Locking-Free Finite Elements for Unilateral Crack Problems in Elasticity

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Abstract. We consider mixed and hybrid variational formulations to the linearized elasticity system in domains with cracks. Inequality type conditions are prescribed at the crack faces which results in unilateral contact problems. The variational formulations are extended to the whole domain including the cracks which yields, for each problem, a smooth domain formulation. Mixed finite element methods such as PEERS or BDM methods are designed to avoid locking for nearly incompressible materials in plane elasticity. We study and implement discretizations based on such mixed finite element methods for the smooth domain formulations to the unilateral crack problems. We obtain convergence rates and optimal error estimates and we present some numerical experiments in agreement with the theoretical results.

Key words: crack problems, variational inequalities, smooth domain method, mixed finite elements, a priori estimates

AMS subject classification: 65N30,74M15,35J85

1. Introduction

We consider the equilibrium problem for a linear elastic body occupying the domain \( \Omega_c = \Omega \setminus \Gamma_c \), \( \Omega \subset \mathbb{R}^2 \), with the interior crack \( \Gamma_c \) and \( \partial \Omega = \Gamma_D \cup \Gamma_N \), \( \Gamma_D \neq \emptyset \). Given a volume force \( f : \Omega_c \to \mathbb{R}^2 \) and a traction \( g : \Gamma_N \to \mathbb{R}^2 \), find \( u = (u_1, u_2) \), and \( \sigma = (\sigma_{ij}), i, j = 1, 2 \), such that

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\[ -\text{div} \sigma = f \quad \text{in} \ \Omega, \]  
\[ C^{-1} \sigma - \varepsilon(\mathbf{u}) = 0 \quad \text{in} \ \Omega_c, \]  
\[ \mathbf{u} = 0 \quad \text{on} \ \Gamma_D, \]  
\[ \sigma \mathbf{n} = 0 \quad \text{on} \ \Gamma_N, \]  
\[ [\mathbf{u}] \mathbf{n} \geq 0, \quad [\sigma \mathbf{n}] = 0, \quad \sigma_n [\mathbf{u}] = 0 \quad \text{on} \ \Gamma_c, \]  
\[ \sigma_n \leq 0, \quad \sigma_t = 0 \quad \text{on} \ \Gamma_c^\pm. \]  

\( \mathbf{u} \) is the displacement field and \( \sigma \) is the tensor of constraints. \([\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-\) denotes the jump of the displacement field \( \mathbf{u} \) across \( \Gamma_c \), and the signs \( \pm \) indicate the positive and negative directions of the normal \( \mathbf{n} \). We have used the following standard notations

\[ \sigma_n = \sigma_{ij} n_i n_j, \quad \sigma_t = \sigma - \sigma_n \mathbf{n} = \{\sigma_{ij}\}_{i=1}^2, \quad \sigma \mathbf{n} = \{\sigma_{ij} n_j\}_{i=1}^2, \]  
\[ \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i, j = 1, 2, \quad \varepsilon(\mathbf{u}) = (\varepsilon_{ij})_{i,j=1}^2, \]  
\[ \sigma = \lambda \text{tr}(\varepsilon(\mathbf{u})) I_d + 2 \mu \varepsilon(\mathbf{u}). \]

\( \lambda \) and \( \mu \) are the Lamé constants. The fourth order tensor \( C \) is symmetric and satisfies the ellipticity condition

\[ c_{ijkl} \xi_{ji} \xi_{kt} \geq c_0 |\xi|^2, \quad \forall \xi_{ij} = \xi_{ij}, \quad c_0 > 0. \]  

We use the summation convention over repeated indices.

The boundary conditions of unilateral type on \( \Gamma_c^\pm \) (1.5)-(1.6) describe the mutual nonpenetration between the crack faces. The weak formulation of problem (1.1)-(1.6) yields an elliptic variational inequality, we refer the reader for the mathematical analysis of such problems to [15, 25] and references therein. Such boundary conditions when prescribed at the external boundary of Lipschitz

![Figure 1: A crack in the reference configuration](image-url)
domains are called the frictionless Signorini conditions. They are encountered in many engineering
and application domains. Both their mathematical analysis [15, 25] and their numerical analysis
[19, 18, 17, 24, 31] and more recently [5, 7, 3, 6, 21, 14, 26, 27, 22] have been widely studied but
still remain a source of many challenging problems. The unilateral crack problems are a particular
class where the Signorini conditions are prescribed at the crack faces [23, 4, 29, 30].

The mixed variational formulation to problem (1.1)-(1.6) in displacement-stress fields could
be extended to the whole domain including the crack [23, 4, 30]. Therefore, with this extended
formulation, we work in the Lipschitzian domain and the difficulties due to the presence of the
crack as geometric object are reduced. For instance this means more flexibility for the meshes and
the use of standard tools of the numerical analysis in Lipschitzian domains.

In another hand, for nearly incompressible materials, i.e. when $\lambda \rightarrow +\infty$, a numerical locking
appears empeaching the convergence of the standard finite element approximation of the displace-
ment. It is well known that resorting to the mixed finite element eliminates the locking phe-
omenon. Thus, the objective of the finite element discretizations that we propose in this paper is
twofold: providing an efficient tool to solve unilateral crack problems in the linear elasticity with
the smooth domain formulation and deriving locking-free methods to solve such problems.

The outline of the paper is as follows: In Section 2 we establish the variational formulation and
show that it yields a well posed problem. Section 3 is devoted to the description of the discretization
methods and the analysis of the discrete problems. The error estimates are established in Section
4. In Section 5 we give some implementation details and we present some numerical experiments
to show the efficiency of the approach.

## 2. Variational formulation

Stable mixed finite element for the elasticity problem are usually obtained by relaxing the sym-
metry of the stress tensor [28, 10] which leads to the use of the so-called Hellinger-Reissner
modified functional of the elasticity system. This yields to modify the elasticity system (1.1)-
(1.6) as follows: We seek the displacement $u : \Omega \rightarrow \mathbb{R}^2$, the stress field $\sigma : \Omega \rightarrow \mathbb{R}^2$ and
$\gamma : \Omega \rightarrow M_{\text{skew}}^{2 \times 2} := \{\eta \in \mathbb{R}^{2 \times 2} : \eta + \eta^t = 0\}$ such that
\begin{align}
C\sigma - \nabla u + \gamma &= 0 \text{ in } \Omega_c, \tag{2.1} \\
\sigma - \sigma^T &= 0 \text{ in } \Omega_c, \tag{2.2}
\end{align}
in place of (1.2) hold. We introduce the functional space
\[
X(\Omega_c) = \{\sigma \in L^2(\Omega_c; \mathbb{R}^{2 \times 2}), \text{div } \sigma \in L^2(\Omega_c; \mathbb{R}^2), \sigma \nu = 0 \text{ on } \Gamma_N\}
\]
equipped with the norm
\[
\| \sigma \|_{X(\Omega_c)} = \left( \| \sigma \|^2_{L^2(\Omega_c; \mathbb{R}^{2 \times 2})} + \| \text{div } \sigma \|^2_{L^2(\Omega_c; \mathbb{R}^2)} \right)^{1/2},
\]
and the closed convex set
\[ K(\Omega_c) = \{ \sigma \in X(\Omega_c), [\sigma \nu] = 0 \text{ on } \Gamma_c, \sigma_t = 0 \text{ on } \Gamma_c, \sigma_\nu \leq 0 \text{ on } \Gamma_c^+ \}. \]

Define \( V(\Omega_c) = L^2(\Omega_c, \mathbb{R}^2) \), and introduce \( W(\Omega_c) = L^2(\Omega_c, M_{skew}^{2 \times 2}) \).

The mixed variational formulation of the modified system in \( \Omega_c \) reads:

Find \( (\sigma, u, \gamma) \in K(\Omega_c) \times V(\Omega_c) \times W(\Omega_c) \) such that

\[
\begin{cases}
  a(\sigma, \tau - \sigma) + b(\tau - \sigma; u, \gamma) \geq 0, & \tau \in K(\Omega_c), \\
  b(\sigma; v, \eta) = -L(v), & (v, \eta) \in V(\Omega_c) \times W(\Omega_c),
\end{cases}
\]

with

\[
\begin{align*}
  a(\sigma, \tau) &= (C^{-1} \sigma, \tau) = \int_{\Omega_c} C^{-1} \sigma : \tau \, dx, \\
  b(\sigma; u, \gamma) &= (\text{div} \sigma, u) + (\sigma, \gamma) = \int_{\Omega_c} \text{div} \sigma \cdot u \, dx + \int_{\Omega_c} \sigma : \gamma \, dx, \\
  L(v) &= (f, v) = \int_{\Omega_c} f \cdot v \, dx.
\end{align*}
\]

The antisymmetric part of tensor \( \tau \) will be denoted by \( as(\tau) = \tau_{2,1} - \tau_{1,2} \).

We define the functional space

\[ Z(\Omega_c) = \{ \sigma \in X(\Omega_c); b(\tau; v, \eta) = 0, \forall (v, \eta) \in V(\Omega_c) \times W(\Omega_c) \}, \]

and we introduce the subset of \( K(\Omega_c) \) defined by

\[ \chi = \{ \tau \in K(\Omega_c); -\tau \in K(\Omega_c) \}. \]

The following ellipticity condition holds:

\[ \forall \tau \in Z(\Omega_c), a(\tau, \tau) \geq \alpha \| \tau \|^2_{X(\Omega_c)}, \]

as well as the Brezzi-Babuska inf-sup condition [2], there exists a constant \( \beta \geq 0 \) such that

\[ \forall (v, \eta) \in V(\Omega_c) \times W(\Omega_c), \sup_{\tau \in \chi} \frac{b(\tau; v, \eta)}{\| \tau \|_{X(\Omega_c)}} \geq \beta \| (v, \eta) \|_{V(\Omega_c) \times W(\Omega_c)}, \]

with

\[ \| (v, \eta) \|_{V(\Omega_c) \times W(\Omega_c)} = \| v \|_{L^2(\Omega_c; \mathbb{R}^2)} + \| \eta \|_{L^2(\Omega_c; \mathbb{R}^{2 \times 2})}. \]

This properties yield the following result (see [29] for details)

**Proposition 1.** There exists a unique solution \( (\sigma, u, \gamma) \in K(\Omega_c) \times V(\Omega_c) \times W(\Omega_c) \) of problem (2.3). Moreover, one has

\[ u \in H^1_{0, \Gamma_D}(\Omega_c, \mathbb{R}^2), \ as(\gamma) = \text{curl} \ u, \]

and

\[ \| \sigma \|_{X(\Omega_c)} + \| u \|_{H^1(\Omega_c, \mathbb{R}^2)} \leq C \| f \|_{L^2(\Omega_c, \mathbb{R}^2)}. \]
2.1. The extended (smooth domain) variational formulation

The constraint corresponding to the jump condition of the stress tensor's normal component across \( \Gamma_c \) implies that \( \sigma \in X(\Omega) \) (see [16]). Then, following [25, 4, 30]) we can suppress this constraint in the definition of \( K(\Omega_c) \) by replacing \( X(\Omega_c) \) with \( X(\Omega) \), so that we can extend the variational formulation to the whole smooth domain \( \Omega \). Set

\[
K = K(\Omega) = \{ \sigma \in X(\Omega), \sigma_\nu \leq 0, \sigma_t = 0 \text{ on } \Gamma_c \},
\]

and denote by \( V \) (resp. \( W \)) the space \( V(\Omega) \) (resp. \( W(\Omega) \)). The mixed variational formulation extended to \( \Omega \) consists in finding \((\sigma, u, \gamma) \in K \times V \times W\) such that

\[
\begin{cases}
a(\sigma, \tau - \sigma) + b(\tau - \sigma; u, \gamma) \geq 0, \quad \tau \in K, \\
b(\sigma; v, \eta) = -L(v), \quad (v, \eta) \in V \times W,
\end{cases}
\]

where integrals are extended to \( \Omega \).

The ellipticity of the bilinear form \( a(\cdot, \cdot) \) and the inf-sup condition verified by \( b(\cdot, \cdot) \) still hold [29]. The existence and uniqueness of the solution of Problem (2.4) follows as in proposition 1. Even if the extended formulation results in a similar problem as in (2.3), we only have

\[
\text{as } (\gamma) = \text{curl } u \text{ in } \mathcal{D}'(\Omega_c),
\]

\[
C\sigma - \nabla u + \gamma = 0 \quad \text{in } \mathcal{D}'(\Omega_c).
\]

2.2. Hybrid formulation

For practical purposes, either one can solve problem (2.4) which leads to solve high-dimension optimization problems, or the hybrid formulation obtained by expressing the constraints on \( \Gamma_c \) with Lagrange multipliers that we give now.

We introduce

\[
M = H^{\frac{1}{2}}_{00}(\Gamma_c, \mathbb{R}^2),
\]

and

\[
M_+ = \left\{ \mu \in H^{\frac{1}{2}}_{00}(\Gamma_c), \mu \geq 0 \right\}.
\]

We also define the bilinear forms \( d_t(\cdot, \cdot) \), resp. \( d_n(\cdot, \cdot) \), on \( X \times M \), resp. on \( X \times M_+ \),

\[
d_t(\tau, \lambda_t) = <(\tau_t, \mu_t) >_{\frac{1}{2},00,\Gamma_c},
\]

and

\[
d_n(\tau, \lambda_n) = <(\tau_n, \mu_n) >_{\frac{1}{2},00,\Gamma_c},
\]

where \(<(\cdot, \cdot) >_{\frac{1}{2},00,\Gamma_c}\) denotes the duality product between \( H^{\frac{1}{2}}_{00}(\Gamma_c) \) and its dual space.
The duality product defined by
\[
\langle \tau, \varphi \rangle = \langle \tau_{11}, \varphi_1 \rangle + \langle \tau_{12}, \varphi_2 \rangle, \quad \forall \varphi = (\varphi_1, \varphi_2) \in M, \quad \varphi_i \nu_i = 0.
\]

The hybrid formulation reads:
Find \((\sigma, u, \gamma, \lambda_t, \lambda_n) \in X \times V \times W \times M \times M_+\), with \(\lambda_t, \nu_i = 0\), such that
\[
\begin{align*}
& a(\sigma, \tau) + b(\tau; u, \gamma) + d_t(\tau, \lambda_t) + d_n(\tau, \lambda_n) = 0, \quad \forall \tau \in X, \\
& (b - \sigma; v, \eta) = -L(v), \quad \forall (v, \eta) \in V \times W, \\
& d_t(\sigma, \mu_t) = 0, \quad \forall \mu_t = (\mu_1, \mu_2) \in M, \quad \mu_i \cdot \nu_i = 0, \\
& d_n(\sigma, \mu_n - \lambda_n) \leq 0, \quad \forall \mu_n \in M_+.
\end{align*}
\]
(2.5)

We get ([29])

**Proposition 2.** There exists a unique solution \((\sigma, u, \gamma, \lambda_t, \lambda_n) \in X \times V \times W \times M \times M_+\) of problem (2.5). Moreover, one has
\[
as(\gamma) = \text{curl} \, u, \quad \lambda_n = [u] \nu, \quad \text{and} \quad \lambda_t = u \quad \text{on} \quad \Gamma_c.
\]

3. Discrete problems

Now we deal with the discretization of problem (2.4) and (2.5). The discretization is based on PEERS finite element, introduced by Arnold, Brezzi and Douglas in [2], and the low order BDM finite element, introduced by Brezzi, Douglas and Marini [10], and analysed by Stenberg [28].

We denote by \(T_h\) a triangulation of \(\Omega\) made of elements which are triangles with a maximum size \(h\) satisfying the usual admissibility assumption, i.e. the intersection of two different elements is either empty, a vertex, or a whole edge. In addition, \(T_h\) is assumed regular, i.e. the ratio of the diameter of any element \(T \in T_h\) to the diameter of its largest inscribed ball is bounded by a constant \(\sigma\) independent of \(T\) and \(h\). We will assume that the endpoints of \(\Gamma_c\) are vertices of the triangulation. The set of nodes on \(\Gamma_c\) are denoted by \(c_1 = x_0, x_1, \ldots, x_{I-1}, x_I = c_2\) and we set \(t_i = ]x_{i-1}, x_i[\).

**Remark 3.** We can assume that on the crack \(\Gamma_c\) we have a 1-D triangulation completely independent of the triangulation \(T_h\) which yields a discretization with high flexibility as regards the triangulation step. However, in this case, we may loose the uniform inf-sup condition on the bilinear form \(d(., .)\). For the PEERS-based discretization this does not affect the error estimates but for the BDM elements the non uniform inf-sup condition reduces the convergence rate. Nevertheless, using the stabilization technique as in [21] allows to recover the full accuracy also in the BDM-Based discretization. In this work, we will not follow this direction and we will assume that the triangulation on \(\Gamma_c\) is the trace of the triangulation \(T_h\).

For all \(T \in T_h\), we denote by \(P_k(T)\) the space of polynomials with total degree less than \(k\).
3.1. The PEERS’s finite element

Let $RT_0$ be the Raviart-Thomas space defined as

$$RT_0 = P_0^2 + (x, y)P_0.$$ 

Recall that on each $T \in T_h$, the associated non-normalized bubble function is

$$b_T(x) = \prod_{i=1}^{3} \lambda_i(x), \forall x \in T,$$

where $\lambda_i (i = 1, \ldots, 3)$ are the barycentric coordinates on $T$.

We consider $b_T$ extended by zero out of $T$, and define the space generated by $\text{rot } b_T$:

$$A = \{ \text{rot } b_T; T \in T_h \},$$

and we set

$$B_0(T) = \{ \tau / (\tau_{i1}, \tau_{i2}) \in A; i = 1, 2 \}.$$

For the approximation of the stress tensor, we take $X_h = \{ \sigma_h \in X; \sigma_{h|T} \in RT_0(T)^2 \oplus B_0(T), \forall T \in T_h \}$, and for the displacement, the approximation space is

$$V_h = \{ \mathbf{v}_h \in L^2(\Omega)^2; \mathbf{v}_{h|T} \in (P_0(T))^2, \forall T \in T_h \}.$$ 

The definition of the discrete convex cone $K_h$ is identical to that of $K$ with obvious adaptation to the discrete case. We define

$$W_h = \{ \gamma_h \in W \cap C^0(\overline{\Omega}, M_{\text{skew}}^2); \gamma_{h|T} \in P_1(T; M_{\text{skew}}^2), \forall T \in T_h \}.$$ 

Since the approximation of $\sigma_h$ is based on the low level Raviart-Thomas finite element, we choose for the Lagrange multiplier space an approximation with the piecewise-constant functions. We set

$$W_{h0}^0(\Gamma_c) = \{ \mu_h, \mu_{h|t_i} \in P_0(t_i), 0 \leq i \leq I - 1 \},$$ 

and we define

$$M_h = W_{h0}^0(\Gamma_c, \mathbb{R}^2)$$

and

$$M_{h+} = \{ \mu_h \in W_{h0}^0(\Gamma_c), \mu_h \geq 0 \}.$$ 

We define by $M_h$ the convex cone $M_h \times M_{h+}$, and $\lambda_h = (\lambda_{th}, \lambda_{nh})$. Note that $M_{h+} \subset M_+$. We denote by $E$ the set of edges $e$ of triangles $T \in T_h$ which are internal to $\Omega_c$, and by $J(u)$ the jump $u^+ - u^-$ across $e$. For the discrete problem analysis, we use the following mesh-dependent norms:

$$\| \sigma \|_{0,h}^2 = \frac{1}{E^2} \left( \| \sigma \|_{0}^2 + \sum_{e \in E_h} h_e \| \sigma \nu \|_{0,e}^2 \right), \sigma \in X_h,$$
\[ \| \mathbf{u} \|_{1,h}^2 = \sum_{T \in T_h} \| \mathbf{u} \|_{1,T}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \| J(\mathbf{u}) \|_{0,e}^2, \quad \mathbf{u} \in V_h, \]

where for brevity we have noted \( \| \sigma \|_{0,h} = \| \sigma \|_{L^2(\Omega; \mathbb{R}^{2 \times 2})}, \) \( \| \sigma \nu \|_{0,e} = \| \sigma \nu \|_{L^2(e; \mathbb{R}^2)} \) and \( \| \mathbf{u} \|_{1,T} = \| \mathbf{u} \|_{H^1(T; \mathbb{R}^2)}. \)

**Remark 4.**

1. Norm \( \| \cdot \|_{1,h} \) is used for the BDM finite element’s analysis. For PEERS’s finite element, one have to modify it according to [57, section 4]. Since we only recall brad lines of Stenberg’s paper, we leave out details.

2. for the BDM finite element, we conform to changing introduced by [18] in the original proof of the inf-sup condition.

In other words, we will write the inf-sup condition with the norm \( \| \sigma \|_{0,h} \) and \( \| (\mathbf{u}, \gamma) \|_{V_h \times W_h}^2 = \| \mathbf{u} \|_{1,h}^2 + \| \gamma \|_{0}^2. \)

### 3.1.1. The discrete problem

For the mixed formulation (2.3), the discrete problem reads:

\[
\begin{cases}
  a(\sigma_h, \tau_h - \sigma_h) + b(\tau_h - \sigma_h; \mathbf{u}_h, \gamma_h) \geq 0, \quad \tau_h \in K_h, \\
  b(\sigma_h; \mathbf{v}_h, \eta_h) = -L(\mathbf{v}_h), \quad (\mathbf{v}_h, \eta_h) \in V_h \times W_h.
\end{cases}
\]

(3.1)

Note that bilinear form \( a(\cdot, \cdot) \) is coercive only on the subspace \( Z_h \). So, with

\[ Z_h = \{ \tau \in X_h; b(\tau_h; \mathbf{v}_h, \eta_h) = 0, \quad \forall (\mathbf{v}_h, \eta_h) \in V_h \times W_h \}, \]

one has

\[ a(\sigma, \sigma) \geq C \| \sigma \|_{0,h}^2, \quad \forall \sigma \in Z_h. \]

Moreover, one has ([57])

\[ \exists \beta > 0, \quad \sup_{\tau \in X_h} \frac{b(\sigma; \mathbf{v}, \eta)}{\| \tau \|_{X_h}} \geq \beta \| (\mathbf{v}, \eta) \|_{V_h \times W_h}, \quad \forall (\mathbf{v}, \eta) \in V_h \times W_h. \]

These two properties and the standard theory of the mixed variational formulations yield

**Proposition 5.** Problem (3.1) admits a unique solution \( (\sigma_h, \mathbf{u}_h, \gamma_h) \in K_h \times V_h \times W_h. \)

Again, the hybrid formulation can be written as: Find \( (\sigma_h, \mathbf{u}_h, \gamma_h, \lambda_{th}, \lambda_{nh}) \in X_h \times V_h \times W_h \times M_h \times M_{h+}, \) with \( \lambda_{th}, \lambda_{nh} = 0, \) such that

\[
\begin{cases}
  a(\sigma_h, \tau_h) + b(\tau_h; \mathbf{u}_h, \gamma_h) + d_t(\tau_h, \lambda_{th}) + d_n(\tau_h, \lambda_{nh}) = 0, \quad \forall \tau_h \in X_h, \\
  b(\sigma_h; \mathbf{v}_h, \eta_h) = -L(\mathbf{v}_h), \quad (\mathbf{v}_h, \eta_h) \in V_h \times W_h, \\
  d_t(\sigma_h, \mu_{th}) = 0, \quad \forall \mu_{th} = (\mu_{th,1}, \mu_{th,2}) \in M_h, \quad \mu_{th,1} \nu_1 = 0, \\
  d_n(\sigma_h, \mu_{nh} - \lambda_{nh}) \leq 0, \quad \forall \mu_{nh} \in M_{h+}.
\end{cases}
\]

(3.2)
or
\[
\begin{cases}
  a(\sigma_h, \tau_h) + c(\tau_h; (u_h, \gamma_h, \lambda_{th}, \lambda_{nh})) = 0, \quad \forall \tau_h \in X_h, \\
  c(\tau_h; (v_h, \eta_h, \mu_{th}, \mu_{nh})) = -L(v_h), \\
  \forall (v_h, \eta_h, \mu_{th}, \mu_{nh}) \in V_h \times W_h \times M_h \times M_{h^+}, \quad \mu_{th,i} \nu_i = 0.
\end{cases}
\tag{3.3}
\]

For \(\mu_h \in W^{0}_h(\Gamma_c)\), we introduce the mesh depending norm
\[
\| \mu_h \|_{L^2(\Omega)}^2 = \sum_{i=1}^{I-1} h_{t_i} \| \mu_h \|_{L^2(t_i)}^2.
\]

The following result is proved in the (scalar) case of an elastic membrane in [6]:
\[
\exists \delta > 0, \sup_{\tau \in X_h} \frac{d_n(\tau, \mu_n)}{\| \tau \|_{X_h}} \geq C \| \mu_n \|_{L^2} \geq \delta h^{-\frac{1}{2}} \| \mu_n \|_{L^2(\Gamma_c)}, \quad \forall \mu_n \in W^{0}_h(\Gamma_c).
\tag{3.4}
\]

No difficulty arises when extending to the elasticity system. We can prove a similar result for the bilinear form \(d(\cdot, \cdot)\) defined on \((X_h \times V_h \times W_h) \times W^{0}_h(\Gamma_c)^3\) by
\[
d((\tau, \nu, \eta), \mu) = d_c((\tau, \nu, \eta), \mu_t) + d_n((\tau, \nu, \eta), \mu_n), \quad \mu = (\mu_t, \mu_n).
\]

Existence and unicity of solution \((\sigma_h, u_h, \gamma_h)\) of (3.1) derive from Proposition (2.1). So, to prove existence and unicity for the saddle point of formulation (3.2), it suffices to check that
\[
\{ \mu_h = (\mu_{th}, \mu_{nh}) \in W^{0}_h(\Gamma_c)^3, \quad d((\tau_h, \nu_h, \eta_h), \mu_h) = 0, \quad \forall (\tau, \nu, \eta) \in X_h \times V_h \times W_h \} = \{0\},
\]
which is obvious.

**Proposition 6.** Problem (3.2) admits a unique solution \((\sigma_h, u_h, \gamma_h, \lambda_{th}, \lambda_{nh}) \in X_h \times V_h \times W_h \times M_h \times M_{h^+}\).

### 3.2. The zero order BDM finite element

We define
\[
R_T = \{ v \in V; v(x, y) = (a, b) + c(-y, x), \quad a, b, c \in \mathbb{R} \}.
\]

For the space of approximation of the constraints we take
\[
X_h = \{ \sigma_h \in X; \sigma_h = \sigma^1_h + \sigma^2_h + \sigma^3_h, \quad \sigma^1_h \in \mathcal{P}_1^{2,2}, \quad \sigma^2_h \in \mathcal{P}_1^{2,2}, \quad \sigma^3_h \in B_0(T), \quad \forall T \in T_h \},
\]
and for the displacement
\[
V_h = \{ v_h \in \left( C(\Omega) \right)^2; v_{h|T} \in R_T, \forall T \in T_h \}.
\]

For the antisymmetric tensor, take
\[
W_h = \{ \gamma_h \in W; \gamma_{h|T} \in \mathcal{P}_1(T; M_{skew}^{2,2}), \forall T \in T_h \}.
\]

For the Lagrange multipliers, it is always possible to take piecewise constant functions like in the PEERS element case, but we know ([6,7]) that piecewise affine functions provide better results when the solutions are more regular. We will choose the space \(W^1_h(\Gamma_c)\) to define \(M_h\) and \(M_{h^+}\). We do not write the discrete problems which are the same with (3.1) and (3.2). We have
Proposition 7. The discrete problem corresponding to the BDM finite elements admits a unique solution \((\sigma_h, u_h, \gamma_h, \lambda_{th}, \lambda_{nh}) \in X_h \times V_h \times W_h \times M_h\), with \(M_h = M_h^+ \times M_h^+\).

### 3.3. Error estimation

The error analysis is similar for the PEERS or BDM-based discretizations and is performed in details in [29]. We summarize in this section the steps of the analysis and the main results. Tiedious but standard computations yield the following abstract error estimates: let \((\sigma; u, \gamma, \lambda, \lambda_n)\) denote the solution of problem (2.5), and let \((\sigma_h; u_h, \gamma_h, \lambda_{th}, \lambda_{nh}, h)\) be the solution of the discrete problem (3.2) obtained with the PEERS or BDM elements, there exist constants \(C_1\) and \(C_2\) independent of \(h\), such that

\[
\|
\sigma - \sigma_h \|_{L^2(\Omega, \mathbb{R}^2 \times 2)} \leq C_1 \left\{ \inf_{\tau_h \in \chi_h} \| \sigma - \tau_h \|_{L^2(\Omega, \mathbb{R}^2 \times 2)} + \| -b((\sigma - \tau_h, (u - u_h, \gamma - \gamma_h)) + L(u - v_h) - d_t(\sigma - \tau_h, \lambda_t - \lambda_{th}) \\
+ d_t(\sigma - \sigma_h, \lambda_t - \mu_{th}) + d_n(\sigma - \sigma_h, \lambda_n - \mu_{nh}) \\
- d_n(\sigma - \tau_h, \lambda_n - \lambda_{nh}) - d_n(\sigma_h, \lambda_n - \mu_{nh}) - d_n(\sigma_h, \mu_{nh}) \right\}^{\frac{1}{2}}.
\]

Moreover, in the case of the PEERS elements, we have

\[
\| (u - u_h, \gamma - \gamma_h) \|_{V_h \times W_h} + \| \lambda_t - \lambda_{th} \|_{L^2(\Gamma_c, \mathbb{R}^2)} + \| \lambda_n - \lambda_{nh} \|_{L^2(\Gamma_c)} \leq C_2 \left( \| \sigma - \sigma_h \|_{L^2(\Omega, \mathbb{R}^2 \times 2)} + \inf_{(u_h, \gamma_h, \lambda_{th}, \lambda_{nh}) \in V_h \times W_h \times M_h} \left( \| (u - v_h, \gamma - \eta_h) \|_{V_h \times W_h} \\
+ \| \lambda_t - \mu_{th} \|_{H_{00}^1(\Gamma_c, \mathbb{R}^2)} + \| \lambda_n - \mu_{nh} \|_{H_{00}^1(\Gamma_c, \mathbb{R}^2)} \right) \right) + \| \lambda_t - \lambda_{th} \|_{L^2(\Omega, \mathbb{R}^2)}.
\]

In order to bound the terms appearing in these abstract error estimates, we introduce and recall the main properties for some useful approximation operators. We consider the projection operator \(\pi_h^0 : L^2(\Gamma_c) \rightarrow W_h^0(\Gamma_c)\) [59, 34]. This operator satisfies the following estimates: For all function \(\varphi \in H^\nu(\Gamma_c)\), with \(\nu = \frac{1}{2}\) or \(\nu = 1\), there exists a constant \(c > 0\), independent of \(h\), such that

\[
\| \varphi - \pi_h^0 \varphi \|_{L^2(\Gamma_c)} \leq c h^\nu \| \varphi \|_{H^\nu(\Gamma_c)}.
\]
Moreover, if \( \varphi \in L^2(\Gamma_c) \), then
\[
\| \varphi - \pi_h^0 \varphi \|_{H^{-1/2}(\Gamma_c)} \leq ch^{1/2} \| \varphi - \pi_h^0 \varphi \|_{L^2(\Gamma_c)}.
\] (3.11)

Note that for \( \varphi \geq 0 \), one has \( \pi_h^0 \varphi \in M_h^0 \). Recall the following results ([57], Lemma 3.1):

With \( \| \cdot \|_k = \| \cdot \|_{H^k(\Omega, \mathbb{R}^{2,2})} \) one has
\[
\| \sigma \|_0 \leq \| \sigma \|_{0,h} \leq C \| \sigma \|_0, \quad \forall \sigma \in X_h,
\] (3.12)
and
\[
\inf_{\tau \in X_h} \| \sigma - \tau \|_{0,h} \leq C h^{k+1} \| \sigma \|_{k+1}, \quad \forall \sigma \in H^{k+1}(\Omega, \mathbb{R}^{2,2}) \cap X.
\] (3.13)

We have the following estimates (see [29]):

**Proposition 8.** Let \((\sigma; u, \gamma, \lambda, \lambda_n)\) denote the solution of problem (2.5). Suppose that \( u \in H^2_{\text{loc}}(\Omega_c) \), \( \sigma \in H^1_{\text{loc}}(\Omega_c) \) and that \( \text{div} \sigma \in H^1(\Omega_c) \). Let \((\sigma_h; u_h, \gamma_h, \lambda_h, \lambda_{hn})\) be the solution of the discrete problem (3.2) obtained with the PEERS finite elements. We have the following estimate:
\[
\| \sigma - \sigma_h \|_0 \leq C(\sigma, u) \left( h \| u - u_h, \gamma - \gamma_h \|_{V_h \times W_h} + \| \lambda - \lambda_{th} \|_{L^2(\Gamma_c, \mathbb{R}^2)} + \| \lambda_n - \lambda_{nh} \|_{L^2(\Gamma_c)} + C(\lambda) h^{1/2} \right),
\] (3.14)
\[
\| (u - u_h, \gamma - \gamma_h) \|_{V_h \times W_h} + \| \lambda - \lambda_{th} \|_{L^2(\Gamma_c, \mathbb{R}^2)} + \| \lambda_n - \lambda_{nh} \|_{L^2(\Gamma_c)} \leq C \left( h^{-1/2} \| \sigma - \sigma_h \|_0 + C(\lambda) h^{1/2} \right).
\] (3.15)

\( C(\sigma, u) \) is a constant depending on \( \| u \|_{H^2_{\text{loc}}(\Omega_c)} \) and \( \| \sigma \|_{H^1_{\text{loc}}(\Omega_c)}^4 \).

Assembling the previous estimates, we have

**Theorem 9.** Let \((\sigma; u, \gamma, \lambda, \lambda_n)\) be the solution of problem (2.5). Suppose that \( u \in H^2_{\text{loc}}(\Omega_c) \), \( \sigma \in H^1_{\text{loc}}(\Omega_c, \mathbb{R}^{2,2}) \) and that \( \text{div} \sigma \in H^1(\Omega_c) \). Let \((\sigma_h; u_h, \gamma_h, \lambda_h, \lambda_{hn})\) be the solution of the discrete problem (3.2), with choices for \( M_h \) and \( M_{h+} \) given for PEERS’s finite elements. We have the following convergence rate:
\[
\| \sigma - \sigma_h \|_{L^2(\Omega, \mathbb{R}^{2,2})} + \| (u - u_h, \gamma - \gamma_h) \|_{V_h \times W_h} + \| \lambda - \lambda_{th} \|_{L^2(\Gamma_c, \mathbb{R}^2)} + \| \lambda_n - \lambda_{nh} \|_{L^2(\Gamma_c)} \leq C(\sigma, u, \gamma, \lambda) h^{1/2}.
\] (3.16)

\( C(\sigma, u, \gamma, \lambda) \) is a constant depending on \( \| u \|_{H^2_{\text{loc}}(\Omega_c)} \), \( \| \sigma \|_{H^1_{\text{loc}}(\Omega_c)} \) and \( \| \text{div} \sigma \|_{H^1(\Omega_c)} \).

Similarly, we obtain for the BDM finite elements:

**Proposition 10.** Let \((\sigma; u, \gamma, \lambda, \lambda_n)\) denote the solution of problem (2.5). Suppose that \( u \in H^2_{\text{loc}}(\Omega_c) \), \( \sigma \in H^1_{\text{loc}}(\Omega_c, \mathbb{R}^{2,2}) \), and that \( \text{div} \sigma \in H^1(\Omega_c) \). Let \((\sigma_h; u_h, \gamma_h, \lambda_{th}, \lambda_{nh})\) be the solution of the discrete problem (3.2) obtained with the BDM finite elements. We have the following
estimate:

\[ \| \sigma - \sigma_h \|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C(\sigma, u) \left( h \| (u - u_h, \gamma - \gamma_h) \|_{\mathbb{V}_h \times \mathbb{W}_h} + \| \lambda_t - \lambda_{th} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} + h \| \lambda_n - \lambda_{nh} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} + h^3 \right), \]  

(3.17)

\[ \| (u - u_h, \gamma - \gamma_h) \|_{\mathbb{V}_h \times \mathbb{W}_h} + \| \lambda_t - \lambda_{th} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} + \| \lambda_n - \lambda_{nh} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} \leq C(h^{-\frac{1}{2}} \| \sigma - \sigma_h \|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} + C(u)h). \]  

(3.18)

\( C(u, \sigma) \) is a constant depending on \( \| u \|_{H^2_{\text{loc}}(\Omega_c)}, \| \sigma \|_{H^1_{\text{loc}}(\Omega_c)} \) and \( \| \text{div} \sigma \|_{H^1(\Omega_c)}^2 \).

**Theorem 11.** Let \( (\sigma; u, \gamma, \lambda, \lambda_n) \) be the solution of problem (2.5). Suppose that \( u \in H^2_{\text{loc}}(\Omega_c), \sigma \in H^1_{\text{loc}}(\Omega_c; \mathbb{R}^{2 \times 2}) \) and that \( \text{div} \sigma \in H^1(\Omega_c) \). Let \( (\sigma_h; u_h, \gamma_h, \lambda_h, \lambda_{hn}) \) be the solution of the discrete problem (2.5), with choices for \( M_h \) and \( M_{h+} \) given for PEERS’s finite elements. We have the following convergence rate:

\[ \| \sigma - \sigma_h \|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} + \| (u - u_h, \gamma - \gamma_h) \|_{\mathbb{V}_h \times \mathbb{W}_h} + \| \lambda_t - \lambda_{th} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} + h \| \lambda_n - \lambda_{nh} \|_{H^1_0(\Gamma_c; \mathbb{R}^2)} \leq C(u, \sigma)h^\frac{3}{4}. \]  

(3.19)

\( C(u, \sigma) \) is a constant depending on \( \| u \|_{H^2_{\text{loc}}(\Omega_c)}, \| \sigma \|_{H^1_{\text{loc}}(\Omega_c)} \) and \( \| \text{div} \sigma \|_{H^1(\Omega_c)}^2 \).

Note that the difference in the rate of convergence between the PEERS and the BDM-based discretization is due to the choice of the Lagrange multipliers spaces (piecewise-constant functions in the first case, and piecewise affine functions in the second one). This difference only appears when Lagrange multipliers are smoother (when cracks are straight lines, for example).

## 4. Numerical implementation

The details of the implementation are given in [29], we will give in this section a brief review of the method. In order to perform the computations, we will use a commonly used method for problems involving the space \( \dot{H}(\text{div}; \Omega) \) which consists in relaxing the continuity of the normal traces of the constraint tensor [1, 28]. Such a method leads to a higher dimension but sparse linear system. The continuity of normal traces is considered by means of Lagrange multipliers.

With this implementation, the computations are carried out locally on each element and the Lagrange multipliers (for relaxing the continuity) contain additional informations that can be used to get a better approximation of the displacement with post-processing scheme such as the one introduced in [28] which provides a best approximation of \( u \).

For the method, we replace the approximation space \( X_h \) by space \( X_h^* \), that we define as follows:

Let \( \mathcal{R}^* \) be the space of vectorial functions whose restrictions to \( T \) belong to \( RT_0(T) \oplus B_0(T) \).

Set \( X_h^* = \{ \tau_h; (\tau_{i1}, \tau_{i2}) \in \mathcal{R}^*, \forall T \in T_h \}. \)
Set

\[ X_h^* = \{ \tau_h \in X; \tau_h|_{T_i} \in X_h^*(T), \forall T \in T_h \}. \]

Let denote by \( E_h \) the set of internal edges of \( T_h \), we introduce the discrete spaces of Lagrangian multipliers

\[ Q_h = \{ \lambda : \lambda|_{E} \in \mathcal{P}_0(E)^2, \ E \subseteq \mathcal{E} \}, \]

\[ N_h = \{ l : l|_{E} \in \mathcal{P}_0(E), \ E \subseteq \Gamma_N, \ l|_{E} = 0 \ for \ E \subseteq \Gamma_D \}. \]

Introducing Lagrange multipliers ensures the continuity of the normal trace of the stress tensor \( \sigma \) on the inner boundaries, so that \( \sigma \in H(div, \Omega) \).

The modified system reads: Find \((\sigma_h, u_h, \gamma_h, \lambda_{th}, \lambda_{nh}, \alpha_h, \beta_h) \in X_h^* \times V_h \times W_h \times M_h \times M_{h^+} \times Q_h \times N_h\) such that

\[
\begin{align*}
    a(\sigma_h, \tau_h) + b(\tau_h; u_h, \gamma_h) &\quad - \sum_{E \in E_h} \int_E [\tau_h \cdot \nu] \alpha_h \, ds + \sum_{E \subseteq \Gamma_N} \int_E \tau_h \cdot \nu \cdot \beta_h \, ds \\
    + d(\tau_h, \lambda_{th}) + d_n(\tau_h, \lambda_{nh}) &\quad = 0, \ \forall \tau_h \in X_h^*, \\
    b(\sigma_h; \nu_h, \eta_h) &\quad = -(f, \nu_h), \ \forall (\nu_h, \eta_h) \in V_h \times W_h, \\
    \sum_{E \in E_h} \int_E [\sigma_h \cdot \nu] m_{hi} \, ds &\quad = 0, \ \forall m_{hi} \in Q_h, \\
    \sum_{E \subseteq \Gamma_N} \int_E \sigma_h \cdot \nu \cdot \ell_{hi} \, ds &\quad = 0, \ \forall \ell_{hi} \in N_h, \\
    d_l(\sigma_h, \mu_{th}) &\quad = 0, \ \forall \mu_{th} = (\mu_{th1}, \mu_{th2}) \in M_h, \ \mu_{hi} \cdot \nu_{hi} = 0, \\
    d_n(\sigma_h, \mu_{nh} - \lambda_{nh}) &\quad \leq 0, \ \forall \mu \in M_{h^+},
\end{align*}
\] (4.1)

To solve this discrete problem we observe that \((\sigma_h, u_h, \gamma_h, \lambda_{th}, \lambda_{nh})\) is the saddle-point of the associated Lagrangian functional. Thus, we will use an Uzawa type algorithm to determine this saddle-point. Let \( V, U \) denote the vectors with the entries given by the values of the functions \((\tau_h, \nu_h, \eta_h, m_{hi}, \ell_{hi}, \mu_{th})\) and \((\sigma_h, u_h, \gamma_h, \alpha_h, \beta_h, \lambda_{th})\), respectively. Let \( M \) and \( \Lambda \) be the vectors with the entries given by the values of \( \mu_{nh} \) and \( \lambda_{nh} \), respectively, for the different choices of the space \( M_h \) (according to the choice of the PEERS or BDM element). The saddle-point formulation in the finite dimensional setting reads:

Find \((U, \Lambda)\) defined by the max-min condition

\[
\max_{SM \geq 0} \left( \min_{V, F} \frac{1}{2} V K V - \frac{1}{2} F V F + (\nu V L) S M, \right),
\] (4.2)

where \( K \) denotes the stiffness matrix

\[
\begin{pmatrix}
B & C & D & E & F & H \\
C' & 0 & 0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 \\
E' & 0 & 0 & 0 & 0 & 0 \\
F & 0 & 0 & 0 & 0 & 0 \\
H & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

with \( B, C, D, \) are the matrices associated to the bilinear forms \( a(\cdot, \cdot) \), \( b(\cdot, \cdot) \), \( E, F \) the matrices associated to the relaxation of the continuity and the Neumann boundary condition and \( H \) is associated to \( d_l(\cdot, \cdot) \). The matrix \( L \) is associated to the bilinear form \( d_n(\cdot, \cdot) \). \( F \) is the vector corresponding to the external loading and the matrix \( S \) expresses the sign conditions for the multipliers.
Let \( m \) denote the number of nodes on \( \Gamma_C \). We define \( (\psi_k)_k, 1 \leq k \leq m \) to be the finite element basis associated to \( W^1_h(\Gamma_C) \) and \( (\varphi_k)_k, 1 \leq k \leq m \), to be the finite element basis associated to \( W^0_h(\Gamma_C) \).

**Remark 12.** Note that \( \psi_1 \), respectively \( \psi_{m-1} \), is constant in \( t_0 \), respectively is constant in \( t_{m-1} \), and zero elsewhere.

If \( M_h \) is \( = M^0_h \) or \( = M^1_h \), then \( S \) is given by the identity matrix, else \( M_h = M^*_h \), and \( S_{ij} = \int_{\Gamma_C} \psi_i \psi_j \, d\tau, \, 1 \leq i, j \leq m - 1 \).

The details for the computations of these matrices are given in [29], we just recall briefly the expression of the coupling matrix \( L \) which is defined in the following way

- If \( M_h = M^1_h \) or \( M_h = M^*_h \), then
  \[
  (L)_{ij} = \int_{\Gamma_C} \psi_j (\eta_i \cdot \mathbf{n}) \cdot \mathbf{n} \, d\tau, \quad 1 \leq i \leq N, \quad 1 \leq j \leq m - 1.
  \]

- If \( M_h = M^0_h \), then
  \[
  (L)_{ij} = \int_{\Gamma_C} \varphi_j (\eta_i \cdot \mathbf{n}) \cdot \mathbf{n} \, d\tau, \quad 1 \leq i \leq N, \quad 1 \leq j \leq m - 1,
  \]

where we denote by \( N \) the dimension of the space \( X_h \) in both cases: PEERS or BDM-based discretization and \( (\eta_i)_i, 1 \leq i \leq N \) a basis of this space. We emphasis that all matrices evaluations are performed at local level (see [29] for details).

The solution \((U, \Lambda)\) of (4.2) satisfies the saddle-point conditions and we have

\[
U = K^{-1}(F - LS\Lambda). \quad (4.3)
\]

Therefore, for \( \Phi = S\Lambda \), the saddle-point problem (4.2) can be rewritten as a quadratic programming problem

\[
\min_{\Phi \geq 0} \left( \frac{1}{2} t^t \Phi^t L K^{-1} L \Phi - t^t \Phi^t L K^{-1} F + \frac{1}{2} t^t F K^{-1} F \right). \quad (4.4)
\]

If \( \overline{\Phi} \) is the solution of (4.4) then \( \Lambda = S^{-1}\overline{\Phi} \). The solution \( U \) is obtained by solving (4.3).

**4.0.1. Example 1: Elasticity system with \( \nu = 0.29 \)**

Let us consider a rectangle \( \Omega = (0, 4) \times (0, 1) \) with the crack \( \Gamma_c = (0, 4) \times \{ \frac{1}{2} \} \). Young’s modulus \( E \) (resp. the Poisson’s ratio \( \nu \)) equals 206900 (resp. 0.29). The external forces are \( f = 1 \) and \( g = 0 \). The notation \( dof \) stands for the total number of degrees of freedom, while \( N_{L_2} \) is the \( L^2 \)-norm.

In table 1, we represent the \( L^2 \)-norm of the relative error of the displacement’s first component \( u_1 \), expressed as a function of the total number of degrees of freedom.
Figure 2: $\Gamma_c = (0, 4) \times \{\frac{1}{2}\}$

<table>
<thead>
<tr>
<th>$dof$</th>
<th>83</th>
<th>162</th>
<th>461</th>
<th>819</th>
<th>937</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{L^2}$</td>
<td>0.227449873</td>
<td>0.0844177121</td>
<td>0.0489419458</td>
<td>0.0394980417</td>
<td>0.0209586395</td>
</tr>
</tbody>
</table>

Table 1: $\| u_1 - u_{1,h} \|_{L^2(\Omega)}$

<table>
<thead>
<tr>
<th>$dof$</th>
<th>83</th>
<th>162</th>
<th>461</th>
<th>819</th>
<th>937</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{L^2}$</td>
<td>0.318122037</td>
<td>0.144441467</td>
<td>0.0821550301</td>
<td>0.0752721434</td>
<td>0.0529486423</td>
</tr>
</tbody>
</table>

Table 2: $\| u_2 - u_{2,h} \|_{L^2(\Omega)}$

<table>
<thead>
<tr>
<th>$dof$</th>
<th>83</th>
<th>162</th>
<th>461</th>
<th>819</th>
<th>937</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{L^2}$</td>
<td>0.425346615</td>
<td>0.239523559</td>
<td>0.0669139203</td>
<td>0.05465566</td>
<td>0.0448078919</td>
</tr>
</tbody>
</table>

Table 3: $\| \sigma_{t1} - \sigma_{t1,h} \|_{L^2(\Omega)}$

Table 4: $\| \sigma_{\nu} - \sigma_{\nu,h} \|_{L^2(\Omega)}$

Table 2 represents the $L^2$-norm of the relative error of the displacement’s second component $u_2$, also expressed as a function of the total number of degrees of freedom.

Table 3 represents the $L^2$-norm of the relative error of the first constraints tangential component $\sigma_{t1}$ expressed as a function of the total number of degrees of freedom.

Table 4 represents the $L^2$-norm of the relative error of the constraints normal component $\sigma_{\nu}$ expressed as a function of the total number of degrees of freedom.
Table 1, Table 2, Table 3 and Table 4 summerize results of our computations, where $u_1$, $u_2$, $\sigma_{t1}$ and $\sigma_v$ stand for the reference values obtained on the finest mesh. Figure 3, respectively Figure 4, represents the relative error for the two components of the displacement field, respectively the tangential components of constraints tensor, as a function of the degrees of freedom in log-log scale. The convergence rates for the relative $L^2$ error norm of the x-displacement $u_1$ is close to 0.81 while those for the y-displacement $u_2$ is close to 0.65 (Figure 3). On Figure 4, the convergence rate for the first tangential component $\sigma_{t,x}$ is about 0.93, and for the second one, $\sigma_{t,y}$, is about 0.78.

In both cases, the numerical rates are close to the values predicted for the BDM element. The same holds true with PEERS element.

4.0.2. Example 2: The nearly incompressible case $\nu = 0.49$

A second test is provided, with an internal crack $\Gamma_c = (1, 3) \times \left\{ \frac{1}{2} \right\}$. The Young’s modulus $E$ (resp. the Poisson’s ratio $\nu$) equals 206900 (resp. 0.49). The external forces are $f = 1$ and $g = 0$.

Figure 5, respectively figure 6, represents the relative error for the two components of the displacement field, respectively the tangential components of constraints tensor, as a function of the degrees of freedom in log-log scale. The numerical rates of convergence still agree with the theoretical estimates in the nearly incompressible case.

Figure 7 and figure 8, represent the isovalues of $\sigma_v$ in each case (i.e. with $\nu = 0.29$ and 0.49). Both examples considered in this section show the efficiency of the approach particularly for the nearly incompressible case. More realistic examples in the linearized elasticity are under consideration.

References

Figure 4: Relative error in the $L^2$-norm for constraints as functions of the *dof*

Figure 5: Relative error in the $L^2$-norm for displacements as functions of the discretization size


Figure 6: Relative error in the $L^2$-norm for constraints as functions of the discretization size

Figure 7: Isovalues for $\sigma_{\nu}$. Example 1


Figure 8: Isovalues for $\sigma_{\nu}$. Example 2


