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Enlarged Asymptotic Compensation in Discrete Distributed Systems

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Abstract. This work concerns an enlarged analysis of the problem of asymptotic compensation for a class of discrete linear distributed systems. We study the possibility of asymptotic compensation of a disturbance by bringing asymptotically the observation in a given tolerance zone $\mathcal{C}$. Under convenient hypothesis, we show the existence and the unicity of the optimal control ensuring this compensation and we give its characterization.

Key words: enlarged asymptotic compensation, control, observation, tolerance zone
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1. Problem statement

We consider a class of linear distributed systems [4-8] described by the following discrete equation:

$$(S_d)\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k\ ; \quad k \geq 0, \\ z_0 \in X, \end{cases} \tag{1.1}$$

where $\phi \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$, z_k, $f_k \in X$ and $u_k \in U$ are respectively the state, the disturbance, the control at step k. X and U are supposed to be Hilbert spaces. The system (1.1) is completed by the output equation

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$$(E_d)\ \ y_k = Cz_k\ ;\ k \geq 0, \tag{1.2}$$

where $C \in \mathcal{L}(X,Y)$. Y is the observation space, a Hilbert space.
In the case where $f = 0$ and $u = 0$, the observation at step N is given by $y_N = C\phi^N z_0$, but if $f \neq 0$, generally $y_N = C\phi^N z_0 + \sum_{i=0}^{N-1} C\phi^{N-1-i} f_i \neq C\phi^N z_0$. Then we show how to find a convenient input operator B ensuring asymptotically the reduction of the effect of any disturbance by bringing the observation in a given tolerance zone $\mathcal{C}$, i.e. such that for any $f = (f_k)_{k\geq 0} \in l^2(X)$, there exists a control $u = (u_k)_{k\geq 0} \in l^2(U)$ satisfying:

$$\sum_{k\geq 0} C\phi^k Bu_k + \sum_{k\geq 0} C\phi^k f_k \in \mathcal{C}, \tag{1.3}$$

where $\mathcal{C}$ is a convex, closed and nonempty subset of Y. This leads to the following definition.

Definition 1. *A disturbance $f \in l^2(X)$ is said to be $\mathcal{C}$-remediable asymtotically if there exists a control $u \in l^2(U)$ such that*

$$K^{\infty}_{C,d}u + R^{\infty}_{C,d}f \in \mathcal{C},$$

where $K^{\infty}_{C,d}$ and $R^{\infty}_{C,d}$ are respectively the operators defined by

$$K^{\infty}_{C,d}u = \sum_{k\geq 0} C\phi^k Bu_k \qquad and \qquad R^{\infty}_{C,d}f = \sum_{k\geq 0} C\phi^k f_k.$$

If the system is exponentially stable, i.e. $\parallel \phi^k \parallel \leq Me^{-k\omega}\ ;\ k \geq 0$, $M > 0$ and $\omega > 0$, then $K^{\infty}_{C,d}$ and $R^{\infty}_{C,d}$ are well defined, but this condition is not necessary [3].

2. Enlarged asymptotic remediability with minimum energy

We consider the following problem (P_d) of enlarged asymptotic compensation with minimum energy

$$(P_d)\left\{\begin{array}{l} \min J(u) \qquad \text{with } J(u) = \|u\|^2_{l^2(U)}\,, \\ \qquad \text{subject to } K^{\infty}_{C,d}u + R^{\infty}_{C,d}f \in \mathcal{C}. \end{array}\right. \tag{2.1}$$

We suppose that the disturbance f is $\mathcal{C}$-remediable asymptotically, the problem (P_d) is then well defined and admits a unique solution v^* in the set of admissible controls defined by

$$U_{ad} = \left\{u \in l^2(U)/K^{\infty}_{C,d}u + R^{\infty}_{C,d}f \in \mathcal{C}\right\}.$$

The considered problem will be resolved using an extension of the Hilbert Uniqueness Method (HUM) and a penalization method. Indeed, for $\theta \in Y' \equiv Y$, let

$$\|\theta\|_{F^{\infty}_d} = \left(\sum_{k\geq 0} \left\|B^*\left(\phi^*\right)^k C^*\theta\right\|^2_{U'} dt\right)^{\frac{1}{2}}.$$

$\|.\|_{F_d^\infty}$ is a semi-norm on Y. If $Ker((R_{C,d}^\infty)^*) = \{0\}$, then $\|.\|_{F_d^\infty}$ is a norm on Y if and only if $(S_d) + (E_d)$ is weakly remediable asymptotically [1, 2, 3]. Practically, this may be interpreted as a weak remediability at a sufficiently large step N. We consider

$$F_d^\infty = \overline{Y}^{\|\cdot\|_{F_d^\infty}}.$$

F_d^∞ is a Hilbert space with the inner product

$$\langle \theta, \sigma \rangle_{F_d^\infty} = \sum_{k \geq 0} \left\langle B^* \left(\phi^*\right)^k C^* \theta, B^* \left(\phi^*\right)^k C^* \sigma \right\rangle \; ; \; \forall \, \theta, \; \sigma \in F_d^\infty,$$

and the operator $\Lambda_{C,d}^\infty$ defined on Y by

$$\Lambda_{C,d}^\infty \, \theta = \sum_{k \geq 0} C \phi^k B B^* \left(\phi^*\right)^k C^* \theta = K_{C,d}^\infty (K_{C,d}^\infty)^* \theta,$$

admits a unique extension as an isomorphism from F_d^∞ to $(F_d^\infty)'$ such that

$$\left\langle \Lambda_{C,d}^\infty \, \theta, \sigma \right\rangle_Y = \langle \theta, \sigma \rangle_{F_d^\infty} \; ; \; \forall \, \theta, \sigma \in F_d^\infty \quad and \quad \left\| \Lambda_{C,d}^\infty \theta \right\|_{(F_d^\infty)'} = \|\theta\|_{F_d^\infty} \; ; \; \forall \, \theta \in F_\infty.$$

The following main result shows how to find the optimal control ensuring the enlarged asymptotic compensation of a disturbance f.

Theorem 2. *If $\mathcal{C}$ is a closed convex subset of Y with nonempty interior and if*

$$\overset{\circ}{\mathcal{C}} \cap (R_{C,d}^\infty f + (F_d^\infty)') \neq \emptyset,$$

then

i) *There exists a unique θ_f in F_d^∞ such that*

$$\Lambda_{C,d}^\infty \theta_f + R_{C,d}^\infty f \in \mathcal{C}, \tag{2.2}$$

and satisfying

$$\left\langle \theta_f, y - \Lambda_{C,d}^\infty \theta_f - R_{C,d}^\infty f \right\rangle \geq 0 \; ; \; \forall y \in \mathcal{C} \cap (R_{C,d}^\infty f + (F_d^\infty)'). \tag{2.3}$$

ii) *The control u_{θ_f} defined by*

$$u_{\theta_f} = \left(K_{C,d}^\infty\right)^* \theta_f, \tag{2.4}$$

is the unique solution of the problem (P_d). Moreover, u_{θ_f} is optimal with

$$\left\| u_{\theta_f} \right\|_{l^2(U)}^2 = \|\theta_f\|_{F_d^\infty}^2 .$$

Proof.
Unicity: It can be easily proved by the Hilbert Uniqueness Method.
Existence: Its proof is based on a combination of HUM and a penalization method. The initial problem (P_d) is firstly approximated by a finite time problem (P_N), then for N sufficiently large, we consider a convenient criterion depending on another parameter $\alpha > 0$. We establish intermediary and convergence results respectively when $\alpha \longrightarrow 0$ and $N \longrightarrow +\infty$, which lead to the solution of the problem (P_d). First, let us remark that the condition

$$\overset{\circ}{\mathcal{C}} \cap (R^{\infty}_{C,d} f + (F^{\infty}_d)') \neq \emptyset,$$

implies that there exists N_0 large enough such that the convex $\mathcal{C}$ is reached at any step $N \geq N_0$. For $N \geq N_0$, we consider the following minimization problem

$$(P_N) \begin{cases} \min \|u\|^2, \\ \text{subject to } \; y^N_{u,f} = K^N_{C,d} u + R^N_{C,d} f \in \mathcal{C}, \end{cases} \tag{2.5}$$

where $K^N_{C,d} u = \sum_{k=0}^{N-1} C\phi^k B u_k$ and $R^N_{C,d} f = \sum_{k=0}^{N-1} C\phi^k f_k$. For $\theta \in Y' \equiv Y$. Let

$$\|\theta\|^2_{F_{N,\alpha}} = \sum_{k=0}^{N-1} \left\| B^* (\phi^*)^k C^* \theta \right\|^2_U + \frac{\alpha^2}{N} \|\theta\|^2_Y ,$$

where $\|\cdot\|_{F_{N,\alpha}}$ is a norm for any $N \geq 1$ and any $\alpha > 0$. Let us note that for N large enough and α sufficiently small, we have for $\theta \in Y$

$$\|\theta\|^2_{F_{N,\alpha}} \simeq \sum_{k=0}^{N-1} \left\| B^* (\phi^*)^k C^* \theta \right\|^2_U = \|\theta\|^2_{F_{N,0}} ,$$

where $\|.\|_{F_{N,0}}$ is a norm under the usual condition of weak asymptotic remediability at a sufficiently large step N. We consider the space

$$F_{N,\alpha} = \overline{Y}^{\|\cdot\|_{F_{N,\alpha}}}.$$

$F_{N,\alpha}$ is a Hilbert space with the inner product

$$\langle \theta, \sigma \rangle_{F_{N,\alpha}} = \sum_{k=0}^{N-1} \left\langle B^* (\phi^*)^k C^* \theta, B^* (\phi^*)^k C^* \sigma \right\rangle_U + \frac{\alpha^2}{N} \langle \theta, \sigma \rangle_Y ,$$

and the operator $\Lambda^N_{C,d}$ defined on Y by

$$\Lambda^N_{C,d} \theta = \sum_{k=0}^{N-1} C\phi^k B B^* (\phi^*)^k C^* \theta \; + \frac{\alpha^2}{N} \theta,$$

admits a unique extension as an isomorphism $F_{N,\alpha} \longrightarrow F'_{N,\alpha}$ satisfying

$$\left\langle \Lambda^N_{C,d}\theta, \sigma \right\rangle_Y = \langle \theta, \sigma \rangle_{F_{N,\alpha}} \; ; \; \forall\, \theta, \sigma \in F_{N,\alpha}.$$

For $N \geq N_0$, we consider the following criterion

$$J_{N,\alpha}(y, v) = \frac{1}{\alpha} \left\| y^N_{v,f} - y \right\|^2 + \|v\|^2 \, ,$$

with $y \in Y$, $v \in U^N$ and $\alpha > 0$. We consider the following minimization problem

$$(P_{N,\alpha}) \left\{ \begin{array}{c} \min J_{N,\alpha}(y, v) \\ y \in \mathcal{C} \text{ and } v \in U^N. \end{array} \right. \tag{2.6}$$

The problem $(P_{N,\alpha})$ admits a solution $(y_{N,\alpha}, v_{N,\alpha})$ characterized by

$$\langle b_{N,\alpha}, y - y_{N,\alpha} \rangle \geq 0 \; ; \forall y \in \mathcal{C}, \tag{2.7}$$

$$v_{N,\alpha} = B^*(\phi^*)^k C^* b_{N,\alpha} \; ; k = 0, \cdots, N-1 \tag{2.8}$$

where $b_{N,\alpha}$ is given by

$$b_{N,\alpha} = \frac{1}{\alpha}(y_{N,\alpha} - y^N_{v_{N,\alpha},f}). \tag{2.9}$$

The sequence $(y_{N,\alpha}, v_{N,\alpha}, b_{N,\alpha})_{\alpha>0}$ is bounded in $Y \times U^N \times F_{N,0}$. Then we can extract a convergent subsequence, its limit (y^*_N, v^*_N, b_N) is characterized by

1- $y^*_N = y^N_{v^*_N,f}$.
2- v^*_N is a solution of (P_N) given by (2.5).
3- The control $v^*_N = \left(v^*_{N,0}, ..., v^*_{N,N-1}\right) \in U^N$ is given by

$$v^*_{N,k} = B^*(\phi^*)^k C^* b_N \; ; k = \; 0, ..., N-1, \tag{2.10}$$

4- The element b_N is characterized by

$$\left\langle b_N, y - y^N_{v^*_N,f} \right\rangle \geq 0 \; ; \forall y \in \mathcal{C} \cap (R^N_{C,d} f + F'_{N,0}). \tag{2.11}$$

Now, concerning the convergence when $N \longrightarrow +\infty$, the sequence $(y^*_N, v^*_N, b_N)_N$ is bounded in $Y \times l^2(U) \times F_{N_0,0}$. Then we can extract a convergent subsequence, its limit (y^*, v^*, b) is characterized by

1- The control $v^* = (v^*_k)_{k\geq 0} \in l^2(U)$ is given by

$$v^*_k = B^*(\phi^*)^k C^* b \; ; k \geq \; 0, \tag{2.12}$$

2- v^* is a solution of (P_d),
3- $y^* = y^\infty_{v^*,f}$.

4- The element b is characterized by

$$\left\langle b, y - y^{\infty}_{v^*,f} \right\rangle \geq 0\,;\, \forall y \in \mathcal{C} \times (R^{\infty}_{C,d} f + (F^{\infty}_{d})'). \tag{2.13}$$

The result derive immediately with $\theta_f = b$ and $u_{\theta_f} = v^*$.

Let us note finally, that the system (1.1) augmented by the output (1.2) can be considered as a discrete version of a continuous system. Hence, for example in the case of a one dimension diffusion system with a Dirichlet boundary condition, all the operators and the considered problem are well defined. Consequently, the obtained results may be applied. But in the case of a Neumann boundary condition, generally K^{∞}_{d} and R^{∞}_{d} are not well defined. However, with a convenient choice of the input and output operators, the considered approach remain true. The results are similar in a higher space dimension (for example if Ω is a rectangle or a parallepiped).

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