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Bilevel Approach of a Decomposed Topology Optimization Problem

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Abstract. In topology optimization problems, we are often forced to deal with large-scale numerical problems, so that the domain decomposition method occurs naturally. Consider a typical topology optimization problem, the minimum compliance problem of a linear isotropic elastic continuum structure, in which the constraints are the partial differential equations of linear elasticity. We subdivide the partial differential equations into two subproblems posed on non-overlapping sub-domains. In this paper, we consider the resulting problem as multilevel one and show that it can be written as one level problem.

Key words: topology optimization, domain decomposition method, compliance, multilevel optimization
AMS subject classification: 74P15, 35Q93, 65K10, 78M50

1. Introduction

The topology optimization has for objective to find an optimal shape without any a priori assumption about its topology. Therefore, it is not yet largely widespread in the industry, the principal reason is that the resulting problem is a large scale optimization problem. However, the parallel computers knew a great evolution, in particular in computing power and storage capacity and Domain decomposition method is a valuable approach when solving partial differential equation problems on parallel computers. Applying this method to the minimum compliance problem, we

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have obtained the following optimization problem [1]:

$$\begin{cases} \min\limits_{\rho_1,\rho_2} l(u_1,u_2,g^*), \\ \min\limits_{g} \mathcal{J}_\delta(u_1,u_2,g), \\ a_{\rho_1}(u_1,v_1) = (f,v_1)_{\Omega_1} + (g,v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d, \\ a_{\rho_2}(u_2,v_2) = (f,v_2)_{\Omega_2} - (g,v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d, \end{cases} \tag{1.1}$$

with the constraints on the variables ρ_1, ρ_2: $\sum_{i=1}^{2}\int_{\Omega_i}\rho_i(x)d\Omega_i \leq V$, $0 < \rho_{min} \leq \rho_i(x) \leq 1$ for all $x \in \Omega_i$ for $i = 1,2$ which is an optimization problem that is constrained by another optimization problem parameterized in the design variables ρ_1 and ρ_2 corresponding to the sub-domains Ω_1 and Ω_2 respectively and Γ_0 is the interface between the two sub-domains. Such a problem is considered as bilevel optimization problem [2]. In fact, on one hand, the functions $l(u_1,u_2,g)$ and $\mathcal{J}_\delta(u_1,u_2,g)$ can't be minimized simultaneously, so, bicriteria optimization is no more suitable, and the hierarchical nature of the two levels is a natural justification for this choice.

One often tool used to reformulate the bilevel programming problem as one level problem are the Karush-Kuhn-Tucker (KKT) conditions [3]. If the regularity condition is satisfied for the lower level problem, then the KKT conditions are necessary optimality conditions. They are also sufficient in the case when the lower level problem is convex, this is the approach used in this paper.

2. Mathematical analysis of our bilevel problem

Consider the lower level problem:

$$\begin{aligned} &\min_{g} \mathcal{J}_\delta(u_1,u_2,g), \\ &a_{\rho_1}(u_1,v_1) = (f,v_1)_{\Omega_1} + (g,v_1)_{\Gamma_0} \quad \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d, \\ &a_{\rho_2}(u_2,v_2) = (f,v_2)_{\Omega_2} - (g,v_2)_{\Gamma_0} \quad \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d, \end{aligned} \tag{2.1}$$

without inequality constraints in the lower level problem, the resulting single level problem after the KKT transformation lacks complementarity conditions which introduce non-convexities even if the original problem is linear. In our case, the convex nature of the inner problem (2.1) enable us to apply the Karush-Kuhn-Tucker approach to transform the bilevel problem (1.1) into a single one and derive the KKT optimality conditions. For the lower level problem (2.1), the following Lagrangian is defined:

$$\begin{aligned} \mathcal{L}_{low}(u_1,u_2,g,\lambda_1,\lambda_2) &= \mathcal{J}_\delta(u_1,u_2,g) - a_{\rho_1}(u_1,\lambda_1) + (f,\lambda_1)_{\Omega_1} + (g,\lambda_1)_{\Gamma_0} \\ &\quad - a_{\rho_2}(u_2,\lambda_2) + (f,\lambda_2)_{\Omega_2} - (g,\lambda_2)_{\Gamma_0}, \end{aligned}$$

where $(u_1,u_2,g,\lambda_1,\lambda_2) \in H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d \times L^2(\Gamma_0)^d \times H^1_{\Gamma_1}(\Omega_1)^d \times H^1_{\Gamma 2}(\Omega_2)^d$. Then, a necessary and sufficient condition for g^* to be an optimal solution to the inner level problem is given that: $\dfrac{\partial \mathcal{L}_{low}}{\partial g}(u_1^*,u_2^*,g^*,\lambda_1^*,\lambda_2^*) = 0$ gives the optimality condition. For $i = 1,2$,

$\dfrac{\partial \mathcal{L}_{low}}{\partial u_i}(u_1^*, u_2^*, g^*, \lambda_1^*, \lambda_2^*) = 0$ gives the adjoint, or co-state equations $\dfrac{\partial \mathcal{L}_{low}}{\partial \lambda_i}(u_1^*, u_2^*, g^*, \lambda_1^*, \lambda_2^*) = 0$ gives the state equation in each sub-domain that is respectively:

$$\begin{array}{rcll} (g^*, r)_{\Gamma_0} & = & -\dfrac{1}{\delta}(\lambda_1^* - \lambda_2^*, r)_{\Gamma_0}, & \forall r \in L^2(\Gamma_0)^d, \\ a_{\rho_1}(\xi, \lambda_1^*) & = & (u_1^* - u_2^*, \xi)_{\Gamma_0}, & \forall \xi \in H^1_{\Gamma_0}(\Omega_1)^d, \\ a_{\rho_2}(\xi, \lambda_2^*) & = & -(u_1^* - u_2^*, \xi)_{\Gamma_0}, & \forall \xi \in H^1_{\Gamma_0}(\Omega_2)^d, \\ a_{\rho_1}(u_1^*, v_1) & = & (f, v_1)_{\Omega_1} + (g^*, v_1)_{\Gamma_0}, & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d, \\ a_{\rho_2}(u_2^*, v_2) & = & (f, v_2)_{\Omega_2} - (g^*, v_2)_{\Gamma_0}, & \forall v_2 \in H^1_{\Gamma 2}(\Omega_2)^d. \end{array} \tag{2.2}$$

The equations (2.2) are the Kuhn-Tucker necessary optimality conditions for the problem (2.1), and under the convexity assumptions of the Lagrangian function, they are sufficient conditions for (u_1^*, u_2^*, g^*) to be an optimal solution of the problem (2.1). It follows that a necessary conditions for $(\bar{\rho}_1, \bar{\rho}_2, \bar{u}_1, \bar{u}_2, \bar{\lambda}_1, \bar{\lambda}_2, \bar{g})$ to be an optimal solution of the bilevel problem (1.1), $(\bar{u}_1, \bar{u}_2, \bar{\lambda}_1, \bar{\lambda}_2, \bar{g})$ must satisfy the above conditions at fixed $(\rho_1, \rho_2) = (\bar{\rho}_1, \bar{\rho}_2)$, thus the bilevel programming problem (1.1) is transformed into a single level problem of the form:

$$\begin{array}{rcll} & & \min\limits_{\rho_1, \rho_2} l(u_1, u_2, g), & \\ a_{\rho_1}(u_1, v_1) & = & (f, v_1)_{\Omega_1} + (g, v_1)_{\Gamma_0}, & \forall v_1 \in H^1_{\Gamma_1}(\Omega_1)^d, \\ a_{\rho_2}(u_2, v_2) & = & (f, v_2)_{\Omega_2} - (g, v_2)_{\Gamma_0}, & \forall v_2 \in H^1_{\Gamma_2}(\Omega_2)^d, \\ a_{\rho_1}(\xi, \lambda_1) & = & (u_1 - u_2, \xi)_{\Gamma_0}, & \forall \xi \in H^1_{\Gamma_0}(\Omega_1)^d, \\ a_{\rho_2}(\xi, \lambda_2) & = & -(u_1 - u_2, \xi)_{\Gamma_0}, & \forall \xi \in H^1_{\Gamma_0}(\Omega_2)^d, \\ (g, r)_{\Gamma_0} & = & -\dfrac{1}{\delta}(\lambda_1 - \lambda_2, r)_{\Gamma_0}, & \forall r \in L^2(\Gamma_0)^d, \end{array} \tag{2.3}$$

with the constraints on ρ_1 and ρ_2 cited bellow.

3. The optimality system

In this section, we derive an optimality system by applying a Lagrange multiplier rule. Define the Lagrange function for the single level problem (2.3) as follows ($\iota = 1,\ 2$):
$\mathcal{L}(u_\iota, \rho_\iota, g, \bar{u}_\iota, \lambda_\iota, \tilde{\lambda}_\iota, \tilde{r}, \Lambda, \lambda_\iota^+(x), \lambda_\iota^-(x)) = l(u_1, u_2, g) - (a_{\rho_1}(u_1, \bar{u}_1) - (f, \bar{u}_1)_{\Omega_1} - (g, \bar{u}_1)_{\Gamma_0}) - (a_{\rho_2}(u_2, \bar{u}_2) - (f, \bar{u}_2)_{\Omega_2} + (g, \bar{u}_2)_{\Gamma_0}) - (a_{\rho_1}(\lambda_1, \tilde{\lambda}_1) - (u_1 - u_2, \tilde{\lambda}_1)_{\Gamma_0}) - (a_{\rho_2}(\lambda_2, \tilde{\lambda}_2) + (u_1 - u_2, \tilde{\lambda}_2)_{\Gamma_0}) - ((g, \bar{r})_{\Gamma_0} + \frac{1}{\delta}(\lambda_1 - \lambda_2, \bar{r})_{\Gamma_0}) + \Lambda(\sum_{\iota=1}^2 \int_{\Omega_\iota} \rho_\iota(x) d\Omega_\iota - V) + \sum_{\iota=1}^2 \int_{\Omega_\iota} \lambda_\iota^+(x)(\rho_\iota(x) - 1) d\Omega_\iota + \sum_{\iota=1}^2 \int_{\Omega_\iota} \lambda_\iota^-(x)(\rho_{min} - \rho_\iota(x)) d\Omega_\iota,$
where, $\bar{u}_\iota$ are the Lagrange multipliers for the equilibrium constraints, $\tilde{\lambda}_\iota$ and $\tilde{r}$ are also Lagrange multipliers corresponding to the co-state equations and optimality condition for the lower level problem, the Lagrange multipliers Λ, $\lambda_\iota^+(x)$ and $\lambda_\iota^-(x)$ are related to the constraints in ρ_ι, $\iota = 1, 2$. The KKT optimality condition for the problem (2.3) can be formally interpreted as stationary

points of $\mathcal{L}(\cdots)$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u_\iota} &= 0 \Rightarrow u_\iota = \bar{u}_\iota, && && (3.1)\\
\frac{\partial \mathcal{L}}{\partial \lambda_\iota} &= 0 \Rightarrow a_{\rho_\iota}(\tilde{\lambda}_\iota, \check{\lambda}_\iota) = (-1)^\iota \frac{1}{\delta}(\tilde{r}, \check{\lambda}_\iota)_{\Gamma_0}, && \forall \check{\lambda}_\iota \in H^1_{\Gamma_i}(\Omega_\iota)^d, && (3.2)\\
\frac{\partial \mathcal{L}}{\partial g} &= 0 \Rightarrow (2g_0, u_1 - u_2)_{\Gamma_0} = (g_0, \tilde{r})_{\Gamma_0}, && \forall g_0 \in L^2(\Gamma_0)^d, && (3.3)
\end{aligned}
$$

the two last equations yields

$$\tilde{\lambda}_\iota = \lambda_\iota, \tag{3.4}$$

then combining (3.1) and (3.4), we have the following optimality condition for ρ_ι:

$$\frac{\partial \mathcal{L}}{\partial \rho_\iota} = 0 \Rightarrow p\rho_\iota(x)^{p-1} E^0_{ijkl}(\varepsilon_{ij}(u_\iota)\varepsilon_{kl}(u_\iota) + \varepsilon_{ij}(\lambda_\iota)\varepsilon_{kl}(\lambda_\iota)) = \Lambda + \lambda^+_\iota - \lambda^-_\iota, \tag{3.5}$$

for intermediate densities, namely, bound constraints are not active $(\rho_{min} < \rho_\iota < 1)$, the corresponding multipliers $(\lambda^-_\iota, \lambda^+_\iota)$ are equal to zero and the optimality condition (3.5) simplifies to: $p\rho_\iota(x)^{p-1} E^0_{ijkl}(\varepsilon_{ij}(u_\iota)\varepsilon_{kl}(u_\iota) + \varepsilon_{ij}(\lambda_\iota)\varepsilon_{kl}(\lambda_\iota)) = \Lambda$. Then we have the following update scheme for the density ρ_i [4]:

$$\rho_{\iota,k+1} = \begin{cases} \max\{(1-\zeta)\rho_{\iota,k}, \rho_{min}\} & \text{if} \quad \rho_{\iota,k} B^\eta_k \leq \max\{(1-\zeta)\rho_{\iota,k}, \rho_{min}\}, \\ \min\{(1+\zeta)\rho_{\iota,k}, 1\} & \text{if} \quad \min\{(1+\zeta)\rho_{\iota,k}, 1\} \leq \rho_{\iota,k} B^\eta_k, \\ \rho_{\iota,k} B^\eta_k & \quad\quad \text{otherwise}. \end{cases} \tag{3.6}$$

$\rho_{\iota,k}$ denotes the value of the density variable in the sub-domain Ω_ι at iteration step k, and B_k is given by the expression $B_k = \Lambda_k^{-1} p\rho_\iota(x)^{p-1} E^0_{ijkl}(\varepsilon_{ij}(u_{\iota,k})\varepsilon_{kl}(u_{\iota,k}) + \varepsilon_{ij}(\lambda_{\iota,k})\varepsilon_{kl}(\lambda_{\iota,k}))$ where $u_{\iota,k}$ and $\lambda_{\iota,k}$ are determined from known equations, η is a tuning parameter and ζ a move limit, their values are chosen by numerical experiment. The Lagrange multiplier for the volume constraint, Λ_k is determined at each iteration using a bisection algorithm. We use the gradient method to solve the finding system.

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