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Hypercyclicity of Semigroups is a Very Unstable Property

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Abstract. Hypercyclicity of C_0-semigroups is a very unstable property: We give examples to show that adding arbitrary small constants or a bounded rank one operator to the generator of a hypercyclic semigroup can destroy hypercyclicity. Also the limit of hypercyclic semigroups (even in operator norm topology) need not be hypercyclic, and a hypercyclic semigroup can be the limit of nonhypercyclic ones. Hypercyclicity is not inherited by the Yosida approximations. Finally, the restriction of a hypercyclic nonnegative semigroup in a Banach lattice to the positive cone may be far from hypercyclic.

Key words: hypercyclic semigroups, perturbation
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1. Introduction and main results

Throughout this paper, let X be a real or complex (infinite dimensional) Banach space. A family $S(t)$ of continuous linear operators on X is called a strongly continuous semigroup iff it satisfies

$$\begin{aligned}
&S(0) = \mathrm{id},\\
&S(t+s) = S(t)S(s) \quad \text{for all nonnegative s,t,}\\
&S(t)x \quad \text{depends continuously on } t \text{ for all } x.
\end{aligned}$$

For a concise treatment of strongly continuous semigroups we refer to [7]. In the last years we see an increasing interest in the analysis and applications of irregularly behaved solutions. An

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extreme case of instability and irregular behavior is observed in the case of hypercyclic semigroups. Just to mention a few recent references on this topic and its applications we cite [1], [2], [3], [6], [8], [9].

Definition 1. *The semigroup $S(t)$ is called hypercyclic iff there exists some vector x such that its orbit $\{S(t)x \mid t \geq 0\}$ is dense in X. Any such vector x is said to be a hypercyclic vector for $S(t)$.*

This paper is a collection of examples which show that hypercyclicity is a very fragile property, which is destroyed by very small perturbations.

Our first example shows that hypercyclicity may be destroyed even by bounded perturbations of rank one:

Example 2. *For any hypercyclic semigroup $S(t)$ with infinitesimal generator A there exists a bounded linear operator B such that the semigroup generated by $A + B$ is not hypercyclic.*

The (surprisingly simple) details of this result as well as the details of the following examples will be given in Section 2..

It is clear that for any hypercyclic semigroup $S(t)$ generated by A the perturbation $A+\lambda\,\mathrm{id}$ may generate a non-hypercyclic semigroup. For instance, take λ sufficiently negative, so that $e^{\lambda t}S(t)$ is exponentially stable. However, it is surprising that there may be an interval such that $e^{\lambda t}S(t)$ is hypercyclic for all λ in the whole interval with the exception of isolated points:

Example 3. *There exists a semigroup $S(t)$ which is not hypercyclic, such that for all sufficiently small $\lambda > 0$ the semigroups $e^{\lambda t}S(t)$ and $e^{-\lambda t}S(t)$ are hypercyclic.*

Moreover, limits of hypercyclic semigroups need not be hypercyclic, and limits of non-hypercyclic semigroups may be hypercyclic. Thus, in the sense of Kato-Trotter convergence, neither the hypercyclic semigroups nor the non-hypercyclic semigroups form an open set.

Example 4. *There exists a semigroup $S(t)$ which is not hypercyclic, and a sequence of hypercyclic semigroups $(S_n(t))$, such that $\|S_n(t) - S(t)\|$ converges to 0 uniformly for t in compact intervals.*

In fact, even the Yosida approximation of a hypercyclic semigroup need not inherit hypercyclicity:

Example 5. *There exists a a hypercyclic semigroup $S(t)$ with infinitesimal generator A with the following properties: For all $\lambda > 1$, the Yosida approximations $A_\lambda = \lambda A(\lambda - A)^{-1}$ generate semigroups which are not hypercyclic.*
In particular, a hypercyclic semigroup may be the limit of non-hypercyclic semigroups (in the Kato-Trotter sense).

In order to motivate the next property, let us consider a situation which occurs frequently in applications. The semigroup under investigation describes the time evolution of a physical or biological process which is characterized by a densitiy function. Typically, hypercyclicity can be shown by spectral methods in a complex space. However, physical or biological interest centers only on nonnegative solutions. Thus, consider a (real) Banach lattice X with a nonnegative semigroup $S(t)$, and let $\tilde{X}$ be its complexification. The extension of $S(t)$ to $\tilde{X}$ is called $\tilde{S}(t)$. It is easily seen that hypercyclicity of $\tilde{S}$ on $\tilde{X}$ implies that S is hypercyclic on X. However, this does not imply hypercyclicity of the restriction of S to the positive cone:

Example 6. *There exists a Banach lattice X with positive cone X^+ and a hypercyclic semigroup $S(t)$ which is nonnegative in the sense that $S(t)(X^+) \subset X^+$, however such that the trajectories contained in the positive cone are nondecreasing. In particular, X^+ does not contain a dense trajectory.*

2. The examples in detail

2.1. Example 2

Recall that a semigroup $T(t)$ cannot be hypercyclic if the point spectrum of A^* is nonempty ([5]). So, let A be the infinitesimal generator of a hypercyclic semigroup on a Banach space X. Let $\langle \cdot, \cdot \rangle$ denote the pairing between the dual space X^* and X. We choose some $v \in X$ and $u^* \in D(A^*)$ such that $\langle u^*, v \rangle = -1$. We define a bounded rank one operator B by

$$Bx := Ax + \langle A^* u^*, x \rangle v.$$

Then $A + B$ (defined on $D(A)$) is again the infinitesimal generator of an analytic semigroup on X. However, since for all $x \in D(A)$

$$\langle u^*, (A+B)x \rangle = \langle u^*, Ax \rangle + \langle A^* u^*, x \rangle \langle u^*, v \rangle = 0$$

we infer that u^* is an eigenvector of $(A+B)^*$ with eigenvalue 0. Therefore B cannot be the generator of a hypercyclic semigroup on X.

2.2. Translation semigroups

Translation semigroups are a good source of examples, since for these semigroups there exists a particularly easy characterization for hypercyclicity.

Definition 7. *For a function $f : \mathbb{R} \to \mathbb{R}$ or $f : [0,\infty) \to \mathbb{R}$ and $t \geq 0$ we define the left translation $T(t)f$ by*

$$[T(t)f](\tau) = f(t+\tau) \qquad \text{for } \tau \in \mathbb{R} \text{ or } \tau \in [0,\infty), \text{ respectively.}$$

We introduce suitable Banach spaces such that this left translation forms a semigroup:

Definition 8. *A measurable function $\rho : \mathbb{R} \to (0,\infty)$ is called an admissible weight function on $\mathbb{R}$ iff there exist some $M \geq 1$ and $\omega \geq 0$ such that for all $t \in \mathbb{R}$ and $h \geq 0$ we have*

$$\rho(t+h) \geq Me^{-\omega h}\rho(t) \qquad \text{and} \qquad \rho(t-h) \geq Me^{-\omega h}\rho(t).$$

A measurable function $\rho : [0,\infty) \to (0,\infty)$ is called an admissible weight function on $[0,\infty)$ iff there exist some $M \geq 1$ and $\omega \geq 0$ such that for all $t \geq 0$ and $h \geq 0$ we have

$$\rho(t+h) \geq Me^{-\omega h}\rho(t).$$

Definition 9. *Let ρ be an admissible weight function on $\mathbb{R}$. For $p \in [1,\infty)$, we define*

$$\begin{aligned} L^p_\rho(\mathbb{R}) &= \Big\{f : \mathbb{R} \to \mathbb{R} \;\Big|\; f \text{ measurable and } \int_{\mathbb{R}} \rho(t)|f(t)|^p\,dt < \infty\Big\}, \\ \|f\|^p_{p,\rho} &= \int_{\mathbb{R}} \rho(t)|f(t)|^p\,dt. \end{aligned}$$

An analogous definition is given for $L^p_\rho([0,\infty))$.

Definition 10. *Let ρ be an admissible weight function on $\mathbb{R}$. We define*

$$\begin{aligned} \mathcal{C}_{0,\rho}(\mathbb{R}) &= \Big\{f \in \mathcal{C}(\mathbb{R},\mathbb{R}) \;\Big|\; \lim_{|t|\to\infty,\, t\in\mathbb{R}} \rho(t)|f(t)| = 0\Big\}, \\ \|f\|_{\infty,\rho} &= \sup_{t\in\mathbb{R}} \rho(t)|f(t)|, \end{aligned}$$

An analogous definition is given for $\mathcal{C}_{0,\rho}([0,\infty))$.

On these spaces, hypercyclicity of the left translation admits an easy characterization:

Theorem 11 ([5]).

Let ρ be an admissible weight function on $\mathbb{R}$, and let X be one of the spaces $\mathcal{C}_{0,\rho}(\mathbb{R})$ or $L^p_\rho(\mathbb{R})$. The family $T(t)$ of left translations is a strongly continuous semigroup on X. Moreover, $T(t)$ is hypercyclic iff there exists a sequence $t_n \to \infty$ such that $\rho(t_n) \to 0$ and $\rho(-t_n) \to 0$.

Let ρ be an admissible weight function on $[0,\infty)$, and let X be one of the spaces $\mathcal{C}_{0,\rho}([0,\infty))$ or $L^p_\rho([0,\infty))$. The family $T(t)$ of left translations is a strongly continuous semigroup on X. Moreover, $T(t)$ is hypercyclic iff there exists a sequence $t_n \to \infty$ such that $\rho(t_n) \to 0$.

2.3. Example 3

We shall construct an admissible weight function ρ such that the left translation $T(t)$ is not hypercyclic on $X = L^p_\rho(\mathbb{R})$, but for all sufficiently small $\lambda > 0$, the semigroups $e^{\lambda t}T(t)$ and $e^{-\lambda t}T(t)$ are hypercyclic.

We first choose some $\mu \in \mathbb{R}$ and set $\sigma(t) := \exp(\mu t)\rho(t)$, $Y := L^p_\sigma$ and $S(t)$ the left translation semigroup on Y.

Then the operator

$$F : \begin{cases} X & \to Y \\ [F(f)(s)] & := e^{-\frac{\mu s}{p}} f(s) \end{cases}$$

is an isometric isomorphism and we have

$$F^{-1}S(t)F = e^{-\frac{\mu t}{p}}T(t).$$

As a consequence, if we put $\mu := \lambda p$, we see that $S(t)$ and hence also $e^{-\lambda t}T(t)$ are hypercyclic on Y and X, respectively, iff there exists a sequence $t_n \to \infty$ such that both, $\sigma(t_n) = e^{p\lambda t_n}\rho(t_n)$ and $\sigma(-t_n) = e^{-p\lambda t_n}\rho(-t_n)$ converge to 0 as $n \to \infty$.

After this preparation, we are ready to construct the desired weight function $\rho(t)$:

For $n = 1, 2, 3, \cdots$ we put $s_n = 2 \cdot 9^n$ and $t_n = 6 \cdot 9^n$. This implies in particular that

$$1 = \frac{s_1}{2} < 3\frac{s_1}{2} = \frac{t_1}{2} < 3\frac{t_1}{2} = \frac{s_2}{2} < 3\frac{s_2}{2} = \frac{t_2}{2} < 3\frac{t_2}{2} = \frac{s_3}{2} < \cdots$$

Now we set

$$\rho(t) = \begin{cases} e^{-\frac{t_i}{2}+|t-t_i|} & \text{for } t \in [\frac{t_i}{2}, 3\frac{t_i}{2}], \\ 1 & \text{for } t \in [\frac{s_i}{2}, 3\frac{s_i}{2}], \\ 1 & \text{for } t \in [-1, 1], \\ e^{-\frac{s_i}{2}+|t+s_i|} & \text{for } t \in [-3\frac{s_i}{2}, -\frac{s_i}{2}], \\ 1 & \text{for } t \in [-3\frac{t_i}{2}, -\frac{t_i}{2}]. \end{cases}$$

Note that ρ is continuous and admissible. The key idea is that whenever $\rho(-t) < 1$ then we have $\rho(+t) = 1$, and hence there is no sequence (u_n) such that $\rho(u_n)$ and $\rho(-u_n)$ both converge to 0. As a consequence, $T(t)$ is not hypercyclic on $X = L^p_\rho(\mathbb{R})$.

Now, choose any $\lambda \in (0, \frac{1}{2p})$. Then

$$e^{pt_i\lambda}\rho(t_i) = e^{(p\lambda - \frac{1}{2})t_i},$$
$$e^{-pt_i\lambda}\rho(-t_i) = e^{-pt_i\lambda}.$$

Both sequences converge to 0 as $i \to \infty$, therefore $e^{\lambda t}T(t)$ is hypercyclic on $L^p_\rho(\mathbb{R})$. On the other hand,

$$e^{-ps_i\lambda}\rho(s_i) = e^{-p\lambda s_i},$$
$$e^{ps_i\lambda}\rho(-s_i) = e^{(p\lambda - \frac{1}{2})s_i}.$$

Again, both sequences converge to 0, thus $e^{-\lambda t}T(t)$ is hypercyclic.

2.4. Example 4

Let X and $T(t)$ be chosen as in Example 3, and let λ_n be any sequence in $(-\frac{1}{2p}, \frac{1}{2p})$ converging to 0. Then the semigroups $e^{\lambda_n t}T(t)$ are all hypercyclic, and obviously they converge to $T(t)$ in the operator norm, uniformly for t in compact intervals. However, as shown above, $T(t)$ is not hypercylic.

2.5. Example 5

Before we construct our example, we prepare some technicalities about the Yosida approximation. For the beginning, let $T(t)$ any C_0-semigroup on some Banach space X and let A be its infinitesimal generator. We approximate $T(t)$ by its Yosida approximation $\exp(A_\lambda t)$ where

$$A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda\,\mathrm{id}$$

which exists for sufficiently large λ.

Lemma 12. *With the notation above, for all* $t > 0$ *and all sufficiently large* λ,

$$\exp(A_\lambda t)x = e^{-\lambda t}x + \int_0^\infty g(\lambda, t, s)T(s)x\,ds$$

where

$$g(\lambda, t, s) := \sum_{j=0}^\infty \frac{\lambda^{2j+2}t^{j+1}s^j}{(j+1)!j!}e^{-\lambda(t+s)}.$$

Moreover, the Laplace transform $h(\lambda,t,\sigma)$ of $g(\lambda,t,s)$ with respect to s satisfies

$$h(\lambda,t,\sigma)=e^{-\lambda t}(e^{\frac{\lambda^2 t}{\lambda+\sigma}}-1).$$

Proof.

$$\begin{aligned}
\exp(A_\lambda t)x &= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{\lambda^{2j}t^j}{j!}(\lambda-A)^{-j}x \\
&= e^{-\lambda t}\left[x+\sum_{j=1}^{\infty}\frac{\lambda^{2j}t^j}{j!}(\lambda-A)^{-j}x\right] \\
&= e^{-\lambda t}\left[x+\sum_{j=1}^{\infty}\frac{\lambda^{2j}t^j}{j!}\int_0^\infty\frac{s^{j-1}}{(j-1)!}e^{-\lambda s}T(s)x\,ds\right] \\
&= e^{-\lambda t}x+\int_0^\infty\sum_{j=0}^{\infty}\frac{\lambda^{2j+2}t^{j+1}s^j}{(j+1)!j!}e^{-\lambda(t+s)}T(s)x\,ds \\
&= e^{-\lambda t}x+\int_0^\infty g(\lambda,t,s)T(s)x\,ds.
\end{aligned}$$

For the Laplace transform of g we obtain

$$\begin{aligned}
h(\lambda,t,\sigma) &= \int_0^\infty e^{-\sigma s}g(\lambda,t,s)\,ds \\
&= \int_0^\infty\sum_{j=0}^{\infty}\frac{\lambda^{2j+2}t^{j+1}s^j}{(j+1)!j!}e^{-\lambda(t+s)}\,e^{-\sigma s}\,ds \\
&= \sum_{j=0}^{\infty}\frac{\lambda^{2j+2}t^{j+1}}{(j+1)!(\lambda+\sigma)^{j+1}}e^{-\lambda t} \\
&= e^{-\lambda t}(e^{\frac{\lambda^2 t}{\lambda+\sigma}}-1).
\end{aligned}$$

□

Lemma 13. *Let $g(\lambda,t,s)$ be as in Lemma 12. By δ we denote Dirac's measure concentrated at 0. Then*

$$\tilde{g}(\lambda,t,\cdot):=g(\lambda,t,\cdot)+e^{-\lambda t}\delta$$

is the density of some nonnegative random variable $Z_{\lambda,t}$. The expectation and variance of $Z_{\lambda,t}$ are given by

$$E(Z_{\lambda,t})=t,\qquad \mathrm{Var}(Z_{\lambda,t})=\frac{2t}{\lambda}.$$

Proof. In fact, let

$$\tilde{h}(\lambda,t,\sigma) = e^{-\lambda t} + h(\lambda,t,\sigma) = e^{-\frac{\lambda t\sigma}{\lambda+\sigma}}$$

be the Laplace transform of $\tilde{g}(\lambda,t,\cdot)$. Then

$$\int_0^\infty \tilde{g}(\lambda,t,s)\,ds = \tilde{h}(\lambda,t,0) = 1.$$

Moreover, for any power k, we can obtain the expectation

$$E(Z^k_{\lambda,t}) = (-1)^k \frac{\partial}{\partial\sigma}\tilde{h}(\lambda,t,0)$$

which yields the expectation and the variance for $Z_{\lambda,t}$. □

Lemma 14. *With g as in Lemma 12, there exists a continuous, nonincreasing function $w : [0,\infty) \to [0,\infty)$, such that $\lim_{t\to\infty} w(t) = 0$, and for all $t > 0$, $\lambda > 0$,*

$$\int_0^\infty g(\lambda,t,s)^2\,ds \le \lambda w(\lambda t).$$

Proof. By Plancherel's Theorem we have

$$\begin{aligned}
&\int_0^\infty g(\lambda,t,s)^2\,ds \\
= \;& \frac{1}{\pi}\int_0^\infty |h(\lambda,t,i\sigma)|^2\,d\sigma \\
= \;& \frac{1}{\pi}\int_0^\infty |e^{-\lambda t}(e^{\frac{\lambda^2 t}{\lambda+i\sigma}} - 1)|^2\,d\sigma \\
& (\text{ setting } \tau := \lambda t \text{ and } \nu := \frac{\sigma}{\lambda}) \\
= \;& \frac{\lambda}{\pi}\int_0^\infty |e^{-\tau}(e^{\frac{\tau}{1+i\nu}} - 1)|^2\,d\nu \\
= \;& \frac{\lambda}{\pi}\int_0^\infty |e^{-\tau}(e^{\frac{\tau-i\nu\tau}{1+\nu^2}} - 1)|^2\,d\nu \\
= \;& \frac{\lambda}{\pi}\int_0^\infty e^{-2\tau}\left(e^{\frac{2\tau}{1+\nu^2}} - 2e^{\frac{\tau}{1+\nu^2}}\cos(\frac{\nu\tau}{1+\tau^2}) + 1\right)\,d\nu \\
= \;& \frac{\lambda}{\pi}\int_0^\infty \left[e^{-2\tau}\left(e^{\frac{\tau}{1+\nu^2}} - 1\right)^2 + 2e^{-2\tau}e^{\frac{\tau}{1+\nu^2}}\left(1 - \cos(\frac{\nu\tau}{1+\nu^2})\right)\right]\,d\nu \\
= \;& \lambda\,[w_1(\tau) + w_2(\tau) + w_3(\tau)]
\end{aligned}$$

with

$$\begin{aligned} w_1(\tau) &= \frac{1}{\pi}\int_0^1 e^{-2\tau}\left(e^{\frac{\tau}{1+\nu^2}}-1\right)^2 d\nu, \\ w_2(\tau) &= \frac{1}{\pi}\int_1^\infty e^{-2\tau}\left(e^{\frac{\tau}{1+\nu^2}}-1\right)^2 d\nu, \\ w_3(\tau) &= \frac{1}{\pi}\int_0^\infty 2e^{-2\tau}e^{\frac{\tau}{1+\nu^2}}\left(1-\cos(\frac{\nu\tau}{1+\nu^2})\right) d\nu. \end{aligned}$$

It is easy to see that the integrands in all three integrals above converge to 0 for fixed ν as $\tau \to \infty$. In the first integral, the integrand can be bounded by the constant 4. The integrand in the second integral is dominated by

$$\begin{aligned} & e^{-2\tau}\left(e^{\frac{\tau}{1+\nu^2}}-1\right)^2 \le e^{-2\tau}\left(e^{\frac{\tau}{1+\nu^2}}\frac{\tau}{1+\nu^2}\right)^2 = e^{-\frac{2\tau\nu^2}{1+\nu^2}}\left(\frac{\tau}{1+\nu^2}\right)^2 \\ & \le \tau^2 e^{-\tau}\left(\frac{1}{1+\nu^2}\right)^2 \le 2\left(\frac{1}{1+\nu^2}\right)^2. \end{aligned}$$

The third integrand is bounded by

$$\begin{aligned} & 2e^{-2\tau}e^{\frac{\tau}{1+\nu^2}}\left(1-\cos\left(\frac{\nu\tau}{1+\nu^2}\right)\right) \le 2e^{-\tau}e^{-\frac{\tau\nu^2}{1+\nu^2}}\left(2\sin^2\left(\frac{\nu\tau}{2(1+\nu^2)}\right)\right) \\ & \le e^{-\tau}\left(\frac{\tau\nu}{1+\nu^2}\right)^4 \le \tau^4 e^{-\tau}\left(\frac{\nu^2}{1+\nu^2}\right)^2\left(\frac{1}{1+\nu^2}\right)^2 \le M\left(\frac{1}{1+\nu^2}\right)^2 \end{aligned}$$

with a suitable constant $M > 0$. Therefore by the dominated convergence theorem, $w_1(\tau)$, $w_2(\tau)$, and $w_3(\tau)$ converge to 0 as $\tau \to \infty$. Now put

$$w(\tau) = \sup_{\sigma>\tau}[w_1(\sigma)+w_2(\sigma)+w_3(\sigma)].$$

□

Lemma 15. *With g as in Lemma 12, the following inequalities hold for all $\lambda, t, u, \alpha > 0$, $\beta \in (0,t)$:*

$$\begin{aligned} & e^{-\lambda t}+\int_0^{t-\beta} g(\lambda,t,s)ds+\int_{t+\beta}^\infty g(\lambda,t,s)ds \le \frac{t}{\beta^2\lambda}, \\ & \int_s^{s+\alpha} g(\lambda,t,s)ds \le \sqrt{\alpha}\sqrt{\lambda w(\lambda t)}. \end{aligned}$$

Proof. Let $Z_{\lambda,t}$ be as in Lemma 13. Chebychev's inequality implies

$$\frac{2t}{\lambda\beta^2} \geq \mathbb{P}(|Z_{\lambda,t} - t| \geq \beta) = \int_{[0,t-\beta]} \tilde{g}(\lambda,t,s)\,ds + \int_{[t-\beta,\infty)} \tilde{g}(\lambda,t,s)\,ds$$
$$= e^{\lambda t} + \int_0^{t-\beta} g(\lambda,t,s)\,ds + \int_{t-\beta}^{\infty} g(\lambda,t,s)\,ds.$$

The second inequality of the lemma is a straightforward consequence of Lemma 14 and Hölder's inequality. □

Lemma 16. *Consider sequences (a_i) and (b_i) converging to infinity such that*

$$\begin{aligned} &0 = b_0 < a_1 < b_1 < a_2 < \ldots \\ &a_i \geq b_{i-1} + 2^{2i} \\ &w(a_i) \leq 2^{-5i} \\ &b_i = a_i + 2^i \end{aligned}$$

and put

$$q(t) := \begin{cases} 2^i & \text{for } t \in [a_i, b_i], \\ 0 & \text{elsewhere}. \end{cases}$$

Then for each fixed $\lambda \geq 1$ we have

$$\lim_{t\to\infty} \int_0^\infty g(\lambda,t,s)q(s+h)ds = 0$$

uniformly for h in compact intervals.

Proof. To show this assertion, we fix some compact interval $[0,T]$ and $\lambda \geq 1$. Let $t > 1$ be sufficiently large and choose i such that

$$a_i \leq t \leq t + T \leq b_{i+1}.$$

Using Lemma 15 several times, we obtain for $0 \leq h \leq T$:

$$\begin{aligned} &\int_0^\infty g(\lambda,t,s)q(s+h)ds \\ \leq\ &\int_0^{b_{i-1}-h} g(\lambda,t,s)2^{i-1}ds + \int_{a_i-h}^{b_i-h} g(\lambda,t,s)2^i ds \\ &+ \int_{a_{i+1}-h}^{b_{i+1}-h} g(\lambda,t,s)2^{i+1}ds + \sum_{k=2}^{\infty} \int_{a_{i+k}-h}^{b_{i+k}-h} 2^{i+k} g(\lambda,t,s)ds \\ \leq\ &2^{i-1}\frac{2t}{(t-b_{i-1}+h)^2\lambda} + 2^i\sqrt{2^i}\sqrt{\lambda w(\lambda t)} \\ &+2^{i+1}\sqrt{2^{i+1}}\sqrt{\lambda w(\lambda t)} + \sum_{k=2}^{\infty} 2^{i+k}\frac{2t}{(a_{i+k}-h-t)^2\lambda}. \end{aligned}$$

Note that $2^{2i}b_{i-1} \leq a_i \leq t$, and therefore $(t - b_{i-1} + h) \geq t(1 - 2^{-2i}) \geq \frac{t}{2}$. Thus

$$\frac{2t}{(t - b_{i-1} + h)^2\lambda} \leq \frac{8t}{\lambda t^2} \leq \frac{8}{\lambda}2^{-2i}.$$

Similarly, we have for $k \geq 2$

$$a_{i+k} - t - h \geq a_{i+k} - b_{i+k-1} \geq 2^{2(i+k)}b_{i+k-1} \geq 2^{2(i+k)}t.$$

Hence

$$\frac{2t}{\lambda(a_{i+k} - h - t)^2} \leq \frac{2}{\lambda}2^{-4(i+k)}.$$

Thus

$$\begin{aligned}
& \int_0^\infty q(s+h)g(\lambda, t, s)ds \\
\leq \quad & 2^{i-1}\frac{8}{\lambda}2^{-2i} + 2^{\frac{3i}{2}}\sqrt{\lambda}\,2^{-\frac{5i}{2}} \\
& +2^{\frac{3(i-1)}{2}}\sqrt{\lambda}\,2^{-\frac{5(i+1)}{2}} + \sum_{k=2}^{\infty} 2^{i+k}\frac{2}{\lambda}2^{-4(i+k)} \\
= \quad & \frac{1}{\lambda}2^{-i+2} + \sqrt{\lambda}(2^{-i} + 2^{-i-1}) + \frac{2}{\lambda}8^{-i-1}\frac{1}{1-\frac{1}{8}}.
\end{aligned}$$

As $t \to \infty$, we have also $i \to \infty$, hence and the right hand side of the last estimate converges to 0 as claimed. □

Construction of Example 5:
Let $0 = a_1 \leq b_1 < a_2 < b_2 \cdots$ be as in Lemma 16. We construct an admissible weight function $\rho : [0, \infty) \to [0, \infty)$ with the following properties

$$\begin{aligned}
& \rho(t) = 1 \quad \text{for } t \in [b_i, a_{i+1}], \\
& 2^{-i} \leq \rho(t) \leq 1 \quad \text{on } [a_i, b_i], \\
& \rho(t) \leq e^h\rho(t+h) \quad \text{for all } t \geq 0,\ h \geq 0, \\
& \liminf_{t\to\infty} \rho(t) = 0.
\end{aligned}$$

This can be easily achieved since the gaps $b_i - a_i$ tend to infinity as $i \to \infty$. Notice that with the function q in Lemma 16 we have for all $t \geq 0$, $h \geq 0$

$$\frac{\rho(h)}{\rho(t)} \leq \frac{1}{\rho(t)} \leq \max(1, q(t)) \leq 1 + q(t).$$

Now fix any (nonzero) $\phi \in \mathcal{C}_{0,\rho}([0,\infty))$, $\lambda \geq 1$, and $T \geq 0$. For all $h \in [0,T]$ we have

$$\begin{aligned}
&\rho(h)\big|[\exp(A_\lambda t)\phi](h)\big| \\
= \;& \rho(h)\left|e^{-\lambda t}\phi(h) + \int_0^\infty g(\lambda,t,s)\phi(s+h)\,ds\right| \\
\leq \;& e^{-\lambda t}\rho(h)\big|\phi(h)\big| + \int_0^\infty g(\lambda,t,s)\frac{\rho(h)}{\rho(s+h)}\rho(s+h)\big|\phi(s+h)\big|\,ds \\
\leq \;& e^{-\lambda t}\|\phi\| + \int_0^\infty g(\lambda,t,s)(1+q(s+h))\|\phi\|\,ds \\
= \;& \left[e^{-\lambda t} + \int_0^\infty g(\lambda,t,s)\,ds + \int_0^\infty g(\lambda,t,s)q(s+h)\,ds\right]\|\phi\| \\
= \;& \left[1 + \int_0^\infty g(\lambda,t,s)q(s+h)\,ds\right]\|\phi\|.
\end{aligned}$$

The last equality holds since $e^{-\lambda t}\delta + g(\lambda,t,\cdot)$ is a probability density function (Lemma 13). Now we infer from Lemma 16 that

$$\limsup_{t\to\infty}\sup_{h\in[0,T]}\rho(h)\left|[\exp(A_\lambda t)\phi](h)\right| \leq \|\phi\|$$

Therefore, for any vector $\psi \in \mathcal{C}_{0,\rho}([0,\infty))$ with

$$\sup_{h\in[0,t]}\rho(h)|\psi(h)| \;\geq\; 2\|\phi\|$$

we have

$$\liminf_{t\to\infty}\|[\exp(A_\lambda t)\phi] - \psi\| \geq \|\phi\|.$$

Consequently, the trajectory of ϕ is not dense, and the semigroups $\exp(A_\lambda t)$ are not hypercyclic.

2.6. Example 6

Our last example will use another standard class of hypercyclic semigroups: Let ρ_k be a sequence in $\mathbb{R}$ with the following properties

$$\rho(k) > 0, \qquad \sup_{k\in\mathbb{N}}\frac{\rho_k}{\rho_{k+1}} < \infty.$$

Let $X := l^p_\rho$ be the set of all sequences (x_n) such that

$$\|(x_k)\|^p := \sum_{k=0}^\infty \rho_k|x_k|^p$$

is finite. With the positive cone

$$X^+ = \big\{(x_n) \in l^p_\rho \mid x_n \geq 0 \text{ for all } n \in \mathbb{N}\big\},$$

the space l^p_ρ is a Banach lattice.

We define the left shift A in l^p_ρ by

$$Ax = y \quad \text{with} \quad y_k = x_{k+1}.$$

Then A is a bounded linear operator, in particular it generates an analytic semigroup $T(t)$. It is known [5] that $T(t)$ is hypercyclic. Since $A(X^+) \subset X^+$, we infer that $T(t)$ is nonnegative in the sense that $T(t)(X^+) \subset X^+$. Moreover, for $x \in X^+$ we have

$$\frac{d}{dt}T(t)x = AT(t)x \in X^+$$

so that $T(t)x \geq T(s)x$ if $t \geq s$. Since for $0 \neq x \in X^+$ the set $\{y \in l^p_\rho \mid y \geq x\}$ is a proper closed subset of X^+, the trajectory $\{T(t)x \mid t \geq 0\}$ is not dense in X^+.

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