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Wentzell Boundary Conditions in the Nonsymmetric Case

A. Favini[a1], G. R. Goldstein[b], J. A. Goldstein[b] and S. Romanelli[c]

[a] Dipartimento di Matematica, Università degli Studi di Bologna, Piazza di Porta S. Donato 5
40126 Bologna, Italy
[b] Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee 38152
[c] Dipartimento di Matematica, Università degli Studi di Bari, via E. Orabona 4, 70125 Bari, Italy

Abstract. Let L be a nonsymmetric second order uniformly elliptic operator with general Wentzell boundary conditions. We show that a suitable version of L generates a quasicontractive semigroup on an L^p space that incorporates both the underlying domain and its boundary. This extends the earlier work of the authors on the symmetric case.

Key words: Wentzell boundary conditions, nonsymmetric elliptic operators, C_0 semigroups, quasidissipative operators
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1. Introduction

In his study of the diffusion limit of the celebrated Wright-Fisher genetic model, W. Feller [4] showed that the limit of the corresponding n-state Markov chain was governed by a Markov process governed by the positive contraction (C_0) semigroup on $C[0,1]$ with generator A defined by

$$Au(x) = x(1-x)u''(x), \qquad x \in [0,1], \quad ' = \frac{d}{dx},$$

$$D(A) = \{u \in C[0,1] : Au \in C[0,1], \quad \lim_{x\to j} Au(x) = 0 \quad \text{for} \quad j = 0,1\}.$$

Boundary conditions such as the one above (i.e. $Au = 0$ on the boundary) were later introduced by A. D. Wentzell [10].

[1]Corresponding author. E-mail: favini@dm.unibo.it

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Now, let

$$Lu := \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}),$$

where $x \in \Omega$, a C^2 bounded domain in $\mathbf{R}^N$, $(a_{ij}(x))$ is a uniformly positive definite real symmetric matrix in $C^1(\overline{\Omega})$, and the modulus of ellipticity is $\alpha_0 > 0$:

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbf{R}^N$. The conormal derivative of u with respect to L at $x \in \partial\Omega$ is

$$\frac{\partial^L u}{\partial \nu} := \sum_{i,j=1}^{N} a_{ij}(\frac{\partial u}{\partial x_j})\nu_i,$$

where $\nu = (\nu_1, ..., \nu_N)$ is the unit outer normal to $\partial\Omega$ at x. Of course, $\frac{\partial^L}{\partial \nu}$ reduces to the usual normal derivative $\frac{\partial}{\partial \nu}$ when L is the Laplacian Δ. The general Wentzell boundary condition for this problem is

$$\alpha(x)Lu + \beta(x)\frac{\partial^L u}{\partial \nu} + \gamma(x)u = 0, \qquad x \in \partial\Omega,$$

where $\alpha, \beta, \gamma \in C^1(\partial\Omega)$, $\alpha > 0$ and $\beta > 0$. Without loss of generality we may take $\alpha \equiv 1$. Let A denote the operator L with this boundary condition. Problems involving A were studied on $C(\overline{\Omega})$ but not on $L^p(\Omega)$, $p < \infty$, because A is never quasi-dissipative on these L^p spaces. (For semigroup background, see [1], [5].)

Beginning in [3] in 2002, we developed an L^p theory for problems involving A. We showed, among other things, that the closure A_p of a suitable realization of A generates a (C_0) quasi-contractive semigroup on

$$X_p = L^p(\Omega, dx) \oplus L^p(\partial\Omega, \frac{dS}{\beta}) = L^p(\overline{\Omega}, \mu)$$

where

$$\mu(E) = \int_{\Omega} \chi_E(x)\, dx + \int_{\partial\Omega} \chi_E(x)\, \frac{dS(x)}{\beta(x)}$$

for E a Borel set and dS denotes the usual surface measure on $\partial\Omega$; here $1 \leq p < \infty$. We view X_p as the closure of $C(\overline{\Omega})$ in the X_p norm for p finite; and we define $X_\infty = C(\overline{\Omega})$. The generation result mentioned above holds on X_p for $1 \leq p \leq \infty$. Analyticity and other results hold as well; see [2] for the most recent results. We note that A_2 is self-adjoint and bounded above (by $\|\gamma_-\|_\infty$) on X_2.

The purpose of this paper is to present an analogue of the above result in the nonsymmetric case. We shall treat the uniformly elliptic case.

2. The main results

As before, let $\Omega \subset\subset \mathbf{R}^N$ be a C^2 domain, $\mathcal{A} = (a_{ij})$ a $C^1(\overline{\Omega})$ real symmetric $N \times N$ matrix function on $\overline{\Omega}$, and β, γ real functions in $C^1(\partial\Omega)$ with $\beta > 0$. Let $b \in C^1(\overline{\Omega}, \mathbf{R}^N)$ and $c \in C(\overline{\Omega}, \mathbf{R})$. Suppose

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2$$

for some $\alpha_0 > 0$ and all $(x, \xi) \in \overline{\Omega} \times \mathbf{R}^N$. Let

$$Mu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} + cu \tag{2.1}$$

with boundary condition

$$Mu + \beta\frac{\partial^L u}{\partial \nu} + \gamma u = 0, \qquad on \quad \partial\Omega \tag{2.2}$$

where L is the symmetric version of M corresponding to replacing each of b, c by zero. Let

$$D_0 = \{u \in C^2(\overline{\Omega}) : \quad (2.2) \quad \text{holds}\},$$

and let A_0 be the restriction of L to D_0. Let

$$X_2 = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta}).$$

Then, for all $u \in D_0$ and all $v \in C^1(\overline{\Omega})$,

$$< A_0u, v >_{X_2} = \int_\Omega (Mu)\overline{v}\, dx + \int_{\partial\Omega} (Mu)\overline{v}\frac{dS}{\beta}$$

$$= -\int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_i}\, dx + \int_{\partial\Omega} (Mu + \beta\frac{\partial^L u}{\partial \nu})\overline{v}\frac{dS}{\beta} + \int_\Omega \sum_{i=1}^{N}(b_i\frac{\partial u}{\partial x_i} + cu)\overline{v}\, dx$$

by the divergence theorem

$$= -\int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_i}\, dx - \int_{\partial\Omega} \gamma\, u\overline{v}\frac{dS}{\beta} + \int_\Omega \sum_{i=1}^{N}(b_i\frac{\partial u}{\partial x_i} + cu)\overline{v}\, dx$$

by (2.2). In particular, for $u \in D_0$,

$$Re < A_0u, u >_{X_2} \leq -\alpha_0 \int_\Omega |\nabla u|^2\, dx - \int_{\partial\Omega} \gamma|u|^2\frac{dS}{\beta} \tag{2.3}$$

$$+Re\int_\Omega (b \cdot \nabla u)\overline{u}\, dx + \int_\Omega c|u|^2\, dx.$$

To see why this holds, let $r = Re\, u$, $s = Im\, u$. Then $\nabla u = \nabla r + i\nabla s$ and

$$Re(\frac{\partial u}{\partial x_j}\frac{\partial \overline{u}}{\partial x_k}) = Re\,(\frac{\partial r}{\partial x_j} + i\frac{\partial s}{\partial x_j})(\frac{\partial r}{\partial x_k} - i\frac{\partial s}{\partial x_k}) = r_{x_j}r_{x_k} + s_{x_j}s_{x_k},$$

whence, using obvious notation,

$$\sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{u}}{\partial x_i} = (\mathcal{A}\nabla u)\cdot\nabla\overline{u} = (\mathcal{A}\nabla r)\cdot\nabla r + (\mathcal{A}\nabla s)\cdot\nabla s \geq \alpha_0|\nabla r|^2 + \alpha_0|\nabla s|^2 = \alpha_0|\nabla u|^2,$$

and (2.3) follows.

Next, by the Cauchy-Schwarz inequality, denoting by

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega |\nabla u|^2 dx$$

we have

$$Re\int_\Omega (b\cdot\nabla u)\overline{u}\,dx \leq \|b\|_\infty\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \|b\|_\infty\{\epsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|u\|^2_{L^2(\Omega)}\}$$

for any $\epsilon > 0$. Consequently,

$$Re < A_0 u, u >_{X_2} \leq (-\alpha_0 + \epsilon\|b\|_\infty)\|\nabla u\|^2_{L^2(\Omega)}$$

$$+\|\gamma_-\|_\infty\,\|u\|^2_{L^2(\partial\Omega),\frac{dS}{\beta})} + \|c_+\|_\infty\,\|u\|^2_{L^2(\Omega)} + \frac{\|b\|_\infty}{4\epsilon}\|u\|^2_{L^2(\Omega)} \leq -\frac{\alpha_0}{2}\|\nabla u\|^2_{L^2(\Omega)} + k\|u\|^2_{X_2}$$

provided $\epsilon > 0$ is chosen so that

$$\epsilon\|b\|_\infty \leq \frac{\alpha_0}{2}$$

and thus

$$k = \|\gamma_-\|_\infty + \|c_+\|_\infty + \frac{\|b\|_\infty}{4\epsilon}.$$

Here w_+ (resp. w_-) is the positive (resp. negative) part of w. Thus A_0 is quasi-dissipative on X_2. The fact that $A_0 - \lambda I$ has dense range for all sufficiently large real λ follows from an application of the Lax-Milgram Lemma as in [3]. Of course in [3] the Lax-Milgram Lemma for symmetric sesquilinear forms was used; now we appeal to the nonsymmetric form (cf. [6], [7], [8], [9]). This completes the proof that A_2, the X_2-closure of A_0, is quasi-m-dissipative. It generates a positive semigroup since $(\lambda I - A_2)^{-1}$ is a positive operator for large $\lambda > 0$. (See [1], [5].)

The extension to X_p, for $1 < p < \infty$, follows the method used in [3], as does the extension to the limiting cases of $p = 1$, $p = \infty$. Since no new ideas are necessary we omit the details. The following summarizes what we have just proved.

Theorem 1. *Let Ω be a bounded C^2 domain in $\mathbf{R}^N$, let $(a_{ij}(x))$ be a real $N \times N$ symmetric uniformly positive definite matrix for $x \in \overline{\Omega}$, which is a C^1 function of $x \in \overline{\Omega}$. Let*

$$b \in C^1(\overline{\Omega}, \mathbf{R}^N), \quad c \in C(\overline{\Omega}, \mathbf{R}^N),$$
$$\beta, \gamma \in C^1(\partial\Omega, \mathbf{R}), \qquad \beta > 0.$$

Define M and D_0 by (2.1)-(2.2) and let A_0 be the restriction of M to D_0. Let

$$X_p = L^p(\Omega, dx) \oplus L^p(\partial\Omega, \frac{dS}{\beta}), \quad 1 \leq p < \infty,$$

and $X_\infty = C(\overline{\Omega})$; and let A_p be the X_p-closure of A_0.

Then A_p is quasi-m-dissipative and generates a strongly continuous positive semigroup on X_p for $1 \leq p \leq \infty$.

This semigroup is analytic, but we defer a precise statement of results of this type to a later paper.

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