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Influence of Vibrations on Convective Instability of Reaction Fronts in Porous Media

H. Aatif, K. Allali* and K. El Karouni

Department of Mathematics, University Hassan II, M.A.C. Laboratory,
P.O. Box 146, FST-Mohammadia, Morocco

Abstract. The aim of this paper is to study the effect of vibrations on convective instability of reaction fronts in porous media. The model contains reaction-diffusion equations coupled with the Darcy equation. Linear stability analysis is carried out and the convective instability boundary is found. The results are compared with direct numerical simulations.

Key words: linear stability analysis, reaction fronts, porous medium, numerical simulations
AMS subject classification: 35K30, 35Q57, 76S05, 76E15

1. Introduction

There are numerous studies that show that high frequency vibrations can influence stability of various convective flows. Vibrations were used to suppress the Rayleigh-Taylor instability in a container [15, 16, 17, 22]. Oscillations can also have a stabilizing effect for low frequencies and a destabilizing effect for high ones [20]. In the case of a porous medium saturated by a fluid, the effect of vertical vibrations on thermal stability of a conductive solution is studied in [26]; for other directions of vibration, depending on the vibrational parameter and the angle of vibration, stabilizing and destabilizing effects are possible [27]. Mechanical and thermal vibrations have also been studied in connection with the Rayleigh-Bénard convection [10, 11], directional solidification [12, 21], and doubly diffusive convection [9]. In spite of numerous results on the influence of vibrations on convective instability, some questions still remain open. In particular, normal vibrations cannot stabilize the conductive state in an unbounded domain [23], while tangential vibration

*Corresponding author. E-mail: allali@fstm.ac.ma

is only effective for vibration frequencies that are not too large [14].

Propagating reaction fronts are intensively studied in relation with numerous physical, chemical, and biological problems (see, e.g., [2, 5, 6, 7, 8, 13, 18, 19]). Convective instability of reaction fronts in porous media without vibrations is studied in [1]. Vibration can have stabilizing or destabilizing effect on reaction fronts depending on its amplitude and frequency [1, 3, 4]. In [1] this question is studied in the case of Navier-Stokes equations.

In this work we consider reaction fronts in a porous medium with the fluid motion described by the Darcy law. The paper is organized as follows. The problem is formulated in Section 2, followed in Section 3 by linear stability analysis. We use a narrow reaction zone approximation developed in combustion theory. At the end of this section we give some numerical results. Direct numerical simulations and comparison with the stability analysis are presented in Section 4.

2. Governing equations

2.1. The model

We consider an upward propagating reaction front in a porous medium filled by an incompressible reacting fluid. We model it with a reaction-diffusion system of equations coupled with the Darcy law. In order to study the influence of vibrations on convective instability of the reaction front, we impose a harmonic oscillation in the vertical direction. Therefore the fluid is submitted to the gravitational acceleration g and the periodic oscillations $\lambda sin(\nu t)\gamma$. Here λ and ν stand respectively for the amplitude and the frequency of vibration, γ is the upward unit vector. We arrive to the following equations:

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \kappa \Delta T + qK(T)\phi(\alpha), \tag{2.1}$$

$$\frac{\partial \alpha}{\partial t} + \mathbf{v}.\nabla \alpha = d\Delta \alpha + K(T)\phi(\alpha), \tag{2.2}$$

$$\mathbf{v} + \frac{K}{\mu}\nabla p = \frac{g\beta K}{\mu}\rho(T - T_0)\gamma(1 + \lambda sin(\nu t)), \tag{2.3}$$

$$\nabla.\mathbf{v} = 0. \tag{2.4}$$

This system is considered in the two-dimensional space. It is supplemented by the following conditions at infinity:

$$T = T_i \, , \alpha = 1 \text{ and } v = 0 \text{ when } y \to +\infty, \tag{2.5}$$

$$T = T_b \, , \alpha = 0 \text{ and } v = 0 \text{ when } y \to -\infty. \tag{2.6}$$

Here T is the temperature, α the depth of conversion, $\mathbf{v} = (v_x, v_y)$ the fluid velocity, p the pressure, κ the coefficient of thermal diffusivity, d the diffusion, q the adiabatic heat release, g the gravity acceleration, ρ is the density, β denotes the coefficient of thermal expansion, μ the viscosity. In addition, T_0 is the mean value of temperature, T_i is an initial temperature while T_b is the temperature of the burned mixture given by $T_b = T_i + q$. The function $K(T)\phi(\alpha)$ is the reaction rate where the temperature dependence is given by the Arrhenius exponent:

$$K(T) = k_0 \exp\Big(-\frac{E}{R_0 T}\Big), \tag{2.7}$$

E is the activation energy, R_0 the universal gas constant and k_0 the pre-exponential factor. For the asymptotic analysis of this problem we will assume that the activation energy is large and will consider zero order reaction for which

$$\phi(\alpha) = \begin{cases} 1 \text{ if } \alpha < 1 \\ 0 \text{ if } \alpha = 1 \end{cases} . \tag{2.8}$$

In the case of direct numerical simulations, it is more convenient to consider the first order reaction, $\phi(\alpha) = 1 - \alpha$. The qualitative behavior of reaction fronts for the zero and first order kinetics is similar.

2.2. The dimensionless model

In order to write down the dimensionless model, we now introduce the spatial variables $x' = \frac{xc_1}{\kappa}$, $y' = \frac{yc_1}{\kappa}$, time $t' = \frac{tc_1^2}{\kappa d}$, velocity $\frac{\mathbf{v}}{c_1}$, pressure $\frac{p\kappa\mu}{K}$ with $c_1 = c/\sqrt{2}$ and frequency $\sigma = \frac{\kappa}{c_1^2}\nu$. Denoting $\theta = \frac{T - T_b}{q}$ and keeping for convenience the same notation for the other variables, we obtain the system

$$\frac{\partial\theta}{\partial t} + v\nabla\theta = \Delta\theta + W_Z(\theta)\,\phi(\alpha), \tag{2.9}$$

$$\frac{\partial\alpha}{\partial t} + v\nabla\alpha = \Lambda\Delta\alpha + W_Z(\theta)\,\phi(\alpha), \tag{2.10}$$

$$v + \nabla p = R_p(\theta + \theta_0)\begin{pmatrix}0\\1\end{pmatrix}(1 + \lambda sin(\sigma t)), \tag{2.11}$$

$$div(v) = 0 \tag{2.12}$$

with the following conditions at infinity:

$$\theta = -1\,,\ \alpha = 0 \text{ and } v = 0 \text{ when } y \to +\infty, \tag{2.13}$$

$$\theta = 0\,,\ \alpha = 1 \text{ and } v = 0 \text{ when } y \to -\infty. \tag{2.14}$$

Here $\Lambda = d/\kappa$ is the inverse of the Lewis number, $R_p = \frac{Kc_1^2P^2R}{\mu^2}$, where R is the Rayleigh number and P the Prandtl number that are defined by $R = \frac{g\beta q\kappa^2}{\mu c_1^3}$ and $P = \frac{\mu}{\kappa}$. In addition, we use the parameters $\delta = \frac{R_0 T_b}{E}$ and $\theta_0 = \frac{T_b - T_0}{q}$. The reaction rate is given by:

$$W_Z(\theta) = Z \exp\Big(\frac{\theta}{Z^{-1} + \delta\theta}\Big), \tag{2.15}$$

where $Z = \dfrac{qE}{R_0 T_b^2}$ stands for Zeldovich number.

The linear stability analysis will be carried out in the case of zero Lewis number ($\Lambda = 0$). This situation corresponds to a liquid mixture. In the direct direct numerical simulations, we will consider sufficiently small but nonzero values of the Lewis number.

3. Linear stability analysis

3.1. Approximation of infinitely narrow reaction zone

To study the problem analytically, we reduce it to a singular perturbation problem where the reaction zone is supposed to be infinitely narrow and the reaction term is neglected outside the reaction zone. This method is called Zeldovich - Frank-Kamenetskii approximation. It is a well-known approach for combustion problems [24, 25]. We will carry out a formal asymptotic analysis with $\epsilon = \dfrac{1}{Z}$ taken as a small parameter, and will obtain a closed interface problem. Let us denote by $\zeta(t, x)$ the location of the reaction zone in the laboratory frame reference. The new independent variable in the direction of the front propagation is given by

$$y_1 = y - \zeta(t, x). \tag{3.1}$$

We introduce new functions θ_1, α_1, $\mathbf{v}_1$, p_1:

$$\begin{aligned} &\theta(t, x, y) = \theta_1(t, x, y_1), \quad \alpha(t, x, y) = \alpha_1(t, x, y_1), \\ &\mathbf{v}(t, x, y) = \mathbf{v}_1(t, x, y_1), \quad p(t, x, y) = p_1(t, x, y_1). \end{aligned} \tag{3.2}$$

We re-write the equations in the form (the index 1 for the independent variables is omitted):

$$\frac{\partial\theta}{\partial t} - \frac{\partial\theta}{\partial y_1}\frac{\partial\zeta}{\partial t} + \mathbf{v}.\widetilde{\nabla}\theta = \widetilde{\Delta}\theta + W_Z(\theta)\phi(\alpha), \tag{3.3}$$

$$\frac{\partial\alpha}{\partial t} - \frac{\partial\alpha}{\partial y_1}\frac{\partial\zeta}{\partial t} + \mathbf{v}.\widetilde{\nabla}\alpha = W_Z(\theta)\phi(\alpha), \tag{3.4}$$

$$\mathbf{v} + \widetilde{\nabla}p = R_p(\theta + \theta_0)(1 + \lambda sin(\sigma t))\gamma, \tag{3.5}$$

$$\frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial y_1}\frac{\partial\zeta}{\partial x} + \frac{\partial v_y}{\partial y_1} = 0, \tag{3.6}$$

where we have set

$$\widetilde{\Delta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_1^2} - 2\frac{\partial\zeta}{\partial x}\frac{\partial^2}{\partial x\partial y_1} + \Big(\frac{\partial\zeta}{\partial x}\Big)^2\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial}{\partial y_1}, \tag{3.7}$$

$$\widetilde{\nabla} = \Big(\frac{\partial}{\partial x} - \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\Big). \tag{3.8}$$

We use matched asymptotic expansions. We look for the outer solution of the problem in the form of expansion

$$\begin{aligned} &\theta = \theta^0 + \epsilon\theta^1 + ..., \quad \alpha = \alpha^0 + \epsilon\alpha^1 + ...,\\ &\mathbf{v} = \mathbf{v}^0 + \epsilon\mathbf{v}^1 + ..., \quad p = p^0 + \epsilon p^1 + ... \end{aligned} \tag{3.9}$$

Here $(\theta^0, \alpha^0, \mathbf{v}^0)$ is a dimensionless form of the basic solution.

In order to obtain jump conditions in the reaction zone we consider the inner problem and we introduce the stretching coordinate $\eta = y_1/\epsilon$, with $\epsilon = 1/Z$. We look for the inner solution in the form of the expansion

$$\begin{aligned} &\theta = \epsilon\tilde{\theta}^1 + ..., \quad \alpha = \tilde{\alpha}^0 + \epsilon\tilde{\alpha}^1 + ...,\\ &\mathbf{v} = \tilde{\mathbf{v}}^0 + \epsilon\tilde{\mathbf{v}}^1 + ..., \quad p = \tilde{p}^0 + \epsilon\tilde{p}^1 + ..., \quad \zeta = \tilde{\zeta}^0 + \epsilon\tilde{\zeta}^1 + ... \end{aligned} \tag{3.10}$$

Substituting these expansion into (3.3)-(3.6), we obtain the first-order inner problem:

$$\Big(1 + \Big(\frac{\partial\tilde{\zeta}^0}{\partial x}\Big)^2\Big)\frac{\partial^2\tilde{\theta}^1}{\partial\eta^2} + \exp\Big(\frac{\tilde{\theta}^1}{1+\delta\tilde{\theta}^1}\Big)\phi(\tilde{\alpha}^0) = 0, \tag{3.11}$$

$$-\frac{\partial\tilde{\alpha}^0}{\partial\eta}\frac{\partial\tilde{\zeta}^0}{\partial\eta} - \frac{\partial\tilde{\alpha}^0}{\partial\eta}\Big(\tilde{v}_x^0\frac{\partial\tilde{\zeta}^0}{\partial x} - \tilde{v}_y^0\Big) = \exp\Big(\frac{\tilde{\theta}^1}{1+\delta\tilde{\theta}^1}\Big)\phi(\tilde{\alpha}^0), \tag{3.12}$$

$$\frac{\partial\tilde{p}^0}{\partial\eta} = 0, \tag{3.13}$$

$$\tilde{v}_x^0 + \frac{\partial\tilde{p}^0}{\partial x} - \frac{\partial\tilde{\zeta}^0}{\partial t}\frac{\partial\tilde{p}^1}{\partial\eta} = 0, \tag{3.14}$$

$$\tilde{v}_y^0 + \frac{\partial\tilde{p}^1}{\partial\eta} = -R_p\theta_0(1+\lambda sin(\sigma t)), \tag{3.15}$$

$$-\frac{\partial\tilde{v}_x^0}{\partial\eta}\frac{\partial\tilde{\zeta}^0}{\partial x} + \frac{\partial\tilde{v}_y^0}{\partial\eta} = 0. \tag{3.16}$$

Then the matching conditions are

$$\eta \to +\infty: \quad \tilde{\theta}^1 \sim \theta^1|_{y_1=0+} + \eta\frac{\partial\theta^0}{\partial y_1}|_{y_1=0+}, \quad \tilde{\alpha}^0 \to 0, \quad \tilde{\mathbf{v}}^0 \to \mathbf{v}^0|_{y_1=0+}, \tag{3.17}$$

$$\eta \to -\infty: \quad \tilde{\theta}^1 \to \theta^1|_{y_1=0-}, \quad \tilde{\alpha}^0 \to 1, \quad \tilde{\mathbf{v}}^0 \to \mathbf{v}^0|_{y_1=0-}. \tag{3.18}$$

From (3.13) we obtain that $\tilde{p}^0$ does not depend on η, which implies that the pressure is continuous through the interface. Next, denoting by s the quantity

$$s = \tilde{v}_x^0\frac{\partial\tilde{\zeta}^0}{\partial x} - \tilde{v}_y^0, \tag{3.19}$$

we obtain from (3.16) that s does not depend on η. Finally from (3.14), (3.15) and (3.19) we easily obtain that $\tilde{v}_x^0$ and $\tilde{v}_y^0$ do not depend on η, which provides the continuity of the velocity through the interface.

We next derive the jump conditions for the temperature from (3.11), in the same way as it is usually done for combustion problems. From (3.12) it follows that $\tilde{\alpha}^0$ is a monotone function and $0 < \tilde{\alpha}^0 < 1$. Since we consider zero-order reaction, we have $\phi(\tilde{\alpha}^0) \equiv 1$. We conclude from (3.11) that $\tilde{\theta}^1$ is also a monotone function. Thus, multiplying (3.11) by $\dfrac{\partial\tilde{\theta}^1}{\partial\eta}$ and integrating, we obtain

$$\Big(\frac{\partial\tilde{\theta}^1}{\partial\eta}\Big)^2\Big|_{\eta=+\infty} - \Big(\frac{\partial\tilde{\theta}^1}{\partial\eta}\Big)^2\Big|_{\eta=-\infty} = -\frac{2}{A}\int_{-\infty}^{\theta^1}\exp(\frac{\tau}{1+\delta\tau})d\tau, \tag{3.20}$$

where we have set

$$A = 1 + \Big(\frac{\partial\tilde{\zeta}^0}{\partial x}\Big)^2. \tag{3.21}$$

Next, subtracting (3.11) from (3.12) and integrating, we obtain

$$\frac{\partial\tilde{\theta}^1}{\partial\eta}\Big|_{\eta=+\infty} - \frac{\partial\tilde{\theta}^1}{\partial\eta}\Big|_{\eta=-\infty} = -\frac{1}{A}\Big(\frac{\partial\tilde{\zeta}^0}{\partial t} + s\Big). \tag{3.22}$$

Using now the matching conditions and truncating the expansion:

$$\theta^0 \approx \theta, \;\; \theta^1|_{y_1=0-} \approx Z\theta|_{y_1=0} \;\; \zeta^0 \approx \zeta, \;\; \mathbf{v} \approx \mathbf{v}^0, \tag{3.23}$$

we obtain the jump conditions

$$\Big(\frac{\partial\theta}{\partial y_1}\Big)^2\Big|_{y_1=0+} - \Big(\frac{\partial\theta}{\partial y_1}\Big)^2\Big|_{y_1=0-} = 2Z\Big(1 + \Big(\frac{\partial\zeta}{\partial x}\Big)^2\Big)^{-1}\int_{-\infty}^{\theta|_{y_1=0}}\exp(\frac{\tau}{Z^{-1}+\delta\tau})d\tau, \tag{3.24}$$

$$\frac{\partial\theta}{\partial y_1}\Big|_{y_1=0+} - \frac{\partial\theta}{\partial y_1}\Big|_{y_1=0-} = -\Big(1 + \Big(\frac{\partial\zeta}{\partial x}\Big)^2\Big)^{-1}\Big(\frac{\partial\zeta}{\partial t} + (v_x\frac{\partial\zeta}{\partial x} - v_y)\Big|_{y_1=0}\Big). \tag{3.25}$$

3.2. Formulation of the interface problem

Let us summarize the interface problem. We have for $y > \zeta$ (in the unburnt medium)

$$\frac{\partial\theta}{\partial t} + \mathbf{v}.\nabla\theta = \Delta\theta, \tag{3.26}$$

$$\alpha \equiv 0, \tag{3.27}$$

$$\mathbf{v} + \nabla p = R_p(\theta + \theta_0)(1 + \lambda sin(\sigma t))\gamma, \tag{3.28}$$

$$\nabla.\mathbf{v} = 0. \tag{3.29}$$

The equations in the burnt medium ($y < \zeta$) lead to the following system:

$$\frac{\partial\theta}{\partial t} + \mathbf{v}.\nabla\theta = \Delta\theta, \tag{3.30}$$

$$\alpha \equiv 1, \tag{3.31}$$

$$\mathbf{v} + \nabla p = R_p(\theta + \theta_0)(1 + \lambda sin(\sigma t))\gamma, \tag{3.32}$$

$$\nabla.\mathbf{v} = 0. \tag{3.33}$$

We finally complete this system by the following jump conditions at the interface $y = \zeta$:

$$[\theta] = 0, \quad \left[\frac{\partial \theta}{\partial y}\right] = \frac{\frac{\partial \zeta}{\partial t}}{1 + \left(\frac{\partial \zeta}{\partial x}\right)^2}, \tag{3.34}$$

$$\left[\left(\frac{\partial \theta}{\partial y}\right)^2\right] = -\frac{2Z}{1 + (\frac{\partial \zeta}{\partial x})^2} \int_{-\infty}^{\theta(\zeta)} \exp\left(\frac{s}{1/Z + \delta s}\right) ds, \tag{3.35}$$

$$[\mathbf{v}] = 0. \tag{3.36}$$

Here we denote by [ ] the quantity

$$[f] = f|_{\zeta - 0} - f|_{\zeta + 0}. \tag{3.37}$$

The above free boundary problem is completed with the conditions at infinity:

$$y \to +\infty, \quad \theta = -1 \text{ and } \mathbf{v} = 0, \tag{3.38}$$

$$y \to -\infty, \quad \theta = 0 \text{ and } \mathbf{v} = 0. \tag{3.39}$$

3.3. Travelling wave solution

In this subsection we perform the linear analysis of the steady-state solution for the interface problem. This problem has a travelling wave solution:

$$\theta(t, x, y) = \theta_s(y - ut), \quad \alpha(t, x, y) = \alpha_s(y - ut) \text{ and } \mathbf{v} = 0, \tag{3.40}$$

where

$$\theta_s(t, y) = \begin{cases} 0 & \text{if } y < 0 \\ e^{-uy} - 1 & \text{if } y > 0 \end{cases}, \tag{3.41}$$

and

$$\alpha_s(t, y) = \begin{cases} 1 & \text{if } y < 0 \\ 0 & \text{if } y > 0 \end{cases}. \tag{3.42}$$

Here the number u stands for the wave speed. It can easily be computed thanks to the jump conditions of the free boundary problem.

We now introduce the coordinates in the moving frame defined by $y_1 = y - ut$. In this referential, the above travelling wave is a stationary solution of the problem

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial y} + \mathbf{v}.\nabla \theta = \Delta \theta, \tag{3.43}$$

$$\mathbf{v} + \nabla p = R_p(\theta + \theta_0)(1 + \lambda sin(\sigma t))\gamma, \tag{3.44}$$

$$\nabla.\mathbf{v} = 0, \tag{3.45}$$

together with the jump condition found in the previous subsection.

We now consider a small perturbation of this stationary solution. For that purpose we consider a perturbation of the reaction front of the form

$$\zeta(t, x) = ut + \xi(t, x), \text{ with } \xi(t, x) = \epsilon_1(t)e^{ikx}. \tag{3.46}$$

We study the stability of the solution of our problem. We look for a solution of the problem in the form of the perturbed stationary solution:

$$\theta = \theta_s + \tilde{\theta}, \ \ v = v_s + \tilde{v}, \tag{3.47}$$

where

$$\begin{aligned} \tilde{\theta}(t, x, y) &= \theta_j(y, t)e^{ikx}, \text{ for } j = 1, 2, \\ \tilde{v}(t, x, y) &= v_j(y, t)e^{ikx}, \text{ for } j = 1, 2. \end{aligned} \tag{3.48}$$

Here the index $j = 1$ corresponds to functions for $z < 0$ and $j = 2$ for $z > 0$.

We exclude the pressure p and the component v_x of the velocity from the interface problem applying two times the operator $curl$. Thus we obtain the following problem:

For the burnt media ($y < 0$):

$$v_1'' - k^2 v_1 = -R_p k^2(1 + \lambda \sin(\sigma t))\theta_1, \tag{3.49}$$

$$\frac{\partial \theta_1}{\partial t} - \theta_1'' - u\theta_1' + k^2\theta_1 = 0. \tag{3.50}$$

For the unburnt media ($y > 0$):

$$v_2'' - k^2 v_2 = -R_p k^2 \theta_2(1 + \lambda \sin(\sigma t)), \tag{3.51}$$

$$\frac{\partial \theta_2}{\partial t} - \theta_2'' - u\theta_2' + k^2\theta_2 = u \exp(-uy)v_2, \tag{3.52}$$

where u stands for the stationary front velocity. Taking into account that

$$\theta\,|_{\xi=\pm 0} = \theta_s(\pm 0) + \xi\theta_s'(\pm 0) + \tilde{\theta}(\pm 0), \tag{3.53}$$

and

$$\frac{\partial \theta}{\partial y}\,|_{\xi=\pm 0} = \theta_s'(\pm 0) + \xi\theta_s''(\pm 0) + \frac{\partial \tilde{\theta}}{\partial y}(\pm 0), \tag{3.54}$$

we obtain the following jump conditions:

$$\theta_2(0,t) - \theta_1(0,t) = u\epsilon_1(t), \tag{3.55}$$

$$\theta_2'(0,t) - \theta_1'(0,t) = -\epsilon_1(t)u^2 - \epsilon_1'(t) + v_1(0,t), \tag{3.56}$$

$$\epsilon_1(t)u^2 + \theta_2'(0,t) = -\frac{Z}{u}\theta_1(0,t), \tag{3.57}$$

$$v_2^{(i)}(0,t) = v_1^{(i)}(0,t) \quad i = 0, 1. \tag{3.58}$$

3.4. Stability boundary

To find the convective instability boundary, we solve numerically the problem (3.49)-(3.52) with the jump conditions (3.55)-(3.58). The numerical accuracy is controlled by decreasing the time and space steps.

For fixed Z and k we vary R_p. If the Rayleigh number R_p is less than a critical value R_c, then solution is decreasing in time. If $R_p > R_c$, it increases, and for $R_p = R_c$ it is periodic in time (Figure 1, right). Similar behavior is observed in the case without vibrations (Figure 1, left). When the Rayleigh number exceed its critical value the perturbation grows in time and when the same parameter is bellow its critical value the perturbation decays. There are no oscillations because the amplitude of vibrations is zero.

Figures 2 shows the critical value of the Rayleigh number as a function of the amplitude of vibrations for different frequencies. If $\lambda = 0$, we obtain the same value $R_c = 26$ as without vibrations [1]. For small positive λ, vibrations stabilize the solution: R_c is an increasing function. For larger λ, vibrations destabilize the solution: R_c is a decreasing function. When we increase the frequency σ, the front becomes more stable.

Figures 3 shows the critical value of the Rayleigh number as a function of the frequency of vibrations for different amplitudes. If $\lambda = 0$, the curve take a constant value $R_c = 26$, it corresponds to no vibration case. If $\lambda \neq 0$, all curves are increasing function, i.e. when the frequency increases the front become more stable. We see also that all curves have a asymptotic behaviour when the frequency is sufficiently large. High frequency can stabilize the front.

4. Direct numerical simulations

4.1. The model with stream function

This subsection is devoted to numerical simulations of our problem with the first order reaction. In the case of large activation energies E, we neglect the small parameter δ. Numerical computations are carried out using the stream function associated to the velocity. Due to incompressibility of the medium we can introduce the stream function ψ defined by the equalities:

$$v = (v_x; v_y) = (\partial_y\psi; -\partial_x\psi). \tag{4.1}$$

We re-write the system (2.9)-(2.14) in the following form:

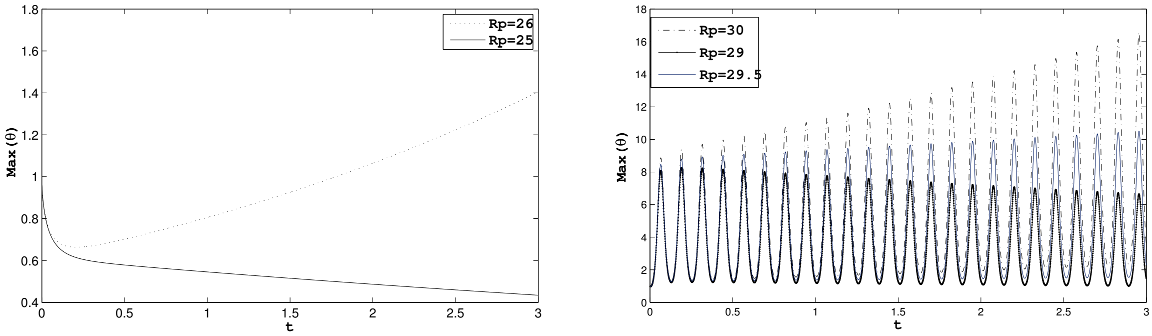

Figure 1: Temperature maximum as a function of time for $k = 3.14$, $Z = 8$, $u = 1.4142$, $\lambda = 0$ (left) and for $k = 3.14$, $Z = 8$, $u = 1.4142$, $\lambda = 5$, $\sigma = 50$ (right).

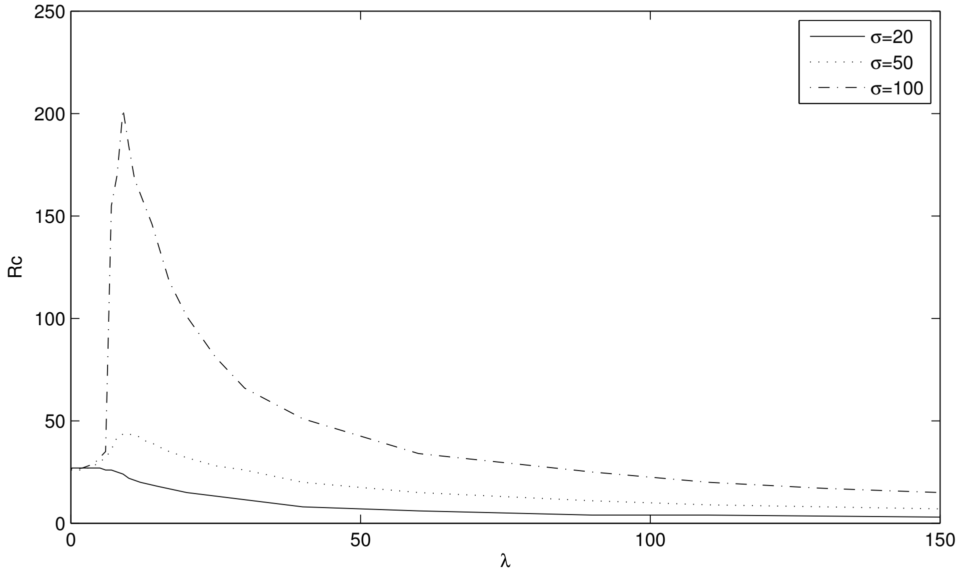

Figure 2: Convective instability boundary: critical Rayleigh number as a function of the amplitude of vibrations for $k = 3.14$, $Z = 8$ and $u = 1.4142$ and for different values of the frequency σ.

$$\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = \Delta\theta + Z^2 \exp\left(Z\theta\right)\left(1-\alpha\right), \tag{4.2}$$

$$\frac{\partial\alpha}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\alpha}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\alpha}{\partial y} = \Lambda\Delta\alpha + Z^2 \exp\left(Z\theta\right)\left(1-\alpha\right), \tag{4.3}$$

$$-\Delta\psi = R_p\frac{\partial\theta}{\partial x}(1+\lambda sin(\sigma t)). \tag{4.4}$$

The conditions at infinity:

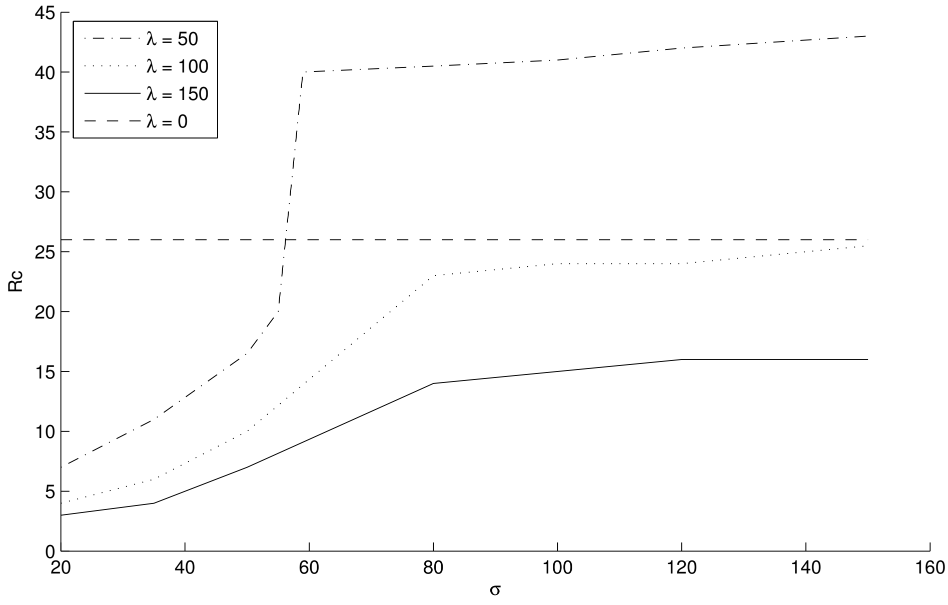

Figure 3: Convective instability boundary: critical Rayleigh number as a function of the frequency of vibrations for $k = 3.14$, $Z = 8$ and $u = 1.4142$ and for different values of the amplitude λ.

$$\theta = 0, \ \ \alpha = 1 \ \text{ when } \ y \to -\infty, \tag{4.5}$$

$$\theta = -1, \ \ \alpha = 0, \ \text{ when } \ y \to +\infty. \tag{4.6}$$

This problem is studied numerically in a finite strip $\Omega = (0; L_x) \times (0; L_y)$ with the noflux boundary conditions for the temperature and concentration and with the zero boundary condition for the function ψ. The initial condition is given by:

$$\theta(0, x, y) = \theta_0(y), \ \alpha(0, x, y) = \alpha_0(y), \ \psi(0, x, y) = 0, \tag{4.7}$$

where the functions θ_0 and α_0 are some step functions.

4.2. Numerical results

We use a semi-implicit finite difference approximation and an alternating direction method for the heat and diffusion equations and the fast Fourier transform for the vorticity equation. The values of parameters are the same as in [1]: $Z = 8$, $\Lambda = 0.1$, $L_x = 2$ and $L_y = 35$. The inverse of the Lewis number is chosen sufficiently small in order to compare with the linear stability analysis where it equals zero. The width of the reactor is chosen in such way that it corresponds the wave number $k = 3.14$ in the linear stability analysis.

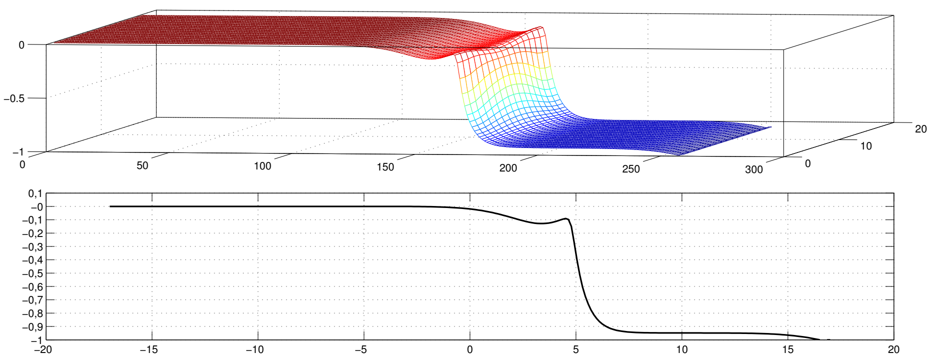

Figure 4: Temperature distribution for $R_p = 27$ at $t = 2$.

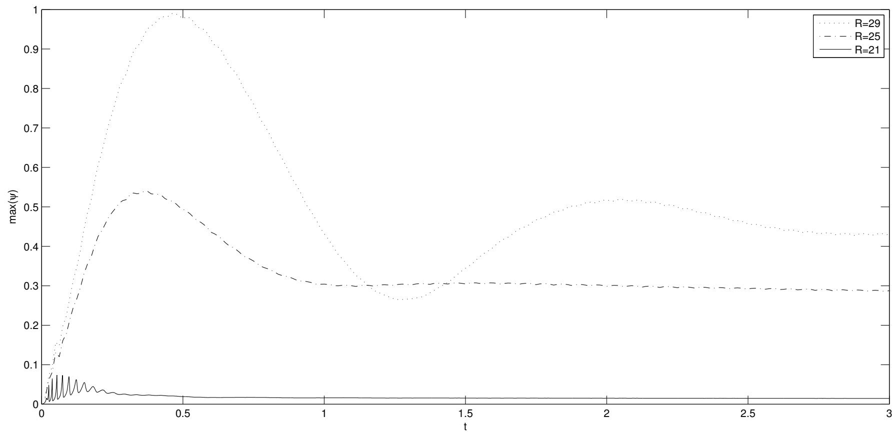

Figure 5: Stream function maximum as function of time for $\lambda = 0$.

The heat release due to the chemical reaction can lead to a nonhomogeneous temperature distribution and to convective instability (Figure 4). Figure 5 shows the stream function maximum as function of time for $\lambda = 0$ and for different values of Rayleigh number. When the Rayleigh number increases, the maximum of the stream function also increases. Beginning from the value $R_p = 21$, the stream function becomes nonzero. Convective rolls appear near the reaction front. (Figure 6). The isotherms in the same figure show that the temperature distribution is not uniform with respect to y and it is not monotone. We note that the critical value of the Rayleigh number found by the linear stability analysis equals $R_c = 26$. So the numerical and analytical results are in a good agreement.

We next study the influence of vibrations. Figure 7 shows that the maximum of the stream function decreases in time for $R_p = 22$, $\lambda = 5$ and $\sigma = 50$. Without vibrations convection persists. Therefore vibrations stabilize the front. Both, direct numerical simulations and linear stability analysis show that vibrations stabilize the front for small amplitude and destabilize the front for high ones.

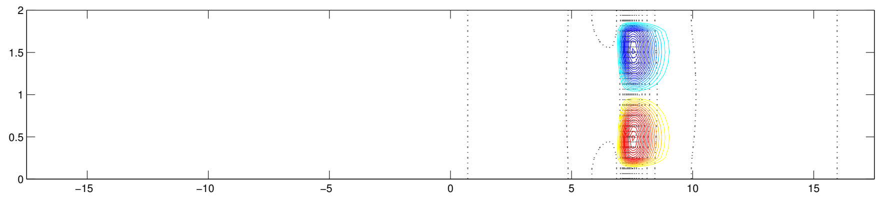

Figure 6: Level lines of the stream function (solid lines) and of the temperature (dashed lines) for $R_p = 22$, $\lambda = 0$ at $t = 3$.

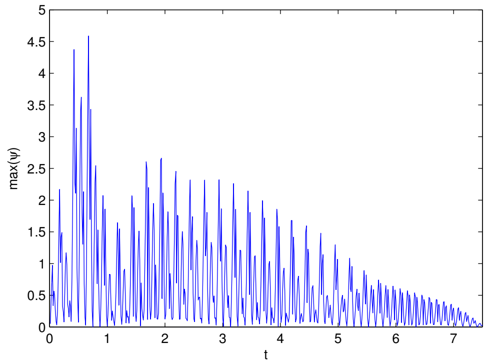

Figure 7: Stream function maximum as function of time for $R_p = 22$, $\lambda = 5$ and $\sigma = 50$.

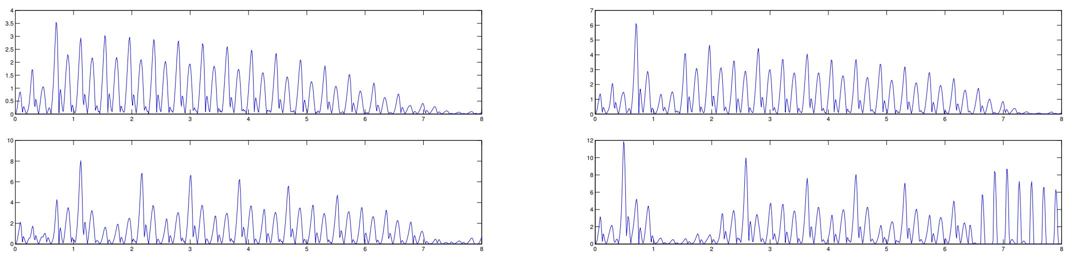

Figure 8: Stream function maximum as function of time for $R_p = 25$ (top left), $R_p = 30$ (top right), $R_p = 35$ (bottom left), $R_p = 40$ (bottom right), $\lambda = 2$ and $\sigma = 30$.

Figure 8 shows the stream function maximum for different values of Rayleigh number and fixed frequency and amplitude of vibrations. The stream function decays for $R_p = 25, 30, 35$, for $R_p = 40$, it does not decay.

Thus, linear stability analysis and direct numerical simulations show that vibrations can influence convective instability of reaction fronts.

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