal{M}_{1,\alpha}(t)\|_1$ for $t \in [0,T]$.

We next derive some estimates for Λ_α.

Lemma 3.3. *For $\alpha \in (0,1)$, $t \in [0,T]$, and $x \in \Omega$, we have*

$$\|\Lambda_\alpha(t)\|_\infty + \int_0^T\int_\Omega \left|\nabla_x D(\Lambda_\alpha)\right|^2\mathrm{d}x\mathrm{d}s + \int_0^T\int_\Omega \frac{D(\Lambda_\alpha)}{\Lambda_\alpha}\left|\nabla_x\Lambda_\alpha\right|^2\mathrm{d}x\mathrm{d}s \ \le\ c_2(T)\ ,\tag{3.25}$$

$$\left\|\widehat{D}(\Lambda_\alpha)(t)\right\|_{W_2^1(\Omega)} + \int_0^T \left\|\partial_t\widehat{D}(\Lambda_\alpha)(s)\right\|_2^2\mathrm{d}s \ \le\ c_2(T)\ ,\tag{3.26}$$

and $\Lambda_\alpha(t,x) \ge \alpha\, e^{-T\,\|\mu\|_\infty}$.

A straightforward consequence of (3.9) and the last assertion of Lemma 3.3 is that

$$D_\alpha(\Lambda_\alpha)(t,x) = D(\Lambda_\alpha)(t,x) \quad \text{for} \quad (t,x)\in(0,T)\times\Omega\ .$$

Proof. Clearly $\Lambda_\alpha \le \mathcal{M}_{1,\alpha}$ and the L_∞-bound for Λ_α is a straightforward consequence of Lemma 3.2. It next follows from (3.19) that $\partial_t \Lambda_\alpha \ge \mathrm{div}_x\big(D_\alpha(\Lambda_\alpha)\,\nabla_x\Lambda_\alpha\big) - \|\mu\|_\infty\,\Lambda_\alpha$ in $(0,T)\times\Omega$ with homogeneous Neumann boundary conditions. As $t \mapsto \alpha\, e^{-t\,\|\mu\|_\infty}$ is a subsolution to the previous equation, the lower bound $\Lambda_\alpha \ge \alpha\, e^{-T\,\|\mu\|_\infty}$ in $(0,T)\times\Omega$ readily follows from (3.12) by the comparison principle.

We next multiply (3.19) by $\Phi_D'(\Lambda_\alpha)$ with Φ_D being defined in (3.2) and integrate over $(0,T)\times\Omega$ to obtain

$$\begin{aligned}
&\int_\Omega \big(\Phi_D(\Lambda_\alpha(t)) - \Phi_D(\Lambda_\alpha^0)\big)\ \mathrm{d}x + \int_0^T\int_\Omega \Phi_D''(\Lambda_\alpha)\,D(\Lambda_\alpha)\ |\nabla_x\Lambda_\alpha|^2\ \mathrm{d}x\mathrm{d}t \\
=\ & \int_0^T\int_\Omega g_\alpha^1\,\Phi_D'(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\ -\ \int_0^T\int_\Omega g_\alpha^2\,\Phi_D'(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\ .
\end{aligned}$$

On the one hand, since Φ_D' is non-positive in $(0,1)$ and $g_\alpha^1 \ge 0$, we infer from (3.22) and the L_∞-estimate on Λ_α that

$$\begin{aligned}
\int_0^T\int_\Omega g_\alpha^1\,\Phi_D'(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\ &\le\ \int_0^T\int_\Omega g_\alpha^1\,\mathbf{1}_{[1,\infty)}(\Lambda_\alpha)\,\Phi_D'(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t \\
&\le\ \int_0^T\int_\Omega \left(e^{a_0/\tau}\,\|u_\alpha\|_\infty + \frac{\|\Lambda_\alpha\|_\infty}{\tau}\right)\,\Phi_D'\big(1+\|\Lambda_\alpha\|_\infty\big)\,\mathrm{d}x\mathrm{d}t \\
&\le\ c(T)\,.
\end{aligned}$$

On the other hand, as $\Phi_D' \ge 0$ on $(1,\infty)$, $\Phi_D' \le 0$ on $(0,1)$, and $r\,|\Phi_D'(r)| \le I_D$ for $r\in[0,1]$, we have

$$\begin{aligned}
-\int_0^T\int_\Omega g_\alpha^2\,\Phi_D'(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\ &\le\ \int_0^T\int_\Omega\int_{a_0}^\infty \mu(a)\,e^{a/\tau}\,u_\alpha(t,a,x)\ \mathrm{d}a\,\mathbf{1}_{(0,1)}(\Lambda_\alpha)\ |\Phi_D'(\Lambda_\alpha)|\ \mathrm{d}x\mathrm{d}t \\
&\le\ \|\mu\|_\infty \int_0^T\int_\Omega \Lambda_\alpha\,\mathbf{1}_{[0,1)}(\Lambda_\alpha)\ |\Phi_D'(\Lambda_\alpha)|\ \mathrm{d}x\mathrm{d}t \\
&\le\ \|\mu\|_\infty\,I_D\,T\,|\Omega|\ .
\end{aligned}$$

Recalling that $\Phi_D(r) \ge -I_D$ for $r\ge 0$ and $D\,\Phi_D'' = (D')^2$, we conclude that

$$\int_0^T\int_\Omega |\nabla_x D(\Lambda_\alpha)|^2\ \mathrm{d}x\mathrm{d}t \le c(T) + \int_\Omega \Phi_D(\Lambda_\alpha(0))\,\mathrm{d}x \le c(T) + \max\left\{I_D, \Phi_D\left(c_0\right)\right\}.$$

Similarly, we multiply (3.19) by $\log\Lambda_\alpha$ and integrate over $(0,T)\times\Omega$: using the non-negativity of g_α^1 and g_α^2, (3.11), and the L_∞-bound on Λ_α we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{D(\Lambda_\alpha)}{\Lambda_\alpha}\ |\nabla_x\Lambda_\alpha|^2\ \mathrm{d}x\mathrm{d}t \\
\le\ & \int_\Omega \Lambda_\alpha^0\ \big(\log\Lambda_\alpha^0\ -\ 1\big)\ \mathrm{d}x - \int_\Omega \Lambda_\alpha(T)\ \big(\log\Lambda_\alpha(T)\ -\ 1\big)\,\mathrm{d}x \\
+\ & \int_0^T\int_\Omega g_\alpha^1\ \log\Lambda_\alpha\,\mathbf{1}_{[1,\infty)}(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega g_\alpha^2\ \log\Lambda_\alpha\,\mathbf{1}_{(0,1)}(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq\ c(T) + T\,|\Omega|\,\|g_\alpha^1\|_\infty\,\log\left(1+\|\Lambda_\alpha\|_\infty\right) + \|\mu\|_\infty \int_0^T\int_\Omega \Lambda_\alpha\,|\log\Lambda_\alpha|\,\mathbf{1}_{(0,1)}(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\\
&\leq\ c(T)
\end{aligned}
$$

as $\|g_\alpha^1\|_\infty$ is bounded uniformly with respect to $\alpha\in(0,1)$ by (3.22) and the L_∞-bound on Λ_α.

We next multiply (3.19) by $2\,\partial_t\widehat{D}(\Lambda_\alpha)$ and integrate over $(0,t)\times\Omega$, $t\in[0,T]$: using (3.11), (3.22), and (3.25) we obtain

$$
\begin{aligned}
&2\int_0^t\int_\Omega D(\Lambda_\alpha)\,|\partial_t\Lambda_\alpha|^2\,\mathrm{d}x\mathrm{d}s\ +\ \left\|\nabla_x\widehat{D}(\Lambda_\alpha)(t)\right\|_2^2\\
\leq\ &\left\|\nabla_x\widehat{D}(\Lambda_\alpha^0)\right\|_2^2\ +\ 2\int_0^t\int_\Omega D(\Lambda_\alpha)\,|\partial_t\Lambda_\alpha|\ \left(e^{a_0/\tau}\,u_\alpha(s,a_0,x)+\left(\frac{1}{\tau}+\|\mu\|_\infty\right)\Lambda_\alpha\right)\mathrm{d}x\mathrm{d}s\\
\leq\ &c(T)\left(1\ +\int_0^t\int_\Omega D(\Lambda_\alpha)\,|\partial_t\Lambda_\alpha|\,\mathrm{d}x\mathrm{d}s\right)\\
\leq\ &\int_0^t\int_\Omega D(\Lambda_\alpha)\,|\partial_t\Lambda_\alpha|^2\,\mathrm{d}x\mathrm{d}s + c(T)\ .
\end{aligned}
$$

Therefore

$$
\int_0^t\int_\Omega D(\Lambda_\alpha)\,|\partial_t\Lambda_\alpha|^2\,\mathrm{d}x\mathrm{d}s\ +\ \left\|\nabla_x\widehat{D}(\Lambda_\alpha)(t)\right\|_2^2\leq c(T)\,,
$$

from which the claim (3.26) follows as $\left|\partial_t\widehat{D}(\Lambda_\alpha)\right|\leq c(T)\,\sqrt{D(\Lambda_\alpha)}\ |\partial_t\Lambda_\alpha|$ by (3.25).

At this point we have gathered the information required to show the strong compactness of $(\Lambda_\alpha)_\alpha$. This is, however, not sufficient to pass to the limit as $\alpha\to 0$ as there is a nonlinear dependence on v_α in (3.14). We now aim at proving the strong compactness of (v_α): this will be achieved by the strong compactness of $(\mathcal{M}_{2,\alpha})_\alpha$ which we show now.

Lemma 3.4. *For $\alpha\in(0,1)$, $t\in[0,T]$, and $\delta\in(0,1)$, we have*

$$
\int_0^T\left(\left\|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2\right\|_2^2+\left\|\partial_t\left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2\right\|_{W^1_{n+1}(\Omega)'}\right)\mathrm{d}t\leq c_3(T,\delta)\ . \tag{3.27}
$$

Proof. We multiply (3.21) by $\left(\mathcal{M}_{2,\alpha}-\delta\right)_+$ and integrate over $(0,T)\times\Omega$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\left\|\left(\mathcal{M}_{2,\alpha}(T)-\delta\right)_+\right\|_2^2+\int_0^T\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x\mathrm{d}t\\
\leq\ &\frac{1}{2}\left\|\left(\mathcal{M}_{2,\alpha}(0)-\delta\right)_+\right\|_2^2+T\,|\Omega|\ \left(\frac{\|\mathcal{M}_{2,\alpha}\|_\infty}{\tau}+\|\mu'\|_\infty\,\|\mathcal{M}_{1,\alpha}\|_\infty\right)\ \left\|\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right\|_\infty\ .
\end{aligned}
$$

As $\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\ \leq\ \mathcal{M}_{2,\alpha}\ \leq\ \|\mu\|_\infty\,\mathcal{M}_{1,\alpha}$ and $\mu\in W^1_\infty(0,\infty)$, we infer from Lemma 3.2 that

$$
\int_0^T\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x\mathrm{d}t\leq c(T)\ . \tag{3.28}
$$

Now, on the one hand, since the support of μ is included in $[a_0,\infty)$, we have $\mathcal{M}_{2,\alpha} \le \|\mu\|_\infty \Lambda_\alpha$ and

$$\left\{(t,x)\in(0,T)\times\Omega\ :\ \mathcal{M}_{2,\alpha}(t,x)\ge\delta\right\}\subset\left\{(t,x)\in(0,T)\times\Omega\ :\ \Lambda_\alpha(t,x)\ge\delta/\|\mu\|_\infty\right\}.$$

Introducing $m_\delta := \min_{[\delta/\|\mu\|_\infty,\infty)} D > 0$ we deduce from (3.25), (3.28), and the previous observation that

$$\begin{aligned}\int_0^T \left\|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2\right\|_2^2 \mathrm{d}t &= 4\int_0^T\int_\Omega \left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2 \left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2 \mathrm{d}x\mathrm{d}t \\ &\le \frac{4}{m_\delta}\int_0^T\int_\Omega \left(\|\mu\|_\infty\Lambda_\alpha-\delta\right)_+^2 D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2 \mathrm{d}x\mathrm{d}t \\ &\le c(T,\delta)\int_0^T\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2 \mathrm{d}x\mathrm{d}t \\ &\le c(T,\delta)\ ,\end{aligned}$$

which proves the first claim in (3.27).

On the other hand, if $\psi\in W^1_{n+1}(\Omega)$, it follows from (3.21) and Lemma 3.2 that

$$\begin{aligned}&\left|\int_\Omega \partial_t\left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2\ \psi\,\mathrm{d}x\right| \\ =&\ 2\left|\int_\Omega \left(\mathcal{M}_{2,\alpha}-\delta\right)_+\ \partial_t\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\ \psi\,\mathrm{d}x\right| \\ \le&\ 2\int_\Omega \left(\mathcal{M}_{2,\alpha}-\delta\right)_+ |\nabla_x\psi|\, D(\Lambda_\alpha)\ |\nabla_x\mathcal{M}_{2,\alpha}|\ \mathrm{d}x \\ +&\ 2\int_\Omega |\psi|\, D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x \\ +&\ \frac{2}{\tau}\,\|\mathcal{M}_{2,\alpha}\|_\infty\,\|\psi\|_1\left\|\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right\|_\infty \\ +&\ 2\left(\|\mu'\|_\infty+\|\mu\|_\infty^2\right)\ \|\mathcal{M}_{1,\alpha}\|_\infty\,\|\psi\|_1\left\|\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right\|_\infty \\ \le&\ 2\,\|\mathcal{M}_{2,\alpha}\|_\infty\ \|\nabla_x\psi\|_2\,\|D(\Lambda_\alpha)\|_\infty^{1/2}\left(\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x\right)^{1/2} \\ +&\ 2\,\|\psi\|_\infty\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x + c(T)\,\|\psi\|_1 \\ \le&\ c(T)\ \left(\|\nabla_x\psi\|_2+\|\psi\|_\infty\right)\left(1+\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x\right)\ .\end{aligned}$$

Owing to the continuous embedding of $W^1_{n+1}(\Omega)$ in $L_\infty(\Omega)$ we conclude that

$$\left\|\partial_t\left(\mathcal{M}_{2,\alpha}-\delta\right)_+^2\right\|_{W^1_{n+1}(\Omega)'}\le c(T)\left(1+\int_\Omega D(\Lambda_\alpha)\left|\nabla_x\left(\mathcal{M}_{2,\alpha}-\delta\right)_+\right|^2\mathrm{d}x\right)\ ,$$

which together with (3.28) implies the second claim in (3.27).

Lemma 3.4 provides the desired compactness for $(\mathcal{M}_{2,\alpha})_\alpha$ with the help of the following lemma.

Lemma 3.5. *Let Q be an open bounded subset of $\mathbb{R}^N$ for some $N \geq 1$ and $p \in [1,\infty)$. We consider a sequence $(z_k)_{k\geq 1}$ of non-negative functions in $L_p(Q)$ and assume that there is a sequence $(Z_j)_{j\geq 1}$ in $L_p(Q)$ such that*

$$\lim_{k\to\infty} \left\| \left(z_k - \frac{1}{j} \right)_+ - Z_j \right\|_p = 0 \quad \text{for all} \quad j \geq 1 \ . \tag{3.29}$$

Then (z_k) converges in $L_p(Q)$ as $k \to \infty$.

Proof. For $i \geq 1$, $j \geq 1$, and $k \geq 1$ we have

$$\begin{aligned} \|Z_i - Z_j\|_p &\leq \left\| \left(z_k - \frac{1}{i} \right)_+ - Z_i \right\|_p + \left\| \left(z_k - \frac{1}{i} \right)_+ - \left(z_k - \frac{1}{j} \right)_+ \right\|_p + \left\| \left(z_k - \frac{1}{j} \right)_+ - Z_j \right\|_p \\ &\leq \left\| \left(z_k - \frac{1}{i} \right)_+ - Z_i \right\|_p + |Q|^{1/p} \left| \frac{1}{i} - \frac{1}{j} \right| + \left\| \left(z_k - \frac{1}{j} \right)_+ - Z_j \right\|_p \ . \end{aligned}$$

Letting $k \to \infty$ and using (3.29) give

$$\|Z_i - Z_j\|_p \leq |Q|^{1/p} \left| \frac{1}{i} - \frac{1}{j} \right| \ ,$$

so that (Z_j) is a Cauchy sequence in $L_p(Q)$ and there is $Z \in L_p(Q)$ such that

$$\lim_{j\to\infty} \|Z_j - Z\|_p = 0 \ . \tag{3.30}$$

Next, for $j \geq 1$ and $k \geq 1$, we have

$$\begin{aligned} \|z_k - Z\|_p &\leq \left\| z_k - \left(z_k - \frac{1}{j} \right)_+ \right\|_p + \left\| \left(z_k - \frac{1}{j} \right)_+ - Z_j \right\|_p + \|Z_j - Z\|_p \\ &\leq \frac{|Q|}{j} + \left\| \left(z_k - \frac{1}{j} \right)_+ - Z_j \right\|_p + \|Z_j - Z\|_p \ , \end{aligned}$$

hence

$$\limsup_{k\to\infty} \|z_k - Z\|_p \leq \frac{|Q|}{j} + \|Z_j - Z\|_p$$

by (3.29). Letting $j \to \infty$ and using (3.30) give the expected convergence.

Finally, to link the limits of $(\Lambda_\alpha)_\alpha$, $(\mathcal{M}_{1,\alpha})_\alpha$ and $(\mathcal{M}_{2,\alpha})_\alpha$ with that of $(u_\alpha)_\alpha$ we need to control the behavior of u_α for large a and report the following result in that direction.

Lemma 3.6. *For $\alpha \in (0,1)$, $t \in [0,T]$, and $A \geq 1$ we have*

$$\int_\Omega \int_A^\infty e^{a/\tau}\, u_\alpha(t,a,x)\,\mathrm{d}a\mathrm{d}x \leq c_5(T)\,\omega_\alpha(A) \quad \textit{with} \quad \omega_\alpha(A) := \int_\Omega \int_{A/2}^\infty e^{a/\tau}\, u_\alpha^0(a,x)\,\mathrm{d}a\mathrm{d}x + \frac{1}{A}\ .$$

Proof. Let $\eta \in C^\infty(\mathbb{R})$ be a fixed non-decreasing function such that $\eta(a) = 0$ for $a \leq 1/2$ and $\eta(a) = 1$ for $a \geq 1$. For $A \geq 1$, we multiply (3.13) by $\eta(a/A)\, e^{a/\tau}$ and integrate over $(0,\infty) \times \Omega$ with the help of (3.17). Since $\eta(0) = 0$ we thus obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \int_0^\infty \eta\left(\frac{a}{A}\right)\, e^{a/\tau}\, u_\alpha(t,a,x)\,\mathrm{d}a\mathrm{d}x \\ &\qquad \leq \int_\Omega \int_0^\infty \left[\frac{1}{A}\,\partial_a \eta\left(\frac{a}{A}\right) + \frac{1}{\tau}\,\eta\left(\frac{a}{A}\right)\right]\, e^{a/\tau}\, u_\alpha(t,a,x)\,\mathrm{d}a\mathrm{d}x\ , \end{aligned}$$

$$\frac{d}{dt}\left(e^{-t/\tau} \int_\Omega \int_0^\infty \eta\left(\frac{a}{A}\right)\, e^{a/\tau}\, u_\alpha(t,a,x)\,\mathrm{d}a\mathrm{d}x\right) \leq \frac{\|\eta\|_{W_\infty^1}}{A} \int_\Omega \mathcal{M}_{1,\alpha}(t,x)\,\mathrm{d}x\ .$$

By virtue of (3.22) the right-hand side of the above differential inequality is bounded by $c(T)/A$ and Lemma 3.6 follows after time integration, taking into account the properties of η.

Proof of Theorem 3.1. Recall that $\left(\widehat{D}(\Lambda_\alpha)\right)_\alpha$ is bounded in $L_\infty((0,T), W_2^1(\Omega))$ and $\left(\partial_t \widehat{D}(\Lambda_\alpha)\right)_\alpha$ is bounded in $L_2((0,T) \times \Omega)$ by (3.26). Owing to the compactness of the embedding of $W_2^1(\Omega)$ in $L_2(\Omega)$ we may apply [15, Corollary 4] to conclude that

$$\left(\widehat{D}(\Lambda_\alpha)\right)_\alpha \quad \text{is relatively compact in} \quad C([0,T], L_2(\Omega))\ . \tag{3.31}$$

A similar argument allows us to deduce from Lemma 3.4 and [15, Corollary 4] the relative compactness of $\left(\left(\mathcal{M}_{2,\alpha} - 1/j\right)_+^2\right)_\alpha$ in $L_2((0,T)\times\Omega)$ for each $j \geq 1$. Since $(\mathcal{M}_{2,\alpha})_\alpha$ is bounded in $L_\infty((0,T) \times \Omega)$ by (3.22), the Lebesgue dominated convergence theorem actually allows us to conclude that

$$\left(\left(\mathcal{M}_{2,\alpha} - 1/j\right)_+\right)_\alpha \quad \text{is relatively compact in} \quad L_2((0,T) \times \Omega) \tag{3.32}$$

for each $j \geq 1$. We then infer from Lemma 3.2, (3.25), and (3.31) that there are a sequence $(\alpha_k)_{k\geq 1}$, $\alpha_k \to 0$, three functions $\ell \in C([0,T], L_2(\Omega)) \cap L_2((0,T), W_2^1(\Omega))$, $d \in L_2((0,T), W_2^1(\Omega))$, and $u \in L_\infty(U)$, and a sequence $(W_j)_{j\geq 1}$ in $L_2((0,T) \times \Omega)$ such that

$$\begin{aligned} (u_{\alpha_k})_k &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L_\infty(U)\ , && (3.33)\\ \left(D(\Lambda_{\alpha_k})\,,\, \widehat{D}(\Lambda_{\alpha_k})\right)_k &\rightharpoonup (d,\ell) \quad \text{in} \quad L_2((0,T), W_2^1(\Omega))\ , && (3.34)\\ \left(\widehat{D}(\Lambda_{\alpha_k})\right)_k &\longrightarrow \ell \quad \text{in} \quad C([0,T], L_2(\Omega))\ , && (3.35)\\ \left(\left(\mathcal{M}_{2,\alpha_k} - 1/j\right)_+\right)_k &\longrightarrow W_j \quad \text{in} \quad L_2((0,T) \times \Omega)\ . \end{aligned}$$

Combining the last convergence and Lemma 3.5 actually give that there is $W \in L_2((0,T)\times\Omega)$ such that

$$(\mathcal{M}_{2,\alpha_k})_k \longrightarrow W \quad \text{in} \quad L_2((0,T) \times \Omega)\ . \tag{3.36}$$

In addition, as the function $\widehat{D}$ is a diffeomorphism from $(0,\infty)$ onto its range with inverse $\widehat{D}^{-1}$, the bound (3.25) and the convergence (3.35) imply that

$$(\Lambda_{\alpha_k})_k \longrightarrow \widehat{D}^{-1}(\ell) \quad \text{in} \quad C([0,T],L^2(\Omega)) \ . \tag{3.37}$$

We now claim that, if χ is a non-negative measurable function such that $\chi(a) \le \Xi\, e^{a/\tau}$ for a.e. $a \ge 0$ and some $\Xi \ge 0$, we have

$$(\mathcal{M}_{\chi,\alpha_k})_k \overset{*}{\rightharpoonup} \mathcal{M}_\chi \quad \text{in} \quad L_\infty((0,T)\times\Omega) \tag{3.38}$$

with

$$\mathcal{M}_{\chi,\alpha}(t,x) := \int_0^\infty \chi(a)\, u_\alpha(t,a,x)\,\mathrm{d}a \quad \text{and} \quad \mathcal{M}_\chi(t,x) := \int_0^\infty \chi(a)\, u(t,a,x)\,\mathrm{d}a$$

for $(t,x) \in (0,T)\times\Omega$. Indeed, consider $\psi \in L_\infty((0,T)\times\Omega)$. For $A>0$ we have

$$\begin{aligned}
& \left| \int_0^T \int_\Omega \left(\mathcal{M}_{\chi,\alpha_k} - \mathcal{M}_\chi\right)(t,x)\,\psi(t,x)\,\mathrm{d}x\mathrm{d}t \right| \\
\le & \left| \int_0^T \int_\Omega \int_0^A \left(u_{\alpha_k} - u\right)(t,a,x)\,\psi(t,x)\,\chi(a)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \right| \\
+ & \left| \int_0^T \int_\Omega \int_A^\infty \left(u_{\alpha_k} - u\right)(t,a,x)\,\psi(t,x)\,\chi(a)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \right| \\
\le & \left| \int_0^T \int_\Omega \int_0^A \left(u_{\alpha_k} - u\right)(t,a,x)\,\psi(t,x)\,\chi(a)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \right| \\
+ & \ \Xi\,\|\psi\|_\infty \int_0^T \int_\Omega \int_A^\infty e^{a/\tau}\ \left(u_{\alpha_k} + u\right)(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \ .
\end{aligned}$$

We then infer from (3.10), Lemma 3.6, and (3.33) by a weak convergence argument that

$$\int_0^T \int_\Omega \int_A^\infty e^{a/\tau}\, u(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \le T\, c_5(T)\,\omega_0(A) \ ,$$

where

$$\omega_0(A) := \int_\Omega \int_{A/2}^\infty e^{a/\tau}\, u^0(a,x)\,\mathrm{d}a\mathrm{d}x + \frac{1}{A} \ .$$

It also follows from (3.10) and Lemma 3.6 that

$$\int_0^T \int_\Omega \int_A^\infty e^{a/\tau}\, u_{\alpha_k}(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \le T\, c_5(T) \left(\int_\Omega \int_0^\infty e^{a/\tau}\ \left|u^0_{\alpha_k} - u^0\right|\,\mathrm{d}a\mathrm{d}x + \omega_0(A) \right) \ ,$$

and thus

$$\limsup_{k\to\infty} \int_0^T \int_\Omega \int_A^\infty e^{a/\tau}\, u_{\alpha_k}(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \le T\, c_5(T)\,\omega_0(A) \ .$$

Consequently, by (3.33),

$$\limsup_{k\to\infty} \left| \int_0^T \int_\Omega \left(\mathcal{M}_{\chi,\alpha_k} - \mathcal{M}_\chi\right)(t,x)\, \psi(t,x)\, \mathrm{d}x\mathrm{d}t \right| \le c(T)\, \Xi\, \|\psi\|_\infty\, \omega_0(A) \ .$$

Since the above inequality holds true for all $A > 0$ and $\omega_0(A) \to 0$ as $A \to \infty$ by (3.4), we may let $A \to \infty$ to conclude that $(\mathcal{M}_{\chi,\alpha_k})_k$ converges weakly towards $\mathcal{M}_\chi$ in $L_1((0,T)\times\Omega)$ as $k \to \infty$. As $0 \le \mathcal{M}_{\chi,\alpha_k} \le \Xi\, \mathcal{M}_{1,\alpha_k}$, $(\mathcal{M}_{1,\alpha_k})_k$ is bounded in $L_\infty((0,T)\times\Omega)$ by (3.22), and since $(0,T)\times\Omega$ has finite measure, the previous L_1-weak convergence implies the claim (3.38).

In particular, we deduce from (3.38) (with $\chi(a) = \mathbf{1}_{[a_0,\infty)}(a)\, e^{a/\tau}$ and $\chi(a) = e^{a/\tau}\, \mu(a)$, respectively) that $(\Lambda_{\alpha_k})_k$ and $(\mathcal{M}_{2,\alpha_k})_k$ converge weakly-$*$ towards Λ and $\mathcal{M}_2$ in $L_\infty((0,T)\times\Omega)$, respectively, with Λ and $\mathcal{M}_2$ given by

$$\Lambda(t,x) := \int_{a_0}^\infty e^{a/\tau}\, u(t,a,x)\, \mathrm{d}a \quad \text{and} \quad \mathcal{M}_2(t,x) := \int_0^\infty e^{a/\tau}\, \mu(a)\, u(t,a,x)\, \mathrm{d}a$$

for $(t,x) \in (0,T)\times\Omega$. Combining this fact with (3.36) and (3.37) leads us to the identities $\Lambda = \widehat{D}^{-1}(\ell)$ and $\mathcal{M}_2 = W$, and we have thus shown that

$$(\Lambda_{\alpha_k})_k \longrightarrow \Lambda \quad \text{in} \quad C([0,T], L_2(\Omega)) \quad \text{and} \quad (\mathcal{M}_{2,\alpha_k})_k \longrightarrow \mathcal{M}_2 \quad \text{in} \quad L_2((0,T)\times\Omega) \ . \tag{3.39}$$

A simple consequence of (3.34) and (3.39) is that $d = D(\Lambda)$ so that

$$D(\Lambda) \in L_2((0,T), W_2^1(\Omega)) \quad \text{and} \quad (D(\Lambda_{\alpha_k}))_k \rightharpoonup D(\Lambda) \text{ in } L_2((0,T), W_2^1(\Omega)) \ . \tag{3.40}$$

Let then v denote the unique solution to

$$\partial_t v(t,x) = \frac{1}{\tau}\left(1 - \xi(v(t,x))\right) v(t,x) + \mathcal{M}_2(t,x) \ , \quad (t,x) \in (0,T)\times\Omega \ , \tag{3.41}$$

with initial condition $v(0) = v^0$. At this stage it is rather easy to deduce from (3.14), (3.41), and the properties of ξ that

$$\frac{d}{dt}\|v_{\alpha_k} - v\|_2^2 \le c_6(T)\left(\|v_{\alpha_k} - v\|_2^2 + \|\mathcal{M}_{2,\alpha_k} - \mathcal{M}_2\|_2^2\right) \ .$$

The strong convergence (3.39) in $L_2((0,T)\times\Omega)$ of $(\mathcal{M}_{2,\alpha_k})_k$ towards $\mathcal{M}_2$, (3.10), (3.11), and the above differential inequality imply that

$$(v_{\alpha_k})_k \longrightarrow v \quad \text{in} \quad C([0,T], L_2(\Omega)) \ . \tag{3.42}$$

Introducing

$$\mathbf{J}_\alpha := D(\Lambda_\alpha)\, \nabla_x u_\alpha \quad \text{and} \quad \mathbf{j}_\alpha := D(\Lambda_\alpha)^{1/2}\, \nabla_x u_\alpha \ ,$$

we infer from (3.13), (3.16), (3.17), and (3.18) that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \int_0^\infty u_\alpha(t,a,x)\,\big(\partial_t\psi + \partial_a\psi\big)(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t + \int_\Omega\int_0^\infty u_\alpha^0(a,x)\,\psi(0,a,x)\,\mathrm{d}a\mathrm{d}x \\ &+ \frac{1}{\tau}\int_0^T\int_\Omega \xi(v_\alpha(t,x))\,v_\alpha(t,x)\,\psi(t,0,x)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega\int_0^\infty \mu(a)\,u_\alpha(t,a,x)\,\psi(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \\ &- \int_0^T\int_\Omega\int_0^\infty \mathbf{J}_\alpha(t,a,x)\,\nabla_x\psi(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.43}$$

for $\psi \in C_c^1([0,T)\times[0,\infty)\times\bar\Omega)$ satisfying $\partial_\nu\psi(t,a,x) = 0$ for $(t,a,x)\in(0,T)\times(0,\infty)\times\partial\Omega$. By (3.23) $(\mathbf{j}_\alpha)_\alpha$ is bounded in $L_2(U,\mathbb{R}^n)$ and so is $\mathbf{J}_\alpha = D(\Lambda_\alpha)^{1/2}\,\mathbf{j}_\alpha$ by (3.25). We may then assume (after possibly extracting a further subsequence) that there is $\mathbf{J}\in L_2(U,\mathbb{R}^n)$ such that

$$(\mathbf{J}_{\alpha_k})_k \rightharpoonup \mathbf{J} \quad \text{in} \quad L_2(U,\mathbb{R}^n)\ . \tag{3.44}$$

Owing to (3.10), (3.33), (3.42), and (3.44), we may pass to the limit as $k\to\infty$ in the weak formulation (3.43) of (3.13) to conclude that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \int_0^\infty u(t,a,x)\,\big(\partial_t\psi + \partial_a\psi\big)(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t + \int_\Omega\int_0^\infty u^0(a,x)\,\psi(0,a,x)\,\mathrm{d}a\mathrm{d}x \\ &+ \frac{1}{\tau}\int_0^T\int_\Omega \xi(v(t,x))\,v(t,x)\,\psi(t,0,x)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega\int_0^\infty \mu(a)\,u(t,a,x)\,\psi(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \\ &- \int_0^T\int_\Omega\int_0^\infty \mathbf{J}(t,a,x)\,\nabla_x\psi(t,a,x)\,\mathrm{d}a\mathrm{d}x\mathrm{d}t \end{aligned}$$

for $\psi \in C_c^1([0,T)\times[0,\infty)\times\bar\Omega)$ satisfying $\partial_\nu\psi(t,a,x) = 0$ for $(t,a,x)\in(0,T)\times(0,\infty)\times\partial\Omega$ as claimed in Theorem 3.1.

It remains to identify $\mathbf{J}$: for that purpose we introduce the sets

$$\mathcal{P} := \big\{(t,x)\in(0,T)\times\Omega\ :\ \Lambda(t,x)>0\big\}\ ,\quad \mathcal{Z} := \big\{(t,x)\in(0,T)\times\Omega\ :\ \Lambda(t,x)=0\big\}\ ,$$

and observe that $\mathbf{J}_\alpha$ may be written

$$\mathbf{J}_\alpha = \nabla_x\left(u_\alpha\,D(\Lambda_\alpha)\right) - u_\alpha\,\nabla_x D(\Lambda_\alpha) \quad \text{in} \quad \mathcal{D}'(U,\mathbb{R}^n)\ . \tag{3.45}$$

It follows at once from (3.25), (3.33), (3.39), and the continuity of D that

$$(u_{\alpha_k}\,D(\Lambda_{\alpha_k}))_k \rightharpoonup u\,D(\Lambda) \quad \text{in} \quad L_2(U)\ . \tag{3.46}$$

Next, we claim that, after possibly extracting a further subsequence (not relabeled), we have

$$\big(\nabla_x\widehat{D}(\Lambda_{\alpha_k})\big)_k \longrightarrow \nabla_x\widehat{D}(\Lambda) \quad \text{in} \quad L_2(U,\mathbb{R}^n) \quad \text{and a.e. in} \quad (0,T)\times\Omega\ , \tag{3.47}$$

and adapt the proof of [5, Eq. (3.22)] to this end. We multiply (3.19) by $\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)$ and integrate over $(0,T)\times\Omega$ to obtain

$$\begin{aligned}\int_0^T\int_\Omega \big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\,\partial_t\Lambda_\alpha\,\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_\Omega \nabla_x\big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\cdot\nabla_x\widehat{D}(\Lambda_\alpha)\,\mathrm{d}x\mathrm{d}t\\ &+ \int_0^T\int_\Omega \big(g_\alpha^1-g_\alpha^2\big)\big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\,\mathrm{d}x\mathrm{d}t\ ,\end{aligned}$$

hence

$$\begin{aligned}\int_0^T\int_\Omega \Big|\nabla_x\big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\Big|^2\,\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_\Omega \nabla_x\big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\cdot\nabla_x\widehat{D}(\Lambda)\,\mathrm{d}x\mathrm{d}t\\ &+ \int_0^T\int_\Omega \big(g_\alpha^1-g_\alpha^2\big)\big(\widehat{D}(\Lambda_\alpha)-\widehat{D}(\Lambda)\big)\,\mathrm{d}x\mathrm{d}t\\ &+ \int_0^T\int_\Omega \big(\widehat{D}(\Lambda)-\widehat{D}(\Lambda_\alpha)\big)\,\partial_t\Lambda_\alpha\,\mathrm{d}x\mathrm{d}t\ .\end{aligned}\tag{3.48}$$

As $\nabla_x\widehat{D}(\Lambda)\in L_2((0,T)\times\Omega)$ by (3.34) and $\widehat{D}(\Lambda)=\ell$ by (3.37) and (3.39) we infer from (3.34) that

$$\lim_{k\to\infty}\int_0^T\int_\Omega \nabla_x\big(\widehat{D}(\Lambda_{\alpha_k})-\widehat{D}(\Lambda)\big)\cdot\nabla_x\widehat{D}(\Lambda)\,\mathrm{d}x\mathrm{d}t=0\ .$$

It next follows from (3.22) and (3.25) that $\|g_\alpha^1\|_\infty+\|g_\alpha^2\|_\infty\le c(T)$. Therefore, by virtue of (3.35), (3.37), and (3.39) we have

$$\lim_{k\to\infty}\int_0^T\int_\Omega \big(g_{\alpha_k}^1-g_{\alpha_k}^2\big)\big(\widehat{D}(\Lambda_{\alpha_k})-\widehat{D}(\Lambda)\big)\,\mathrm{d}x\mathrm{d}t=0\ .$$

We next argue as in [5] (with $\widehat{D}$ instead of $z\mapsto z^m$ and $X=W_2^1(\Omega)$) to show that

$$\lim_{k\to\infty}\int_0^T\int_\Omega \big(\widehat{D}(\Lambda)-\widehat{D}(\Lambda_{\alpha_k})\big)\,\partial_t\Lambda_{\alpha_k}\,\mathrm{d}x\mathrm{d}t=0\ .$$

Taking $\alpha=\alpha_k$ in (3.48) we may therefore pass to the limit as $k\to\infty$ and conclude that (3.47) holds true.

Now, as $\nabla_x D(\Lambda_{\alpha_k})=\big(D'(\Lambda_{\alpha_k})\,\nabla_x\widehat{D}(\Lambda_{\alpha_k})\big)/D(\Lambda_{\alpha_k})$ we deduce from (3.39) and (3.47) that

$$\big(\nabla_x D(\Lambda_{\alpha_k})\big)_k\longrightarrow\frac{D'(\Lambda)}{D(\Lambda)}\,\nabla_x\widehat{D}(\Lambda)=\nabla_x D(\Lambda)\quad\text{a.e. in}\quad\mathcal{P}\ .\tag{3.49}$$

Moreover, by (3.25),

$$\int_{\mathcal{Z}}|\nabla_x D(\Lambda_{\alpha_k})|\,\mathrm{d}x\mathrm{d}t\ =\ \int_{\mathcal{Z}}\frac{|D'(\Lambda_{\alpha_k})|\,\Lambda_{\alpha_k}^{1/2}}{D(\Lambda_{\alpha_k})^{1/2}}\,\frac{D(\Lambda_{\alpha_k})^{1/2}}{\Lambda_{\alpha_k}^{1/2}}\,|\nabla_x\Lambda_{\alpha_k}|\,\mathrm{d}x\mathrm{d}t$$

$$\begin{aligned} &\leq \left(\int_{\mathcal{Z}} \frac{|D'(\Lambda_{\alpha_k})|^2 \, \Lambda_{\alpha_k}}{D(\Lambda_{\alpha_k})} \, \mathrm{d}x\mathrm{d}t \right)^{1/2} \left(\int_0^T \int_\Omega \frac{D(\Lambda_{\alpha_k})}{\Lambda_{\alpha_k}} \, |\nabla_x \Lambda_{\alpha_k}|^2 \, \mathrm{d}x\mathrm{d}t \right)^{1/2} \\ &\leq c(T) \left(\int_{\mathcal{Z}} \frac{|D'(\Lambda_{\alpha_k})|^2 \, \Lambda_{\alpha_k}}{D(\Lambda_{\alpha_k})} \, \mathrm{d}x\mathrm{d}t \right)^{1/2} , \end{aligned}$$

and the right-hand side of the above inequality converges to zero as $k \to \infty$ by (3.1), (3.39), the definition of $\mathcal{Z}$, and the Lebesgue dominated convergence theorem. As $\nabla_x D(\Lambda) = 0$ a.e. in $\mathcal{Z}$ by Stampacchia's theorem, we have shown that, after possibly extracting a further subsequence (not relabeled), $(\nabla_x D(\Lambda_{\alpha_k}))_k$ converges to $\nabla_x D(\Lambda)$ a.e. in $\mathcal{Z}$. Recalling (3.49) we have actually established that $(\nabla_x D(\Lambda_{\alpha_k}))_k$ converges to $\nabla_x D(\Lambda)$ a.e. in $(0,T) \times \Omega$ which, together with (3.40) and the Vitali theorem, implies that $(\nabla_x D(\Lambda_{\alpha_k}))_k$ converges to $\nabla_x D(\Lambda)$ in $L_1((0,T) \times \Omega, \mathbb{R}^n)$. Combining this convergence with (3.33) entails that

$$(u_{\alpha_k} \, \nabla_x D(\Lambda_{\alpha_k}))_k \rightharpoonup u \, \nabla_x D(\Lambda) \quad \text{in} \quad L_1(U, \mathbb{R}^n) \, . \tag{3.50}$$

Thanks to (3.45), (3.46), and (3.50), we conclude that $\mathbf{J} = \nabla_x (u \, D(\Lambda)) - u \, \nabla_x D(\Lambda)$ in $\mathcal{D}'(U, \mathbb{R}^n)$ as claimed in (3.8). In fact, as $\mathbf{J}$ and $u \, \nabla_x D(\Lambda)$ both belong to $L_2(U, \mathbb{R}^n)$, we realize that $u \, D(\Lambda)$ belongs to $L_2((0,T) \times (0,\infty), W_2^1(\Omega))$.

Remark 3.7. *It is actually not necessary to assume that $D \in C^{2-}(\mathbb{R})$ and Theorem 3.1 is valid if $D \in C(\mathbb{R})$, so that it applies to the diffusivity*

$$D(\Lambda) \, = \, D_0 \left(\Lambda - \Lambda_{min} \right)_+^{m-1} + e^{-1/(\varepsilon \, \Lambda)}$$

for $m > 1$ and $\varepsilon > 0$. The proof is nevertheless slightly more technical as the sequence $(D_\alpha)_\alpha$ approximating D cannot coincide with D on some interval and has to be constructed carefully so that the above proof still works.

4. Concluding Remarks

A model accounting for the swarming of the bacteria *Proteus mirabilis* and involving age and spatial variables has been studied. It describes the evolution of small non-moving cells (swimmers) and larger moving cells (swarmers), the latter moving according to Brownian movement $\mathrm{div}_x \big(D(\Lambda) \, \nabla_x u \big)$ with a diffusivity $D(\Lambda)$ depending on the total motile swarmer cell biomass Λ defined in (1.6) and thus, in a nonlocal way (with respect to age), on u. Assuming that the diffusivity is bounded from below by a positive real number, existence and uniqueness of a strong solution have been established in Section 2. It is, however, expected on biological grounds that a certain amount of biomass is required for the motion of swarmers to be initiated, that is, $D(\Lambda)$ is expected to vanish when Λ is below a threshold value $\Lambda_{min} \geq 0$. A step in that direction is made in Section 3 where the existence of a weak solution is obtained for $\Lambda_{min} = 0$. To our knowledge, the more realistic case $\Lambda_{min} > 0$ has not been investigated analytically so far, and we hope to return to this problem and to the formation of regularly spaced concentric terraces as well in the near future.

As a final comment, let us point out that in the model studied in this paper only Brownian motion is responsible for the movement of swarmer cells and describes somehow local displacements. Though, as pointed out in [9] and [6], only swarmer cells of a certain maturity can actively participate in group migration, the so-called "raft building", but nothing prevents young swarmers from being caught up in the flow and thus move with larger swarmers in the rafts. The diffusion term $\mathrm{div}_x(D(\Lambda)\nabla_x u)$, however, reflects active movement of swarmers of any age, i.e. also of young swarmers. It is therefore more realistic to model migration by a drift term along the gradient of biomass, namely, $\mathrm{div}_x\big(u\,E(\Lambda,v)\,\nabla_x\Lambda\big)$ with $E \geq 0$. The velocity $E(\Lambda,v)\,\nabla_x\Lambda$ then points in the direction of increasing biomass density. The swarmer cell density equation including the above two spatial mechanisms then reads

$$\partial_t u + \partial_a u \;=\; \mathrm{div}_x\big(D(\Lambda)\,\nabla_x u + u\,E(\Lambda,v)\,\nabla_x\Lambda\big) \;-\; \mu(a)\,u \qquad (4.1)$$

for $(t,a,x) \in (0,\infty)\times(0,\infty)\times\Omega$ instead of (1.1). The special case $E(\Lambda,v) = D'(\Lambda)$ is actually stated in [6]. Note that the choice $D \equiv 0$ is possible in (4.1) and it would be interesting to see whether similar structures also arise from the model accounting only for drift motion. From a more theoretical viewpoint, the study of (4.1) seems to be more complicated than that of (1.1) because the initial-boundary value problem is no longer diagonal. Nevertheless existence of weak solutions can still be established and will appear elsewhere [11].

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