Math. Model. Nat. Phenom.
Vol. 5, No. 4, 2010, pp. 448-469
DOI: 10.1051/mmnp/20105417

On Threshold Eigenvalues and Resonances for the Linearized NLS Equation

V. Vougalter *

University of Toronto, Department of Mathematics, Toronto, ON, M5S 2E4, Canada

Abstract. We prove the instability of threshold resonances and eigenvalues of the linearized NLS operator. We compute the asymptotic approximations of the eigenvalues appearing from the end-point singularities in terms of the perturbations applied to the original NLS equation. Our method involves such techniques as the Birman-Schwinger principle and the Feshbach map.

Key words: NLS equation, spectral stability, Birman-Schwinger principle, Feshbach map
AMS subject classification: 35Q55, 47J15, 81Q05

1. Introduction

In this work we consider the nonlinear Schrödinger (NLS) equation in three dimensions,

$$i\psi_t = -\Delta\psi + U_0(x)\psi + F_0(|\psi|^2)\psi, \tag{1.1}$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R},\ \psi \in \mathbb{C}$. The problem in other dimensions can be treated similarly. For certain functions $U_0(x)$ and $F_0(|\psi|^2)$, the NLS equation (1.1) possesses solitary wave solutions of the form,

$$\psi = \phi_0(x)e^{i\omega t}, \qquad \omega > 0, \tag{1.2}$$

where $\phi_0(x)$ is an exponentially decreasing solution of the elliptic problem,

$$-\Delta\phi_0 + \omega\phi_0 + U_0(x)\phi_0 + F_0({\phi_0}^2)\phi_0 = 0, \tag{1.3}$$

such that $\phi_0 : \mathbb{R}^3 \to \mathbb{R},\ \phi_0 \in \mathbb{C}^\infty$. The existence of such solutions was proven for a large class of nonlinearities (see e.g. [1], [3], [4], [5], [23], [26]). These solutions are called the ground

*E-mail: vitali@math.toronto.edu

states if they are positive in the whole space $\mathbb{R}^3$ and the excited states if they are sign-indefinite. Linearization of the NLS equation near solution (1.2) leads to the spectral problem for the operator $\mathcal{L}_0 := \sigma_1 \mathrm{H}_0$ considered on $L^2(\mathbb{R}^3, \mathbb{C}^2)$:

$$\mathcal{L}_0 \mathbf{u} = z\mathbf{u}, \tag{1.4}$$

where $\mathbf{u} = (u, w)^T, \sigma_1$ is the standard Pauli matrix and the energy operator H_0 is the second variation of the Hamiltonian functional for the NLS equation (1.1):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{H}_0 = \begin{pmatrix} L_{+,\,0} & 0 \\ 0 & L_{-,\,0} \end{pmatrix}, \tag{1.5}$$

where $L_{\pm,\,0} = -\Delta + \omega + V_{\pm,\,0}$ is the pair of Schrödinger operators with the potentials $V_{\pm,\,0}$ given by

$$V_{+,\,0}(x) = U_0(x) + F_0(\phi_0^2) + 2F_0'(\phi_0^2)\phi_0^2, \qquad V_{-,\,0}(x) = U_0(x) + F_0(\phi_0^2), \tag{1.6}$$

$V_{\pm,\,0} \in C^\infty$ and decay exponentially. The operator $\mathcal{L}_0$ is not self-adjoint since σ_1 and H_0 do not commute in general. The spectral analysis of the operator $\mathcal{L}_0$ plays the crucial role in proving the asymptotic stability of the solitary waves of the NLS equation (see e.g. [8], [7], [25], [16]). Spectral properties of the linearized NLS operator were studied recently in (see e.g. [13], [10], [29], [30], [9]). Its spectrum $\sigma(\mathcal{L}_0)$ is symmetric with respect to the real and imaginary axes. The essential spectrum consists of two branches $(-\infty, -\omega]$ and $[\omega, \infty)$ such that the points $\pm\omega$ are called the *thresholds*, while the eigenvalues could be located anywhere in the complex plane. For the operator $\mathcal{L}_0$ its point spectrum is denoted as $\sigma_p(\mathcal{L}_0)$. The notations $\langle \mathbf{f}, \mathbf{g} \rangle$ are used for the inner product of $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, (f, g) for $f, g \in L^2(\mathbb{R}^3)$, $\|f\|_2$ for the norm of $f \in L^2(\mathbb{R}^3)$, $\langle \mathbf{u}, \mathbf{v} \rangle_N$ for the scalar product of $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ and $|\mathbf{u}|$ for the length of $\mathbf{u} \in \mathbb{C}^3$.

A solution $\mathbf{u}_0$ of problem (1.4) at $z = \omega$ such that $(1 + |x|^2)^{-\frac{s}{2}} \mathbf{u}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, $s > \frac{1}{2}$ and $\nabla \mathbf{u}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^6)$ but $\mathbf{u}_0 \notin L^2(\mathbb{R}^3, \mathbb{C}^2)$ is called the *threshold resonance*. In the paper we extend the results of the work [10] on the bifurcations from a threshold resonance or a simple eigenvalue at $z = \omega$ in three dimensions by estimating these bifurcations in terms of the perturbation applied to the original NLS equation.The main result of the work is the proof of the structural instability of the singularities of both kinds at the end of the essential spectrum $z = \omega$ under generic perturbations and the computation of the asymptotics of the eigenvalues which can appear in the neighborhood of the point $z = \omega$ when such perturbations are applied. We omit the studies near $z = -\omega$ due to the symmetry of the problem.

We consider two types of perturbations to equation (1.1). The first one is when the potential in the NLS is perturbed

$$U(x) = U_0(x) + \varepsilon U_p(x), \ \varepsilon > 0. \tag{1.7}$$

In the work all the unperturbed terms are denoted with the 0 index, the perturbed ones without any indices and the perturbations applied with the p index.

In the second case the perturbation is applied to the nonlinearity term:

$$F(|\psi|^2) = F_0(|\psi|^2) + \varepsilon F_p(|\psi|^2), \ \varepsilon > 0. \tag{1.8}$$

The ε is a small positive parameter and the frequency ω of oscillations of the standing wave solution is assumed to be fixed and $\phi(x)$ is the perturbed solitary wave solution.

Thus the energy operator will be

$$L_+ = -\Delta + \omega + U_0(x) + \varepsilon U_p(x) + 2F_0'(\phi^2)\phi^2 + F_0(\phi^2), \tag{1.9}$$

$$L_- = -\Delta + \omega + U_0(x) + \varepsilon U_p(x) + F_0(\phi^2), \tag{1.10}$$

when perturbation is applied to the potential term in the NLS equation and

$$L_+ = -\Delta + \omega + U_0(x) + 2(F_0'(\phi^2) + \varepsilon F_p'(\phi^2))\phi^2 + F_0(\phi^2) + \varepsilon F_p(\phi^2), \tag{1.11}$$

$$L_- = -\Delta + \omega + U_0(x) + F_0(\phi^2) + \varepsilon F_p(\phi^2), \tag{1.12}$$

when perturbation is applied to the nonlinearity term.

Clearly the kernel of the linearized operator $\sigma_1 \mathrm{H}_0$ contains the element $\boldsymbol{\phi}_0 = (0,\ \phi_0)^T$. The operator $L_{+,\,0}$ possesses zero modes $\nabla\phi_0$ in the translation invariant case which disappear in the presence of a generic potential $U_0(x)$. We make the standard natural assumption of their nonexistence (cf [13]).

Assumption 1. *Ker*$(\sigma_1 H_0) = \{\boldsymbol{\phi}_0\}$.

We can write the energy operator as $\mathrm{H} = \mathrm{H}_0 + \varepsilon V_p + O(\varepsilon^2)$ in the space $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. The explicit form of V_p is given in Proposition 7. According to our assumption for a generic potential $U_0(x)$ the operator $L_{+,\,0}$ has a bounded inverse, $L_{+,\,0}^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ exists and $\|L_{+,\,0}^{-1}\|_{L^2 \to H^2} < \infty$. For the non-invertible cases we refer to [12, 26]. As for the potential and the nonlinear term involved in the NLS equation (1.1) and the perturbations applied to them we assume the following.

Assumption 2. *The potentials $U_0(x) \in C^\infty,\ U_p(x) \in C^\infty$ and are exponentially decreasing, the nonlinear terms $F_0 \in C^\infty,\ F_p \in C^\infty$ and $F_0(0) = 0,\ F_p(0) = 0$.*

Theorem 3. *Let the operator $\mathcal{L}_0$ have a threshold resonance and no eigenvalues at the end of its essential spectrum ω, Assumptions 1 and 2 be satisfied and, NLS equation be perturbed either as (1.7) or (1.8) and the quantity $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle$ does not vanish. Let V_p be defined in Proposition 7. If $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle < 0$ then $\mathcal{L}$ has a simple eigenvalue $z(\varepsilon)$, on the real axis, such that*

$$z(\varepsilon) = \omega - 4\varepsilon^2 \langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle^2 + o(\varepsilon^2).$$

If $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle > 0$, then $\mathcal{L}$ has no eigenvalue in the neighborhood of $z = \omega$. In both cases, the threshold resonance at $z = \omega$ disappears at $\varepsilon \neq 0$.

To formulate our next result we introduce the following auxiliary matrices:

$$(\hat{V}_p)_{i,j} := \langle \mathbf{u}_0^i, V_p \mathbf{u}_0^j \rangle, \ \ 1 \le i, j \le N,$$

where $\{\mathbf{u}_0^i\}_{i=1}^N$ are the linearly independent eigenvectors corresponding to the N-fold degenerate endpoint eigenvalue of problem (1.4) at $z = \omega$ with coinciding algebraic and geometric multiplicities both equal to N and V_p is given in Proposition 7. We assume that the eigenvalues of the hermitian matrix $\hat{V}_p$ are simple since the multiple ones are not generic and do not vanish, i.e.

$$\hat{V}_p \mathbf{v}_p^i = \lambda_p^i \mathbf{v}_p^i, \quad \langle \mathbf{v}_p^i, \mathbf{v}_p^j \rangle_N = \delta_{i,j}, \quad \lambda_p^i \neq \lambda_p^j, \quad i \neq j, \quad i, j = 1, ..., N.$$

Let

$$\hat{B}_{i,j} := \langle \mathbf{u}_0^i, \sigma_1 \mathbf{u}_0^j \rangle, \quad 1 \le i, j \le N.$$

and assume that $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle \neq 0, \quad i = 1, ..., N$. The number of eigenvalues λ_p^i which are positive (negative) such that $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle$ is positive (negative) is being denoted as $N_{B\pm}^{\pm}$. Thus

$$N = N_{B+}^{+} + N_{B-}^{-} + N_{B+}^{-} + N_{B-}^{+}.$$

Let

$$\hat{S}_{k,i} := \int_{\mathbb{R}^3} x_k s^i(x) \mathrm{d}x, \quad 1 \le k \le 3, \quad 1 \le i \le N,$$

where $s^i(x) = V_{+,\,0}(x) u_0^i(x) + V_{-,\,0}(x) w_0^i(x)$. Here $u_0^i(x)$ and $w_0^i(x)$ denote the first and the second components of the vector $\mathbf{u}_0^i(x)$ respectively.

Theorem 4. *Let the operator $\mathcal{L}_0$ have the N-fold degenerate eigenvalue at the end of its essential spectrum ω, Assumptions 1 and 2 be satisfied, the NLS equation be perturbed either as (1.7) or (1.8), the quantities λ_p^i, $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle$, $\hat{S}\mathbf{v}_p^i$, $i = 1, ..., N$ do not vanish. Let V_p be defined in Proposition 7. Then the N-fold degenerate eigenvalue disappears as $\varepsilon \neq 0$. Moreover, we have the following three cases for the spectrum of the perturbed operator $\mathcal{L}$.*

(i) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle < 0$ and $\lambda_p^i > 0$ or $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle > 0$ and $\lambda_p^i < 0$ the eigenvalues move in the negative direction of the real axis, such that

$$z_i(\varepsilon) = \omega + \varepsilon \frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle} + O(\varepsilon^{\frac{3}{2}}), \quad i = 1, ..., N_{B-}^{+} + N_{B+}^{-}.$$

(ii) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle < 0$ and $\lambda_p^i < 0$ the eigenvalues move in the complex plane, such that they become complex conjugate pairs and their asymptotics are given by

$$z_i(\varepsilon) = \omega + \varepsilon \frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle} \pm i \frac{\varepsilon^{\frac{3}{2}}}{24\pi} \frac{|\hat{S}\mathbf{v}_p^i|^2 \lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle^2} \sqrt{\frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle}} + O(\varepsilon^2), \quad i = 1, ..., N_{B-}^{-}.$$

(iii) When $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i \rangle > 0$ and $\lambda_p^i > 0$ the N_{B+}^{+} eigenvalues will disappear from the neighborhood of the endpoint $z = \omega$ of the essential spectrum of the linearized NLS operator.

Remark 5. *A straightforward computation yields that the results of Theorem 3 and Theorem 4 when $N = 1$ are analogous to the results on the bifurcations of a zero resonance and a simple eigenvalue from the edge of the essential spectrum of the linearized NLS operator obtained by Pelinovsky and Cuccagna (see Propositions 3.4 and 4.5 of [10]) with the perturbation potential computed in terms of the perturbation applied to the original NLS equation.*

Remark 6. *When we deal with a soliton, which remains to be the ground state under perturbation for ε small enough, only the bifurcations of the (i) and the (iii) kind in Theorem 4 are possible, since the eigenvalues of the linearized operator are restricted to the real and imaginary axes in this case.*

Bifurcations of resonances at the endpoints were studied in one dimension using the method of Evans functions (see [24], [18], [19]), numerically in one and two dimensions in ([9]). The primary goal of the study of such bifurcations is to show the structural instability of such resonances and eigenvalues, that they disappear when generic perturbations are applied to the NLS equation. Nonexistence of resonances and eigenvalues at the thresholds $\pm\omega$ is one of the key assumptions in the works on the asymptotic stability of the NLS solitons (see [16]). The assumption of nonexistence of eigenvalues embedded in the essential spectrum plays the crucial role in proving the dispersive estimates for the problem ([11]), the existence of stable manifolds for an orbitally unstable NLS ([28]). Another question arising here is whether the endpoint bifurcations lead to the appearance of the eigenvalues lying in the upper half-plane which leads to instability of solitary waves. Theorems 3 and 4 are the statements about the fate of the threshold resonance and the N-fold degenerate eigenvalue when perturbation is applied to the NLS equation, which is the generalization of the work of ([10]), proven via techniques different than the resolvent expansions and the mappings between the weighted Sobolev spaces.

In the next section we investigate how the perturbations to the equation (1.1) entail modifications of the operators $L_{\pm,\,0}$ and thus determine the V_p.

2. Corrections to the energy operator for the perturbed NLS

We have the following statement.

Proposition 7. *Let Assumptions 1 and 2 be satisfied. Then the application of perturbation to the NLS equation (1.1) translates into perturbation of the energy operator, such that in the linear order, in the space $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$*

$$V_{+,p} = U_p(x) - 2\phi_0\{3F_0'(\phi_0^2) + 2F_0''(\phi_0^2)\phi_0^2\}(L_{+,\,0})^{-1}(U_p\phi_0),$$

$$V_{-,p} = U_p(x) - 2\phi_0 F_0'(\phi_0^2)(L_{+,\,0})^{-1}(U_p\phi_0)$$

in the first case and

$$V_{+,p} = -2\phi_0\{3F_0'(\phi_0^2) + 2\phi_0^2 F_0''(\phi_0^2)\}(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + F_p(\phi_0^2) + 2F_p'(\phi_0^2)\phi_0^2,$$

$$V_{-,p} = -2\phi_0 F_0'(\phi_0^2)(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + F_p(\phi_0^2)$$

in the second case.

To evaluate the corrections to the standing solitary wave solution of the NLS and thus to the energy operator of the problem we use the Implicit Function Theorem technique. The perturbed solitary wave solution can be expressed as

$$\phi(x) = \phi_0(x) + \phi_p(x).$$

Let us first consider the case when the potential is being perturbed in the original NLS equation, such that the nonlinear elliptic problem for the standing wave solution is

$$-\Delta\phi + \omega\phi + [U_0(x) + \varepsilon U_p(x)]\phi + F_0(\phi^2)\phi = 0.$$

A straightforward computation using (1.3) leads to the equation for the correction term to the solitary wave solution

$$\phi_p = L_{+,\,0}^{-1}[-\varepsilon\mathcal{K}_1(\phi_0) - \mathcal{K}_2(\phi_0, \phi_p)], \tag{2.1}$$

with $\mathcal{K}_1(\phi_0) = U_p\phi_0$ and $\mathcal{K}_2(\phi_0, \phi_p) = F_0(\phi^2)\phi + \varepsilon U_p\phi_p - 2F_0'(\phi_0^2)\phi_0^2\phi_p - F_0(\phi_0^2)\phi_p - F_0(\phi_0^2)\phi_0$.

We establish the two facts needed to prove the existence of the perturbed solitary wave solution in the appropriate Sobolev space and to evaluate its asymptotics.

Fact 8. *If* $\phi_p \in H^2(\mathbb{R}^3)$ *then* $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_1(\phi_0) + \mathcal{K}_2(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

Fact 9. *If* $\varepsilon,\ \|\phi_p^{(1)}\|_{H^2},\ \|\phi_p^{(1)}\|_{H^2}$ *are small enough, then*

$$\|L_{+,\,0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(1)})] - L_{+,\,0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(2)})]\|_{H^2} \leq \frac{1}{2}\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2}.$$

Proof of Fact 8. Since the perturbation $|U_p(x)| \leq C,\ x \in \mathbb{R}^3$, we have $\|\mathcal{K}_1(\phi_0)\|_2 \leq C\|\phi_0\|_2 < \infty$. Here and further C stands for a finite, positive constant. Thus $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_1(\phi_0)] \in H^2(\mathbb{R}^3)$.

Clearly $|\varepsilon U_p\phi_p| \leq \varepsilon C|\phi_p|$. Let $\eta(\phi) := F_0(\phi^2)\phi$. For this smooth function $\eta(\phi) \in C^\infty(\mathbb{R})$ we have

$$\eta(\phi) - \eta(\phi_0) - \eta'(\phi_0)(\phi - \phi_0) = \int_{\phi_0}^{\phi}\Big(\int_{\phi_0}^{s}\eta''(q)dq\Big)ds,$$

and $\phi_p \in L^\infty(\mathbb{R}^3)$ by means of the Sobolev embedding theorem. Therefore,

$$|F_0(\phi^2)\phi - 2F_0'(\phi_0^2)\phi_0^2\phi_p - F_0(\phi_0^2)\phi_p - F_0(\phi_0^2)\phi_0| \le C|\phi_p|^2, \tag{2.2}$$

which yields the upper bound

$$|\mathcal{K}_2(\phi_0, \phi_p)| \le \varepsilon C|\phi_p| + C|\phi_p|^2.$$

Since $\phi_p \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have $\phi_p \in L^4(\mathbb{R}^3)$. Thus $\mathcal{K}_2(\phi_0, \phi_p) \in L^2(\mathbb{R}^3)$ and, therefore $L_{+,\,0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

□

Proof of Fact 9. Let $\phi_1 := \phi_0 + \phi_p^{(1)}$ and $\phi_2 := \phi_0 + \phi_p^{(2)}$. The norms $\|\phi_p^{(1,2)}\|_\infty$ are finite and sufficiently small by means of the Sobolev embedding theorem. A trivial computation yields

$$\mathcal{K}_2(\phi_0, \phi_p^{(1)}) - \mathcal{K}_2(\phi_0, \phi_p^{(2)}) = F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 + \varepsilon U_p(\phi_p^{(1)} - \phi_p^{(2)}) -$$

$$-2F_0'(\phi_0^2)\phi_0^2(\phi_p^{(1)} - \phi_p^{(2)}) - F_0(\phi_0^2)(\phi_p^{(1)} - \phi_p^{(2)}).$$

Due to the bound $|U_p(x)| \le C, \; x \in \mathbb{R}^3$ we have $\|\varepsilon U_p(\phi_p^{(1)} - \phi_p^{(2)})\|_2 \le \varepsilon C\|\phi_p^{(1)} - \phi_p^{(2)}\|_2$. To estimate the remaining terms in the right side of the identity above we use the formula

$$\int_{\phi_2}^{\phi_1} \Big(\int_{\phi_0}^{s} \eta''(q)dq\Big) ds = \eta(\phi_1) - \eta(\phi_2) - \eta'(\phi_0)(\phi_1 - \phi_2),$$

which implies

$$\|F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 - 2F_0'(\phi_0^2)\phi_0^2(\phi_p^{(1)} - \phi_p^{(2)}) - F_0(\phi_0^2)(\phi_p^{(1)} - \phi_p^{(2)})\|_2 \le \tag{2.3}$$

$$\le C(\|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_2,$$

and therefore

$$\|\mathcal{K}_2(\phi_0, \phi_p^{(1)}) - \mathcal{K}_2(\phi_0, \phi_p^{(2)})\|_2 \le C(\varepsilon + \|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_2.$$

Hence we arrive at

$$\|L_{+,\,0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(1)})] - L_{+,\,0}^{-1}[\mathcal{K}_2(\phi_0, \phi_p^{(2)})]\|_{H^2} \le$$

$$\le \|L_{+,\,0}^{-1}\|_{L^2 \to H^2} C(\varepsilon + \|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2},$$

which completes the proof of Fact 9 due to the smallness of ε and the norms $\|\phi_p^{(1,2)}\|_\infty$.

□

Now we turn to the studies of the situation when the nonlinear term is being perturbed in the NLS equation. Hence the standing wave solution will satisfy the elliptic equation

$$-\Delta\phi + \omega\phi + U_0(x)\phi + [F_0(\phi^2) + \varepsilon F_p(\phi^2)]\phi = 0.$$

We easily obtain the equation for the correction term to the solitary wave solution via (1.3).

$$\phi_p = L_{+,\,0}^{-1}[-\varepsilon\mathcal{K}_3(\phi_0) - \mathcal{K}_4(\phi_0, \phi_p)], \tag{2.4}$$

where $\mathcal{K}_3(\phi_0) = F_p(\phi_0^2)\phi_0$ and $\mathcal{K}_4(\phi_0, \phi_p) = F_0(\phi^2)\phi + \varepsilon F_p(\phi^2)\phi - F_0(\phi_0^2)\phi_0 - F_0(\phi_0^2)\phi_p - F_0'(\phi_0^2)2\phi_0^2\phi_p - \varepsilon F_p(\phi_0^2)\phi_0$. Analogously to the first case when the potential term was being perturbed we prove the two crucial facts.

Fact 10. *If* $\phi_p \in H^2(\mathbb{R}^3)$ *then* $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_3(\phi_0) + \mathcal{K}_4(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

Fact 11. *If* $\varepsilon, \|\phi_p^{(1)}\|_{H^2}, \|\phi_p^{(2)}\|_{H^2}$ *are small enough, then*

$$\|L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(1)})] - L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(2)})]\|_{H^2} \le \frac{1}{2}\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2}.$$

Proof of Fact 10. Since $|F_p(\phi_0^2)| \le C,\ x \in \mathbb{R}^3$, we obtain $\|\mathcal{K}_3(\phi_0)\|_2 \le C\|\phi_0\|_2 < \infty$. Hence $L_{+,\,0}^{-1}[\varepsilon\mathcal{K}_3(\phi_0)] \in H^2(\mathbb{R}^3)$. Let $\eta_p(\phi) := F_p(\phi^2)\phi$. For this smooth function $\eta_p \in C^\infty$ we have $\eta_p(\phi) - \eta_p(\phi_0) = \int_{\phi_0}^{\phi} \eta_p'(s)ds$. By means of the Sobolev embedding theorem $\phi_p \in L^\infty$. Thus $|\varepsilon F_p(\phi^2)\phi - \varepsilon F_p(\phi_0^2)\phi_0| \le \varepsilon C|\phi_p|$, which along with estimate (2.2) yields

$$|\mathcal{K}_4(\phi_0, \phi_p)| \le \varepsilon C|\phi_p| + C|\phi_p|^2.$$

Since the correction to the solitary wave solution $\phi_p \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have $\phi_p \in L^4(\mathbb{R}^3)$. Therefore, $\mathcal{K}_4(\phi_0, \phi_p) \in L^2(\mathbb{R}^3)$ and $L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p)] \in H^2(\mathbb{R}^3)$.

□

Proof of Fact 11. By means of the Sobolev embedding theorem the norms $\|\phi_p^{(1,2)}\|_\infty$ are sufficiently small. An elementary computation yields

$$\mathcal{K}_4(\phi_0, \phi_p^{(1)}) - \mathcal{K}_4(\phi_0, \phi_p^{(2)}) = F_0(\phi_1^2)\phi_1 - F_0(\phi_2^2)\phi_2 + \varepsilon F_p(\phi_1^2)\phi_1 - \varepsilon F_p(\phi_2^2)\phi_2 -$$

$$-F_0(\phi_0^2)(\phi_p^{(1)} - \phi_p^{(2)}) - F_0'(\phi_0^2)2\phi_0^2(\phi_p^{(1)} - \phi_p^{(2)}).$$

A trivial estimate $|\eta_p(\phi_1) - \eta_p(\phi_2)| = |\int_{\phi_2}^{\phi_1} \eta_p'(s)ds| \le C|\phi_p^{(1)} - \phi_p^{(2)}|$, where the function η_p is defined above implies the inequality $\varepsilon\|F_p(\phi_1^2)\phi_1 - F_p(\phi_2^2)\phi_2\|_2 \le C\varepsilon\|\phi_p^{(1)} - \phi_p^{(2)}\|_2$, which along with bound (2.3) gives us

$$\|\mathcal{K}_4(\phi_0, \phi_p^{(1)}) - \mathcal{K}_4(\phi_0, \phi_p^{(2)})\|_2 \le C(\varepsilon + \|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_2.$$

Thus we arrive at the estimate

$$\|L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(1)})] - L_{+,\,0}^{-1}[\mathcal{K}_4(\phi_0, \phi_p^{(2)})]\|_{H^2} \le$$

$$\leq \|L_{+,\,0}^{-1}\|_{L^2\to H^2} C(\varepsilon + \|\phi_p^{(1)}\|_\infty + \|\phi_p^{(2)}\|_\infty)\|\phi_p^{(1)} - \phi_p^{(2)}\|_{H^2},$$

which completes the Proof of Fact 11 due to the smallness of ε and the norms $\|\phi_p^{(1,2)}\|_\infty$.

□

Having established Facts 8-11 enables us to prove Proposition 7.

Proof of Proposition 7. Part I. When the potential term in the NLS equation is being perturbed, by means of Facts 8 and 9 along with the contraction lemma we obtain that the solution ϕ_p to equation (2.1) exists and is unique and has the asymptotics

$$\phi_p(x) = \phi(x) - \phi_0(x) = -\varepsilon L_{+,\,0}^{-1}(U_p\phi_0) + O(\varepsilon^2)$$

in the space $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. A straightforward computation using the identity above gives

$$F_0(\phi^2) = F_0(\phi_0^2) - 2\varepsilon F_0'(\phi_0^2)\phi_0(L_{+,\,0})^{-1}(U_p\phi_0) + O(\varepsilon^2),$$

$$F_0'(\phi^2) = F_0'(\phi_0^2) - 2\varepsilon F_0''(\phi_0^2)\phi_0(L_{+,\,0})^{-1}(U_p\phi_0) + O(\varepsilon^2).$$

Substituting these expressions into formulas (1.9) and (1.10) we obtain

$$L_+ = L_{+,\,0} + \varepsilon U_p(x) - 2\varepsilon\phi_0\{3F_0'(\phi_0^2) + 2F_0''(\phi_0^2)\phi_0^2\}(L_{+,\,0})^{-1}(U_p\phi_0) + O(\varepsilon^2),$$

$$L_- = L_{-,\,0} + \varepsilon U_p(x) - 2\varepsilon\phi_0 F_0'(\phi_0^2)(L_{+,\,0})^{-1}(U_p\phi_0) + O(\varepsilon^2)$$

in $H^2(R^3) \subset L^\infty(\mathbb{R}^3)$, which completes the proof in the first case.

Part II. When the perturbation is being applied to the nonlinear term in the NLS equation, via Facts 10, 11 and the contraction lemma we arrive at the existence of the unique solution to equation (2.4) having the asymptotics

$$\phi_p(x) = \phi(x) - \phi_0(x) = -\varepsilon(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2)$$

in the space $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. Hence

$$F_0(\phi^2) = F_0(\phi_0^2) - 2\varepsilon\phi_0 F_0'(\phi_0^2)(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2),$$

$$F_0'(\phi^2) = F_0'(\phi_0^2) - 2\varepsilon\phi_0 F_0''(\phi_0^2)(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + O(\varepsilon^2),$$

$$\varepsilon F_p(\phi^2) = \varepsilon F_p(\phi_0^2) + O(\varepsilon^2), \qquad \varepsilon F_p'(\phi^2)\phi^2 = \varepsilon F_p'(\phi_0^2)\phi_0^2 + O(\varepsilon^2).$$

A trivial calculation using (1.11) and (1.12) gives

$$L_+ = L_{+,\,0} - 2\varepsilon\phi_0\{3F_0'(\phi_0^2) + 2\phi_0^2 F_0''(\phi_0^2)\}(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + \varepsilon F_p(\phi_0^2) + 2\varepsilon F_p'(\phi_0^2)\phi_0^2 + O(\varepsilon^2),$$

$$L_- = L_{-,\,0} - 2\varepsilon\phi_0 F_0'(\phi_0^2)(L_{+,\,0})^{-1}(F_p(\phi_0^2)\phi_0) + \varepsilon F_p(\phi_0^2) + O(\varepsilon^2)$$

in $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, which completes the proof of Proposition 7.

□

Having proved Proposition 7 we turn to the spectral analysis of the linearized NLS operator in the presence of threshold eigenvalues and resonances.

3. The endpoint solutions, the Birman-Schwinger principle and the Feschbach map for the linearized NLS operator

The spectral problem for the operator $\mathcal{L}$ can be written in the equivalent form via the transformation $\boldsymbol{\psi} = (u+w,\ u-w)^T$:

$$\sigma_3 \mathcal{H} \boldsymbol{\psi} = z\boldsymbol{\psi}, \tag{3.1}$$

where σ_3 is the Pauli matrix and $\mathcal{H}$ is the new energy operator:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} -\Delta + \omega + f(x) & g(x) \\ g(x) & -\Delta + \omega + f(x) \end{pmatrix}, \tag{3.2}$$

such that $V_\pm(x) = f(x) \pm g(x)$. The solution of the spectral problem above $\boldsymbol{\psi}_0 = (\psi_{0,\,1}, \psi_{0,\,2})^T$ at the end of the essential spectrum $z = \omega$ satisfies the system

$$\Delta\psi_{0,\,1} = f_0\psi_{0,\,1} + g_0\psi_{0,\,2}, \quad (\Delta - 2\omega)\psi_{0,\,2} = g_0\psi_{0,\,1} + f_0\psi_{0,\,2}\,. \tag{3.3}$$

We consider the solutions of system (3.3), such that $(1+|x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ with $s > \frac{1}{2}$ in the case of a threshold resonance or $\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ in the case of an endpoint eigenvalue (see below). By inverting the Laplacian, $(\Delta)^{-1} : L^2(\mathbb{R}^3,\ e^{\nu|x|}\mathrm{d}x) \to L^2(\mathbb{R}^3,\ (1+|x|^2)^{-s}\mathrm{d}x)$ with $\nu > 0$ and small enough, $s > \frac{1}{2}$ the first component of the solution of this system can be written as

$$\psi_{0,\,1} = -\frac{C_0}{4\pi|x|} + \chi, \tag{3.4}$$

where

$$C_0 = \int_{\mathbb{R}^3} (f_0(x)\psi_{0,\,1}(x) + g_0(x)\psi_{0,\,2}(x))\mathrm{d}x$$

and $\chi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} - \frac{1}{|x-y|}\Big)(f_0(y)\psi_{0,1}(y) + g_0(y)\psi_{0,2}(y))\mathrm{d}y$ (see [10], also p.122 of [21]). Clearly $\psi_{0,\,2} \in L^2(\mathbb{R}^3)$. Let us show that χ is square integrable as well.

Fact 12. *For the second term in the right side of identity (3.4) we have* $\chi(x) \in L^2(\mathbb{R}^3)$.

Proof. Let us introduce the auxiliary function

$$s(x) := f_0(x)\psi_{0,1}(x) + g_0(x)\psi_{0,2}(x)\ ,$$

which is square integrable since $\psi_{0,1}$ and $\psi_{0,2}$ belong to weighted L^2 spaces and f_0 and g_0 decay exponentially. Hence $\chi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} - \frac{1}{|x-y|}\Big)s(y)dy$. It satisfies the Poisson equation

$$-\Delta\chi = \delta(x)\int_{\mathbb{R}^3} s(y)dy - s(x)\ ,$$

where $\delta(x)$ is the Dirac measure centered at the origin. By applying the Fourier transform to this equation using the fact that $\widehat{s}(0) = \dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3} s(x)dx$ we easily obtain

$$\widehat{\chi}(p) = \frac{\widehat{s}(0) - \widehat{s}(p)}{p^2}.$$

Here and further down the hat symbol stands for the Fourier transform. For proving our statement we use the decomposition

$$\widehat{\chi}(p) = \widehat{\chi}_1(p) + \widehat{\chi}_2(p),$$

with $\widehat{\chi}_1(p) := \dfrac{\widehat{s}(0) - \widehat{s}(p)}{p^2}\chi_{\{|p|\le 1\}}$ and $\widehat{\chi}_2(p) := \dfrac{\widehat{s}(0) - \widehat{s}(p)}{p^2}\chi_{\{|p|>1\}}$, where $\chi_{\{|p|\le 1\}}$ stands for the characteristic function of the unit ball in the Fourier space centered at the origin and $\chi_{\{|p|>1\}}$ for the characteristic function of its complement. The functions $s(x)$ and $|x|s(x)$ have finite norms in $L^1(\mathbb{R}^3)$, which can be easily shown by means of Schwarz inequality using that $f_0(x)$ and $g_0(x)$ decay exponentially and $\psi_{0,1}(x)$ and $\psi_{0,2}(x)$ belong to the weighted L^2 space. Thus we have

$$|\widehat{\chi}_2(p)| \le \Big(\frac{2}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|s(x)|dx\Big)\frac{\chi_{\{|p|>1\}}}{p^2} \in L^2(\mathbb{R}^3).$$

In order to show the square integrability of the remaining term we will make use of the following representation available by means of the Fundamental Theorem of Calculus

$$\widehat{s}(p) = \widehat{s}(0) + \int_0^{|p|}\frac{\partial}{\partial|\eta|}\widehat{s}(|\eta|,\Omega)d|\eta|\ ,$$

where $|\eta|$ denotes the radial variable and Ω stands for the angle variables on the sphere. This implies

$$\widehat{\chi}_1(p) = -\frac{\int_0^{|p|}\frac{\partial}{\partial|\eta|}\widehat{s}(|\eta|,\Omega)d|\eta|}{p^2}\chi_{\{|p|\le 1\}}.$$

Clearly $|\nabla_p\widehat{s}(p)| \le \dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3}|s(x)||x|dx$, which yields

$$|\widehat{\chi}_1(p)| \le \Big(\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|s(x)||x|dx\Big)\frac{1}{|p|}\chi_{\{|p|\le 1\}} \in L^2(\mathbb{R}^3).$$

□

Hence there are two distinct types of solutions for spectral problem (3.1) at the end of the essential spectrum (see e.g. [10]).

I. Threshold resonance

This situation occurs when $C_0 \ne 0$. Therefore the solution $\boldsymbol{\psi}_0$ of system (3.3) such that $(1+|x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3,\mathbb{C}^2)$ with $s > \frac{1}{2}$ decays as $\frac{1}{|x|}$. From representation (3.4) of the first component of the solution $\psi_{0,\,1}$ it follows that the threshold resonance is simple under the assumptions of Theorem 3. Indeed, if there were two of them: $\boldsymbol{\psi}_0^1$ and $\boldsymbol{\psi}_0^2$, we could assume them having the same constant C_0 in formula (3.4), which can be achieved by multiplying the equations of system (3.3) by the appropriate constant. Then their difference $\boldsymbol{\psi}_0^1 - \boldsymbol{\psi}_0^2 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ will be a solution of system (3.3). But the coexistence of a zero resonance and an eigenvalue at the threshold is beyond the scope of our work, which is assumed in Theorem 3. Therefore, $\boldsymbol{\psi}_0^1$ and $\boldsymbol{\psi}_0^2$ coincide.

We normalize $\boldsymbol{\psi}_0$ such that $C_0 = \sqrt{4\pi}$ and in the original variables for our spectral problem the solution is $\mathbf{u}_0 = (u_0, w_0)^T$.

II. Eigenvalue

This is the case when $C_0 = 0$ and, therefore $\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$.

The eigenvalue at the end of the essential spectrum now is assumed to be N-fold degenerate, such that the corresponding eigenvectors are $\boldsymbol{\psi}_0^i = (\psi_{0,\,1}^i, \psi_{0,\,2}^i)^T, \;\; i = 1, ..., N$, for which we have

$$\int_{\mathbb{R}^3} s^i(x)\mathrm{d}x = 0, \; i = 1, ..., N. \tag{3.5}$$

According to the Birman-Schwinger principle (see e.g. p. 88 of [27], p. 312 of [22]) each eigenvalue z of spectral problem (3.1) is correspondent to the eigenvalue -1 of the same multiplicity of the problem

$$K_{\alpha,\,\varepsilon}\mathbf{w} = -\mathbf{w}\;, \tag{3.6}$$

where the Birman-Schwinger operator is defined as

$$K_{\alpha,\,\varepsilon} := |Q_0|^{\frac{1}{2}} R_\alpha Q |Q_0|^{-\frac{1}{2}} \tag{3.7}$$

on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ with the operator-valued terms

$$R_\alpha = \begin{pmatrix} G_\alpha & 0 \\ 0 & G_{\sqrt{2\omega - \alpha^2}} \end{pmatrix}, \; G_\alpha = (-\Delta + \alpha^2)^{-1} \text{ and } G_{\sqrt{2\omega-\alpha^2}} = (-\Delta + 2\omega - \alpha^2)^{-1},$$

the family of potentials

$$Q = Q_0 + \varepsilon Q_p = \begin{pmatrix} f_0 & g_0 \\ g_0 & f_0 \end{pmatrix} + \varepsilon \begin{pmatrix} f_p & g_p \\ g_p & f_p \end{pmatrix}$$

and the solution of spectral problem (3.6)

$$\mathbf{w} = |Q_0|^{\frac{1}{2}}\boldsymbol{\psi}.$$

The spectral parameter α is defined relatively to the spectrum of problem (3.1) via

$$z = \omega - \alpha^2,$$

such that α belonging to the first sheet of the Riemann surface $\{\alpha : \mathrm{Re}\alpha > 0\}$ corresponds to an eigenvalue near ω and $\sigma_p(\mathcal{L})$ is empty in the neighborhood of ω when α belongs to the second sheet of the Riemann surface $\{\alpha : \mathrm{Re}\alpha < 0\}$. For the matrix valued potentials here and further we use the notations in the sense of the spectral calculus

$$Q_0 = signQ_0|Q_0|, \quad Q_0^{\frac{1}{2}} = signQ_0|Q_0|^{\frac{1}{2}}.$$

The Birman-Schwinger operator $K_{\alpha,\,\varepsilon}$ is compact due to the exponential decay of the potentials and of the kernel of the operator R_α in three dimensions. Similarly to the work [20] dealing with the standard Schrödinger operator we expand

$$K_{\alpha,\,\varepsilon} = \sum_{s=0}^{3} \alpha^s K_{s,\,\varepsilon} + O(\alpha^4),$$

with $K_{s,\,\varepsilon} := |Q_0|^{\frac{1}{2}} R_s Q |Q_0|^{-\frac{1}{2}},\ s = 0, ..., 3$, where the kernels of the operators are given by

$$R_0(x,y) = \begin{pmatrix} \frac{1}{4\pi|x-y|} & 0 \\ 0 & \frac{e^{-\sqrt{2\omega}|x-y|}}{4\pi|x-y|} \end{pmatrix}, \quad R_1(x,y) = \begin{pmatrix} -\frac{1}{4\pi} & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_2(x,y) = \begin{pmatrix} \frac{|x-y|}{8\pi} & 0 \\ 0 & \frac{e^{-\sqrt{2\omega}|x-y|}}{8\pi\sqrt{2\omega}} \end{pmatrix}, \quad R_3(x,y) = \begin{pmatrix} -\frac{|x-y|^2}{24\pi} & 0 \\ 0 & 0 \end{pmatrix}$$

and $O(\alpha^4)$ here and below stands for the terms of the order four and higher and analytic in α for α small enough. The expansion above for $K_{\alpha,\,\varepsilon}$ is convergent in the operator norm ($L^2 \to L^2$) due to the exponential decay of the potentials. First we consider the case of a multiple eigenvalue at $z = \omega$ with corresponding eigenfunctions $\boldsymbol{\psi}_0^i,\ i = 1, ..., N$. By letting $\alpha, \varepsilon \to 0$ we obtain the Birman-Schwinger problem at the edge of the essential spectrum of the linearized NLS operator $K_{0,\,0}\mathbf{w}_0^i = -\mathbf{w}_0^i,\ i = 1, ..., N$ with $\mathbf{w}_0^i = |Q_0|^{\frac{1}{2}}\boldsymbol{\psi}_0^i \in L^2$ and $K_{0,\,0} = |Q_0|^{\frac{1}{2}} R_0 Q_0^{\frac{1}{2}}$. This is equivalent to

$$\boldsymbol{\psi}_0^i = -R_0 Q_0 \boldsymbol{\psi}_0^i,\ i = 1, ..., N. \tag{3.8}$$

Let $\mathbf{q}_0^i := (q_{0,\,1}^i, q_{0,\,2}^i)^T = Q_0\boldsymbol{\psi}_0^i,\ i = 1, ..., N$. A trivial calculation using (3.5) yields

$$R_1\mathbf{q}_0^i = 0, \quad \int_{\mathbb{R}^3} q_{0,\,1}^i(x)\mathrm{d}x = \int_{\mathbb{R}^3} s^i(x)\mathrm{d}x = 0, \quad i = 1, ..., N. \tag{3.9}$$

The spectral problem for the operator adjoint to the unperturbed Birman-Schwinger operator is

$$K_{0,\,0}^* \mathbf{w}_0^{i\,*} = -\mathbf{w}_0^{i\,*},\ i = 1, ..., N,$$

with

$$K_{0,\,0}^* = Q_0^{\frac{1}{2}} R_0 |Q_0|^{\frac{1}{2}}$$

and the solutions of the adjoint problem are related to the original ones as

$$\mathbf{w}_0^{i\,*} = s_0^i signQ_0 \mathbf{w}_0^i,\ i = 1, ..., N,$$

where $s_0^i = sign\langle signQ_0\mathbf{w}_0^i, \mathbf{w}_0^i\rangle$. Let us denote the eigenspace of the operator $K_{0,\,0}$ correspondent to the eigenvalue -1 as X_0.We choose the basis of eigenvectors in the subspace X_0 orthonormal with respect to the eigenvectors of the adjoint operator via the Gram-Schmidt procedure, such that

$$\langle \mathbf{w}_0^{i\,*}, \mathbf{w}_0^j\rangle = \delta_{i,\,j}, \quad i,\ j = 1, ..., N. \tag{3.10}$$

We define the projection operator onto the subspace X_0.

Definition 13. *For an arbitrary function* $\mathbf{f} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$

$$P_0\mathbf{f} := \sum_{i=1}^{N}\langle \mathbf{w}_0^{i\,*}, \mathbf{f}\rangle \mathbf{w}_0^i.$$

The operator defined above satisfies $P_0^2 = P_0$ *and its adjoint is given by*

$$P_0^*\mathbf{g} = \sum_{i=1}^{N}\langle \mathbf{w}_0^i, \mathbf{g}\rangle \mathbf{w}_0^{i\,*}$$

for any $\mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$.

This definition enables us to restrict the Birman-Schwinger spectral problem (3.6) to $RanP_0$, the range of the projection P_0, which is the space of N dimensions.

Definition 14. *The Feshbach map (see p.207 of [15], also [2]) for the Birman-Schwinger operator* $K_{\alpha,\,\varepsilon}$ *is the operator* $F : RanP_0 \to RanP_0$*, such that*

$$F := P_0(K_{\alpha,\,\varepsilon} - K_{\alpha,\,\varepsilon}\bar{P}_0(\tilde{K}_{\alpha,\,\varepsilon} + 1)^{-1}\bar{P}_0K_{\alpha,\,\varepsilon})P_0\ ,$$

where $\bar{P}_0 = 1 - P_0$ *and* $\tilde{A} = \bar{P}_0A\bar{P}_0$ *for any operator* $A \neq P_0$.

Note that since -1 is an isolated eigenvalue of the Hilbert-Schmidt operator $K_{\alpha,\,\varepsilon}$, the spectrum of $\tilde{K}_{\alpha,\,\varepsilon} + 1$ is empty in the neighborhood of the origin and therefore, the operator $(\tilde{K}_{\alpha,\,\varepsilon} + 1)^{-1}$ is well defined. We denote the first term in the definition of the Feschbach map as

$$U := P_0K_{\alpha,\,\varepsilon}P_0$$

and the second one as

$$D := P_0K_{\alpha,\,\varepsilon}\bar{P}_0(\tilde{K}_{\alpha,\,\varepsilon} + 1)^{-1}\bar{P}_0K_{\alpha,\,\varepsilon}P_0.$$

Since the Feshbach operator has the isolated eigenvalue -1 of multiplicity N as well as the Birman-Schwinger operator (see p.208 of [15], [2]) we have the spectral problem

$$F\mathbf{v} = -\mathbf{v},$$

where $\mathbf{v} = \sum_{k=1}^{N} c_k \mathbf{w}_0^k$. Thus

$$det(F + I) = 0. \tag{3.11}$$

Hence the implicit dependence of α on ε is given by this equation and in order to study this dependence we need to estimate the matrix elements of the Feshbach operator. Due to the orthogonality relations (3.10) they are given by

$$F_{i,j} = \langle {\mathbf{w}_0^i}^*, F\mathbf{w}_0^j \rangle, \quad 1 \le i, j \le N.$$

Let us first evaluate the matrix elements of the operator U. We have the following lemma.

Lemma 15. *For all $i, j = 1, ..., N$ we have*

$$U_{i,j} = -\delta_{i,j} - 2\varepsilon s_0^i (\hat{V}_p)_{i,j} - 2\alpha^2 s_0^i \hat{B}_{i,j} + \alpha^2 \varepsilon s_0^i \langle \mathbf{q}_0^i, R_2 Q_p \boldsymbol{\psi}_0^j \rangle +$$

$$+ \frac{\alpha^3 s_0^i}{12\pi} (\hat{S}^* \hat{S})_{i,j} + \alpha^3 \varepsilon s_0^i \langle \mathbf{q}_0^i, R_3 Q_p \boldsymbol{\psi}_0^j \rangle + O(\alpha^4),$$

where $O(\alpha^4)$ is analytic in α for α small enough.

Proof. Step I. For the first term of the Feshbach operator we have the expansion

$$U = \sum_{s=0}^{3} \alpha^s U_s + O(\alpha^4) \ ,$$

where $U_s := P_0 K_{s,\,\varepsilon} P_0$ and $K_{s,\,\varepsilon} := |Q_0|^{\frac{1}{2}} R_s Q |Q_0|^{-\frac{1}{2}}, \ s = 0, ..., 3$, the term $O(\alpha^4)$ is analytic in α for α small enough and the expansion converges in the operator norm ($L^2 \to L^2$) due to the exponential decay of the potentials. Let us evaluate the matrix elements for each of the operators U_s.

Step II. Using the expression for the projection operator from Definition 13, relations (3.10), (3.8), the variable change for the eigenfunctions of the spectral problem (3.1) and for the elements of the energy operator we obtain

$$(U_0)_{i,j} = \langle {\mathbf{w}_0^i}^*, P_0 K_{0,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = \langle {\mathbf{w}_0^i}^*, K_{0,\,0} \mathbf{w}_0^j \rangle +$$

$$+ \varepsilon \langle {\mathbf{w}_0^i}^*, |Q_0|^{\frac{1}{2}} R_0 Q_p |Q_0|^{-\frac{1}{2}} \mathbf{w}_0^j \rangle = -\delta_{i,\,j} - 2\varepsilon s_0^i (\hat{V}_p)_{i,j}, \quad 1 \le i, j \le N.$$

Step III. For any operator A such that the quadratic form below is defined, using the definition of the operator $K_{1,\,\varepsilon}$, the formulas relating the vector functions ${\mathbf{w}_0^i}^*$ to $\mathbf{w}_0^i, \ \mathbf{w}_0^i$ to $\boldsymbol{\psi}_0^i$, the definition of $\mathbf{q}_0^i$ and (3.9) we have

$$\langle {\mathbf{w}_0^i}^*, P_0 K_{1,\,\varepsilon} A \mathbf{w}_0^j \rangle = s_0^i \langle R_1 \mathbf{q}_0^i, Q |Q_0|^{-\frac{1}{2}} A \mathbf{w}_0^j \rangle = 0, \ 1 \le i, j \le N.$$

Thus choosing $A = P_0$ yields

$$(U_1)_{i,j} = \langle \mathbf{w}_0^{i\,*}, P_0 K_{1,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = 0, \ 1 \le i, j \le N.$$

Step IV. For the quadratic term in the expansion of the operator U we use the definitions of the terms involved to derive

$$(U_2)_{i,j} = \langle \mathbf{w}_0^{i\,*}, P_0 K_{2,\,\varepsilon} P_0 \mathbf{w}_0^j \rangle = s_0^i \big\{ \langle \mathbf{q}_0^i, R_2 \mathbf{q}_0^j \rangle + \varepsilon \langle \mathbf{q}_0^i, R_2 Q_p \boldsymbol{\psi}_0^j \rangle \big\}, \ 1 \le i, j \le N.$$

Thus it remains to estimate

$$\langle \mathbf{q}_0^i, R_2 \mathbf{q}_0^j \rangle = \frac{1}{8\pi} \big\{ (q_{0,\,1}^i, |x| * q_{0,\,1}^j) + (q_{0,\,2}^i, \frac{1}{\sqrt{2\omega}} e^{-\sqrt{2\omega}|x|} * q_{0,\,2}^j) \big\},$$

where $*$ stands for the convolution. Using the standard argument (see e.g. p.167 of [21]) we write the first term in the right side of the equality above as

$$\frac{1}{8\pi}(q_{0,\,1}^i, |x| * q_{0,\,1}^j) = -\frac{1}{8\pi}\frac{\partial}{\partial t}|_{t=0^+}(q_{0,\,1}^i, e^{-t|x|} * q_{0,\,1}^j).$$

Having the formula for the heat kernel of the root of the Laplacian handy (see e.g. [21]) we easily obtain

$$\widehat{e^{-t|x|}}(p) = \frac{(2\pi)^{\frac{3}{2}}}{\pi^2} \frac{t}{[t^2 + p^2]^2}, \quad p \in \mathbb{R}^3, \quad t > 0.$$

Thus the identities above yield

$$\frac{1}{8\pi}(q_{0,\,1}^i, |x| * q_{0,\,1}^j) = -\frac{\partial}{\partial t}|_{t=0^+}(\widehat{q_{0,\,1}^i}(p), \frac{t}{(t^2+p^2)^2}\widehat{q_{0,\,1}^j}(p)) = -(\widehat{q_{0,\,1}^i}(p), \frac{1}{p^4}\widehat{q_{0,\,1}^j}(p)).$$

By inverting the Fourier transform we obtain

$$-((-\Delta)^{-1} q_{0,\,1}^i, (-\Delta)^{-1} q_{0,\,1}^j).$$

For the quadratic form remaining to estimate via the above mentioned heat kernel with $t = \sqrt{2\omega}$ we derive

$$\frac{1}{8\pi}(q_{0,\,2}^i, \frac{1}{\sqrt{2\omega}} e^{-\sqrt{2\omega}|x|} * q_{0,\,2}^j) = (\widehat{q_{0,\,2}^i}(p), \frac{1}{(2\omega+p^2)^2}\widehat{q_{0,\,2}^j}(p)) =$$
$$= ((-\Delta + 2\omega)^{-1} q_{0,\,2}^i, (-\Delta + 2\omega)^{-1} q_{0,\,2}^j).$$

Therefore

$$\langle \mathbf{q}_0^i, R_2 \mathbf{q}_0^j \rangle = -((-\Delta)^{-1} q_{0,\,1}^i, (-\Delta)^{-1} q_{0,\,1}^j) + ((-\Delta + 2\omega)^{-1} q_{0,\,2}^i, (-\Delta + 2\omega)^{-1} q_{0,\,2}^j). \quad (3.12)$$

By means of the Birman-Schwinger principle at the edge of the essential spectrum of the linearized NLS operator (3.8) we have

$$(-\Delta)^{-1} q_{0,\,1}^i = -\psi_{0,\,1}^i, \quad (-\Delta + 2\omega)^{-1} q_{0,\,2}^i = -\psi_{0,\,2}^i, \quad i = 1, ..., N,$$

such that the right side of (3.12) equals to $-(\psi^i_{0,\,1}, \psi^j_{0,\,1}) + (\psi^i_{0,\,2}, \psi^j_{0,\,2}) = -\langle \boldsymbol{\psi}^i_0, \sigma_3 \boldsymbol{\psi}^j_0 \rangle$, which is identical to $-2\langle \mathbf{u}^i_0, \sigma_1 \mathbf{u}^j_0 \rangle$ via the change of variables

$$\psi^i_{0,\,1} = u^i_0 + w^i_0, \quad \psi^i_{0,\,2} = u^i_0 - w^i_0, \quad i = 1, ..., N$$

given by (3.1). Hence

$$(U_2)_{i,j} = -2s^i_0 \hat{B}_{i,j} + s^i_0 \varepsilon \langle \mathbf{q}^i_0, R_2 Q_p \boldsymbol{\psi}^j_0 \rangle, \; 1 \le i, j \le N.$$

Step V. For the cubic term in the expansion of the operator U using the definitions of the operators and the vector-functions involved in it along with identity (3.9) we easily obtain

$$(U_3)_{i,j} = \langle \mathbf{w}^{i\,*}_0, P_0 K_{3,\,\varepsilon} P_0 \mathbf{w}^j_0 \rangle = s^i_0 \langle \mathbf{q}^i_0, R_3 \mathbf{q}^j_0 \rangle + \varepsilon s^i_0 \langle \mathbf{q}^i_0, R_3 Q_p \boldsymbol{\psi}^j_0 \rangle =$$

$$= \frac{s^i_0}{12\pi} (\hat{S}^* \hat{S})_{i,j} + \varepsilon s^i_0 \langle \mathbf{q}^i_0, R_3 Q_p \boldsymbol{\psi}^j_0 \rangle, \; 1 \le i, j \le N.$$

Collecting all the estimates derived in $Steps\ I - V$ we complete the proof of the lemma.

□

Lemma 16. *For all $i, j = 1, ..., N$ we have*

$$D_{i,j} = \varepsilon^2 (D_0)_{i,j} + \alpha \varepsilon^2 (D_1)_{i,j} + \alpha^2 \varepsilon (D_2)_{i,j} + \alpha^3 \varepsilon (D_3)_{i,j} + O(\alpha^4),$$

where here and further with a slight abuse of notations $(D_k)_{i,j},\ k = 0, ..., 3$ stand for the terms analytic in ε, such that their exact values are unimportant and $O(\alpha^4)$ is analytic in α for α small enough.

Proof. Let us expand each of the three operator valued factors involved in the second term D of the Feshbach map in terms of the parameter α up to the third order inclusively. Thus the first one can be written as

$$P_0 K_{\alpha,\,\varepsilon} \bar{P}_0 = \sum_{s=0}^{3} \alpha^s P_0 K_{s,\,\varepsilon} \bar{P}_0 + O(\alpha^4)$$

and the third one analogously as

$$\bar{P}_0 K_{\alpha,\,\varepsilon} P_0 = \sum_{s=0}^{3} \alpha^s \bar{P}_0 K_{s,\,\varepsilon} P_0 + O(\alpha^4),$$

with the $K_{s,\,\varepsilon}$ operators given in Lemma 15. To expand the middle term we use $\tilde{K}_{\alpha,\,\varepsilon} = \sum_{s=0}^{3} \alpha^s \tilde{K}_{s,\varepsilon} +$ $O(\alpha^4)$ such that $1 + \tilde{K}_{\alpha,\,\varepsilon} = (1 + \tilde{K}_{0,\,\varepsilon})[1 + \sum_{s=1}^{3} \alpha^s M_{s,\,\varepsilon} + O(\alpha^4)]$ and therefore,

$$(\tilde{K}_{\alpha,\,\varepsilon} + 1)^{-1} = (1 + \sum_{s=1}^{3} \alpha^s M_{s,\,\varepsilon} + O(\alpha^4))^{-1} M_{0,\,\varepsilon}\ ,$$

where the operators

$$M_{s,\,\varepsilon} := (1+\tilde{K}_{0,\,\varepsilon})^{-1}\tilde{K}_{s,\,\varepsilon},\ s \geq 1, \quad (1+\tilde{K}_{0,\,\varepsilon})^{-1}, \quad s=0$$

Collecting the terms for α small enough we easily obtain

$$(\tilde{K}_{\alpha,\,\varepsilon}+1)^{-1} = M_{0,\,\varepsilon} - \alpha M_{1,\,\varepsilon}M_{0,\,\varepsilon} + \alpha^2(M_{1,\,\varepsilon}^2 - M_{2,\,\varepsilon})M_{0,\,\varepsilon} +$$

$$+\alpha^3(M_{1,\,\varepsilon}M_{2,\,\varepsilon} + M_{2,\,\varepsilon}M_{1,\,\varepsilon} - M_{3,\,\varepsilon} - M_{1,\,\varepsilon}^3)M_{0,\,\varepsilon} + O(\alpha^4).$$

We multiply out the expansions derived above for each of the three factors contributing to the operator D and estimate the matrix elements $D_{i,j} = \langle \mathbf{w}_0^{i\,*}, D\mathbf{w}_0^j\rangle,\ i,j = 1,...,N.$ After long but straightforward computations using *Step III* of Lemma 15, the basic properties of the projection operators $(I-P_0)\mathbf{w}_0^i = 0$ and $(I-P_0^*)\mathbf{w}_0^{i\,*} = 0,\ i = 1,...,N$ along with identities (3.8) and (3.9) we arrive at the statement of the lemma.

□

Armed with Lemmas 15 and 16 we prove Theorem 4.

Proof of Theorem 4. By means of the lemmas mentioned above equation (3.11) is equivalent to

$$detW(\varepsilon,\alpha) = 0,$$

where

$$W_{i,j}(\varepsilon,\alpha) = \varepsilon(\hat{V}_p)_{i,j} + \alpha^2\hat{B}_{i,j} - \frac{\alpha^3}{24\pi}(\hat{S}^*\hat{S})_{i,j} + \varepsilon^2(D_0)_{i,j} +$$

$$+\alpha\varepsilon^2(D_1)_{i,j} + \alpha^2\varepsilon(D_2)_{i,j} + \alpha^3\varepsilon(D_3)_{i,j} + O(\alpha^4),\ i,j = 1,...,N$$

and $O(\alpha^4)$ is analytic in α for α small enough. The terms proportional to $\alpha^2\varepsilon$ and $\alpha^3\varepsilon$ from $U_{i,j}$ were absorbed by $(D_2)_{i,j}$ and $(D_3)_{i,j}$. By the standard argument of the perturbation theory (see e.g. Chapter 15 of [17]) we obtain

$$\varepsilon\lambda_p^i + \alpha_i^2\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle - \frac{\alpha_i^3}{24\pi}|\hat{S}\mathbf{v}_p^i|^2 + O(\alpha_i^4) = 0, \quad i = 1,...,N.$$

A straightforward computation yields the expression for the spectral parameter

$$\alpha_i = -\frac{\varepsilon}{48\pi}\frac{|\hat{S}\mathbf{v}_p^i|^2\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle^2} \pm \sqrt{\varepsilon}\sqrt{\frac{-\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle}} + O(\varepsilon^{\frac{3}{2}}), \quad i = 1,...,N.$$

Thus we have the following possibilities.

(i) $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle < 0$ and $\lambda_p^i > 0$ or $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle > 0$ and $\lambda_p^i < 0.$ The situation when the spectral parameter has a positive real part corresponds to the eigenvalues bifurcating along the real line having the asymptotics

$$z_i = \omega + \varepsilon\frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle} + O(\varepsilon^{\frac{3}{2}}), \quad i = 1,...,N_{B^-}^+ + N_{B^+}^-.$$

When the spectral parameter moves to the second sheet of the Riemann surface, the eigenvalues near the edge of the essential spectrum of the linearized NLS operator are not being produced.

(ii) $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle < 0$ and $\lambda_p^i < 0$. Hence Re$\alpha_i > 0$ and the eigenvalues bifurcate in the complex plane becoming the complex conjugate pairs having the asymptotics

$$z_i = \omega + \varepsilon \frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle} \pm i \frac{\varepsilon^{\frac{3}{2}}}{24\pi} \frac{|\hat{S}\mathbf{v}_p^i|^2 \lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle^2} \sqrt{\frac{\lambda_p^i}{\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle}} + O(\varepsilon^2), \quad i = 1, ..., N_{B^-}^-.$$

(iii) $\langle \mathbf{v}_p^i, \hat{B}\mathbf{v}_p^i\rangle > 0$ and $\lambda_p^i > 0$. In this case Re$\alpha_i < 0$ and therefore N_{B+}^+ eigenvalues will disappear from the neighborhood of the endpoint of the essential spectrum of the linearized NLS operator under perturbation.

□

Proof of Theorem 3. Though the threshold resonance solution by its definition $(1+|x|^2)^{-\frac{s}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ with $s > \frac{1}{2}$, $\boldsymbol{\psi}_0 \notin L^2(\mathbb{R}^3, \mathbb{C}^2)$, satisfying

$$\boldsymbol{\psi}_0 = -R_0 Q_0 \boldsymbol{\psi}_0$$

analogously to (3.8), the weighted one $\mathbf{w}_0 = |Q_0|^{\frac{1}{2}}\boldsymbol{\psi}_0 \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ and therefore it is the eigenfunction of the unperturbed Birman-Schwinger principle corresponding to the eigenvalue -1. The projection P_0 onto this eigenspace is defined analogously to Definition 13 with $N = 1$. As distinct from the eigenvalue case, for the resonance solution $C_0 = \sqrt{4\pi}$ (see the beginning of Part 3) and we easily obtain

$$R_1 Q_0 \boldsymbol{\psi}_0 = (-\frac{1}{\sqrt{4\pi}}, 0)^T.$$

Using the identities above along with the property of the projection operator $(I - P_0)\mathbf{w}_0 = 0$ we compute the expansions of the first and the second term of the determinant of the Feshbach map analogously to Lemmas 15 and 16. Note that the terms of the form which were studied in Step III of Lemma 15 will not be vanishing in the threshold resonance case. We derive

$$U_{1,1} = \langle \mathbf{w}_0^*, U\mathbf{w}_0\rangle = -1 - 2\varepsilon s_0 \langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle - \alpha s_0 - \frac{\alpha s_0 \varepsilon}{\sqrt{4\pi}} \int_{\mathbb{R}^3} s_p(x) dx + O(\alpha^2)$$

and

$$D_{1,1} = \varepsilon^2 (D_0)_{1,1} + \alpha\varepsilon (D_1)_{1,1} + O(\alpha^2),$$

where $s_p(x) = V_{+,p}(x)u_0(x) + V_{-,p}(x)w_0(x)$ and $s_0 = sign\langle sign Q_0 \mathbf{w}_0, \mathbf{w}_0\rangle$. Then we arrive at

$$W(\varepsilon, \alpha) = \varepsilon\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle + \frac{\alpha}{2} + \varepsilon^2 (D_0)_{1,1} + \alpha\varepsilon (D_1)_{1,1} + O(\alpha^2) = 0.$$

The term proportional to $\alpha\varepsilon$ from $U_{1,1}$ was absorbed by $(D_1)_{1,1}$. Hence in the case of the threshold resonance $\varepsilon = O(\alpha)$ or explicitly

$$\alpha = -2\varepsilon\langle \mathbf{u}_0, V_p \mathbf{u}_0\rangle + O(\varepsilon^2).$$

Therefore, when $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle > 0$ we have Re$\alpha < 0$ and $\sigma_p(\mathcal{L})$ does not contain eigenvalues in the neighborhood of $z = \omega$. In the situation when $\langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle < 0$ and thus Re$\alpha > 0$ the asymptotics of the eigenvalues bifurcating from the edge of the essential spectrum are given by

$$z(\varepsilon) = \omega - 4\varepsilon^2 \langle \mathbf{u}_0, V_p \mathbf{u}_0 \rangle^2 + o(\varepsilon^2).$$

□

Acknowledgement

The author thanks I.M.Sigal for suggesting the problem, for valuable discussions, for partial support via NSERC grant No. 7901, Gang Zhou, M.Merkli and D.Pelinovsky for stimulating discussions.

References

[1] S. Adachi. *A Positive solution of a nonhomogeneous elliptic equation in $\mathbb{R}^N$ with G-invariant nonlinearity*. Comm. PDE., 27 (2002), No. 1-2, 1–22.

[2] V. Bach, J. Fröhlich, I.M. Sigal. *Renormalization group analysis of spectral problems in quantum field theory*. Adv. Math., 137 (1998), No. 2, 205–298.

[3] H. Berestycki, P.-L. Lions. *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., 82 (1983), No. 4, 313–345.

[4] H. Berestycki, P.-L. Lions. *Nonlinear scalar field equations. II. Existence of infinitely many solutions*. Arch. Rational Mech. Anal., 82 (1983), No. 4, 347–375.

[5] H. Berestycki, P.-L. Lions, L. Peletier. *An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$*. Indiana Univ. Math. J., 30 (1981), No. 1, 141–157.

[6] V.S. Buslaev, G.S. Perelman. *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton*. St. Petersburg Math. J., 4 (1993), No. 6, 1111–1142.

[7] V.S. Buslaev, C. Sulem. *On asymptotic stability of solitary waves for nonlinear Schrödinger equations*. Ann. Inst. H. Poincare Anal. Non Lineaire, 20 (2003), No. 3, 419–475

[8] S. Cuccagna. *On asymptotic stability of ground states of NLS*, Rev. Math. Phys., 15 (2003), No. 8, 877–903.

[9] S.-M. Chang, S. Gustafson, K. Nakanishi, T.-P. Tsai. *Spectra of linearized operators for NLS solitary waves*. SIAM J. Math. Anal., 39 (2007), No. 4, 1070–1111.

[10] S. Cuccagna, D. Pelinovsky. *Bifurcations from the endpoints of the essential spectrum in the linearized nonlinear Schrödinger problem*. J. Math. Phys., 46 (2005), No. 5, 053520, 15 pp.

[11] B. Erdogan, W. Schlag. *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II*. J. Anal. Math., 99 (2006), 199–248.

[12] A. Floer; A. Weinstein. *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*. J. Funct. Anal., 69 (1986), No. 3, 397–408.

[13] S. Cuccagna, D. Pelinovsky, V. Vougalter. *Spectra of positive and negative energies in the linearized NLS problem*. Comm. Pure Appl. Math., 58 (2005), No. 1, 1–29.

[14] M. Grillakis. *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*. Comm. Pure Appl. Math., 43 (1990), No. 3, 299–333.

[15] S. Gustafson, I.M. Sigal. Mathematical concepts of quantum mechanics. Springer–Verlag, Berlin, 2003.

[16] Z. Gang, I.M. Sigal. *Asymptotic stability of nonlinear Schrödinger equations with potential*, Rev. Math. Phys., 17 (2005), No. 10, 1143–1207.

[17] P.D. Hislop, I.M. Sigal. Introduction to spectral theory with applications to Schrödinger operators. Springer, 1996.

[18] T. Kapitula, B. Sandstede. *Edge bifurcations for near integrable systems via Evans functions techniques*. SIAM J. Math.Anal., 33 (2002), No. 5, 1117–1143.

[19] T. Kapitula, B. Sandstede. *Eigenvalues and resonances using the Evans functions*. Discrete Contin. Dyn. Syst., 10 (2004), No. 4, 857–869.

[20] M. Klaus, B. Simon. *Coupling constant thresholds in nonrelativistic quantum mechanics. I. Short-range two-body case*. Ann. Phys., 130 (1980), No. 2, 251–281.

[21] E. Lieb, M. Loss. Analysis. Graduate studies in Mathematics, 14. American Mathematical Society, Providence, 1997.

[22] E. Lieb, B. Simon, A. Wightman. Book "Studies in mathematical physics: Essays in Honor of Valentine Bargmann." Princeton University Press, 1976.

[23] K. McLeod. *Uniqueness of positive radial solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^n$. *II*. Trans. Amer. Math. Soc., 339 (1993), No. 2, 495–505.

[24] D. Pelinovsky, Y. Kivshar, V. Afanasjev. *Internal modes of envelope solitons*, Phys. D, 116 (1998), No. 1–2, 121–142.

[25] G. Perelman. *Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations*. Comm. Partial Differential Equations, 29 (2004), No. 7–8, 1051–1095.

[26] W. Strauss. *Existence of solitary waves in higher dimensions*. Comm.Math.Phys., 55 (1977), No. 2, 149–162.

[27] B. Simon. Functional integration and quantum physics. Pure and Applied Mathematics, 86 (1979), Academic Press.

[28] W. Schlag. *Stable manifolds for an orbitally unstable nonlinear Schrödinger equation*. Ann. of Math. (2), 169 (2009), No. 1, 139–227.

[29] V. Vougalter. *On the negative index theorem for the linearized NLS problem*. To appear in Canad. Math. Bull.

[30] V. Vougalter, D. Pelinovsky. *Eigenvalues of zero energy in the linearized NLS problem*. J. Math. Phys., 47 (2006), No. 6, 062701, 13 pp.

[31] M.I. Weinstein. *Modulation stability of ground states of nonlinear Schrödinger equations*. SIAM J. Math. Anal., 16 (1985), No. 3, 472–491.