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Local Parameterization and the Asymptotic Numerical Method

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Abstract. The Asymptotic Numerical Method (ANM) is a family of algorithms, based on computation of truncated vectorial series, for path following problems [2]. In this paper, we present and discuss some techniques to define local parameterization [4, 6, 7] in the ANM. We give some numerical comparisons of pseudo arc-length parameterization and local parameterization on non-linear elastic shells problems.

Key words: asymptotic numerical method, continuation methods, local parameterization
AMS subject classification: 78M35, 74B20, 74G10

1. Introduction

The Asymptotic Numerical Method (ANM) is a family of algorithms for path following problems [2]. The principle is simply to expand the unknown in power series with respect to a path parameter. The range of validity is deduced from the computation of the truncated vectorial series. So, the step lengths are computed a posteriori. Using the evaluation of the series at the end of the interval of validity, we obtain a new starting point and we define, in this way, the ANM continuation procedure. This continuation method, based on ANM, has proved to be an efficient method to compute solution of non-linear partial differential equations [2]. The step lengths depend on the definition of the path parameter. In [2], we have used the pseudo arc-length parameterization. Another choice, the so-called local parameterization [6, 7] has been examined by J. J. Gervais et al [4] in the framework of the ANM. In this paper, we propose new strategies for the choice of the

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local parameterization which can lead to larger step lengths. In the second part, we remind the basis of ANM and we introduce new methods to define the auxiliary equation corresponding to local parameterization. In the third part, these new strategies, for the choice of local parameterization, are applied on some examples from non-linear elastic shells.

2. Local parameterizations in the ANM

Let us consider the following class of non-linear quadratic problems:

$$\mathbf{R}(\mathbf{U},\lambda) = \mathbf{L}(\mathbf{U}) + \mathbf{Q}(\mathbf{U},\mathbf{U}) - \lambda\mathbf{F} = 0, \tag{2.1}$$

where $\mathbf{L}(.)$ and $\mathbf{Q}(.,.)$ are linear and quadratic operators, $\mathbf{F}$ is a given vector and $\mathbf{R}$ is the so-called residual vector. In the case of thin elastic shell equilibrium equations, the unknowns are $\mathbf{U} = (\mathbf{u}, \mathbf{S})$ and λ; $\mathbf{u}$ is the displacement and $\mathbf{S}$ is the second Piola Kirchhoff stress tensor and λ is the load parameter. In this paper, we limit ourselves to the quadratic framework (2.1), more difficult problems can be found in [2]. The ANM has proved to be an efficient method to compute solution path of (2.1). The principle is simply to expand the unknown $(\mathbf{U},\lambda)$ with respect to a path parameter a.

$$\begin{cases} \mathbf{U}(a) &= \mathbf{U}^j + a\mathbf{U}_1 + a^2\mathbf{U}_2 + ... + a^N\mathbf{U}_N, \\ \lambda(a) &= \lambda^j + a\lambda_1 + a^2\lambda_2 + ... + a^N\lambda_N, \end{cases} \tag{2.2}$$

where $(\mathbf{U}^j,\lambda^j)$ is a known and regular solution of (2.1) and N is the order of truncation of the series. In this way, the equation (2.2) defines the solution branch j (one step of the ANM). The definition of the path parameter leads to a well-defined problem. Various choices are possible, the most used in the ANM algorithm, is the pseudo arc-length parameterization (PAL) [2]:

$$a = < \mathbf{T}, \mathbf{V} >, \tag{2.3}$$

where $< .,. >$, is a scalar product, $\mathbf{T} =^t (\mathbf{u}_1, \lambda_1)$ is the tangent vector at the starting point $(\mathbf{U}^j,\lambda^j)$ and $\mathbf{V} = (\mathbf{u} - \mathbf{u}^j, \lambda - \lambda^j)$. Other choices are discussed in this paper. By introducing (2.2) and (2.3) into (2.1) and equating like powers of a, we obtain the following set of linear problems:

$$\begin{aligned} &order\ \ p = 1: \quad \mathbf{L}_t(\mathbf{U}_1) = \lambda_1\mathbf{F}, && < \mathbf{u}_1, \mathbf{u}_1 > + \lambda_1\lambda_1 = 1, \\ &order\ \ p \geq 2: \quad \mathbf{L}_t(\mathbf{U}_p) = \lambda_p\mathbf{F} - \textstyle\sum_{r=1}^{r=p-1} \mathbf{Q}(\mathbf{U}_r, \mathbf{U}_{p-r}), && < \mathbf{u}_1, \mathbf{u}_p > + \lambda_1\lambda_p = 0, \end{aligned} \tag{2.4}$$

where $\mathbf{L}_t(.) = \mathbf{L}(.) + 2\mathbf{Q}(\mathbf{U}^j, .)$ is the tangent operator. The next step is the computation of the terms $(\mathbf{U}_i, \lambda_i)$. The problems (2.4) are discretized by a classical finite element method. Let us note that these problems have the same tangent stiffness matrix and hence the terms $(\mathbf{U}_i, \lambda_i)$ $(1 \leq i \leq N)$ of (2.2) are computed by inverting only one **stiffness** matrix. The last step is the continuation technique [1, 2]. The end of the branch j is the starting point of the next branch $(j+1)$. The step lengths $(a \in [0, a_{maxs}])$ of the analytical representation (2.2) is approximated by

$$a_{maxs}(j,\epsilon_{pol},N)=(\epsilon_{pol}\frac{||\mathbf{u_1}||}{||\mathbf{u_N}||})^{\frac{1}{N-1}},$$

where ϵ_{pol} is a given parameter of tolerance and $||.||$ indicates the standard norm associated with the scalar product. The Padé representation allows to increase the range of validity of the series representation (2.2) [2, 3]. This rational representation can be written in the form:

$$\left\{\begin{array}{lcl} \mathbf{U}(a) &=& \mathbf{U}^j + a\frac{\Delta_{N-2}(a)}{\Delta_{N-1}(a)}\mathbf{U}_1 + a^2\frac{\Delta_{N-3}(a)}{\Delta_{N-1}(a)}\mathbf{U}_2 + ... + a^{N-1}\frac{1}{\Delta_{N-1}(a)}\mathbf{U}_{N-1}, \\ \lambda(a) &=& \lambda^j + a\frac{\Delta_{N-2}(a)}{\Delta_{N-1}(a)}\lambda_1 + a^2\frac{\Delta_{N-3}(a)}{\Delta_{N-1}(a)}\lambda_2 + ... + a^{N-1}\frac{1}{\Delta_{N-1}(a)}\lambda_{N-1}, \end{array}\right. \tag{2.5}$$

where $\Delta_i(a)$ are polynomials of degree (i) with real coefficients: $\Delta_i(a)=1+ad_1+a^2d_2+...+a^id_i$. In the continuation algorithm based on Padé representation, the estimated range of validity $((a\in[0,a_{maxp}]))$ is defined by:

$$\frac{||\mathbf{u}_{N-1}(a_{maxp})-\mathbf{u}_{N-2}(a_{maxp})||}{||\mathbf{u}_{N-1}(a_{maxp})-\mathbf{u}^j||}=\epsilon_{pad},$$

where ϵ_{pad} is a given parameter of tolerance.

The step lengths (a_{maxs} or a_{maxp}) depend on the chosen parameterization. We examine, in this paper, other choices the so-called local parameterization [4, 6, 7]:

$$a=<\mathbf{e}_i,\mathbf{V}>, \tag{2.6}$$

$$<\mathbf{e}_i,\mathbf{V}>-a<\mathbf{e}_i,\mathbf{T}^*>=0. \tag{2.7}$$

With local parameterization, the auxiliary equation (2.3) is replaced by (2.6) as in [7] or by (2.7) as in [6] and in [4], where $\mathbf{e}_i$ is the ith vector of the canonical basis of $\Re^{ndl+1}$, ndl is the degree of freedom of the discretized structure and $\mathbf{T}^*$ is the tangent vector normalized at the point $(\mathbf{u}^j,\lambda^j)$. The vector of projection $\mathbf{e}_i$ must verify the relation $<\mathbf{e}_i,\mathbf{T}^*>\neq 0$. It remains to specify the choice of the index i that will allow a large range of validity. The index i ($1\leq i\leq ndl+1$) is locally determined for each branch j.

Several algorithms are possible. In the first algorithm (Algo1), we choose $i=i_1$ corresponding to the first maximum component in modulus of $\mathbf{T}^*$ in the equation (2.6). In the second algorithm (Algo2), as in [6], we use (2.7) and we determine the indices i_1 and i_2 of the largest and second largest components of $\mathbf{T}^*$ in modulus and we choose the index i_2 instead of i_1 if the three conditions (A1), (A2) and (A3) are realized (see equation (2.8)).

$$\begin{array}{lcl} (A1) : |<T^j,\mathbf{e}_{i_1}>| & > & |<T^{j-1},\mathbf{e}_{i_1}>|, \\ (A2) : |<T^{j-1},\mathbf{e}_{i_2}>| & > & |<T^j,\mathbf{e}_{i_2}>|, \\ (A3) : \frac{|<T^j,\mathbf{e}_{i_2}>|}{|<T^j,\mathbf{e}_{i_1}>|} & > & \mu. \end{array} \tag{2.8}$$

In (2.8), $\mathbf{T}^{j-1}$ is the tangent vector at the previous branch $(j-1)$ and μ is a fixed real number $(0 < \mu < 1)$. In our numerical tests, we take $\mu = 0.1$. In the third algorithm (Algo3), as in [4], we use (2.7) and we choose the index i_2 instead of i_1 if the three conditions (B1), (B2) and (B3) are realized (see equation (2.9)).

$$\begin{aligned} (B1) \quad &: \quad \frac{|<T^j,\mathbf{e}_{i_2}>|}{|<T^{j-1},\mathbf{e}_{i_2}>|} \quad > \quad \frac{|<T^j,\mathbf{e}_{i_1}>|}{|<T^{j-1},\mathbf{e}_{i_1}>|}, \\ (B2) \quad &: \quad \frac{|<T^j,\mathbf{e}_{i_2}>|}{|<T^j,\mathbf{e}_{i_1}>|} \quad > \quad \mu, \\ (B3) \quad &: \quad < T^j,\mathbf{e}_{i_2} >< T^{j-1},\mathbf{e}_{i_2} > \quad > \quad 0. \end{aligned} \tag{2.9}$$

In the fourth algorithm (Algo4), based on the work of [5], we use the auxiliary equation (2.7) and we choose the index i which corresponds to the maximum component in modulus of the sequent vector defined by:

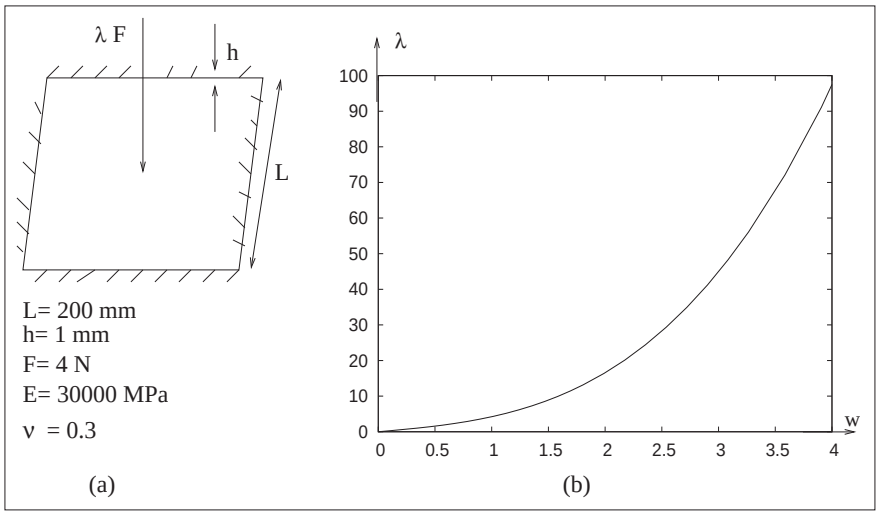
$$\mathbf{V}_{seq} = (\mathbf{u}^j - \mathbf{u}^{j-1}, \lambda^j - \lambda^{j-1}), \tag{2.10}$$

where $(\mathbf{u}^{j-1}, \lambda^{j-1})$ is the starting point of the branch $(j-1)$.

3. Numerical comparison and discussion

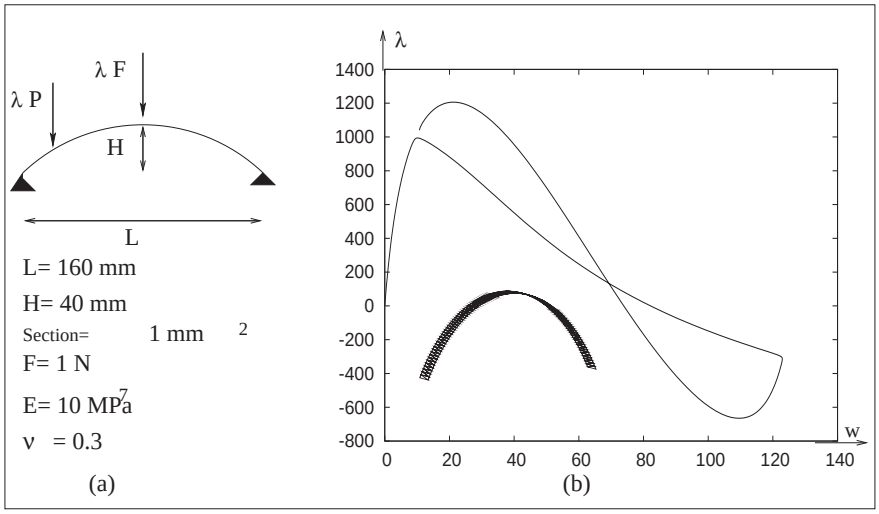
In this section, we consider two numerical examples to test the four algorithms using local parameterizations presented in Section 2. In this computation, the number of steps (number of branch j) is the measure of the efficiency of the four algorithms. These algorithms will be compared to the pseudo arc-length parameterization algorithm (PAL). We consider, as a first numerical example, the bending of an elastic square plate. The plate is embedded at the four edges and subjected to a force at its center (see figure 1-a). The structure is discretized in 32 finite elements of 18 degrees of freedom (Discrete Kirchhoff Triangle: DKT18). In order to evaluate the influence of the truncation order, we examine three choices of the order $N = 10$, 20, 30 for the two representations (2.2) and (2.5). The load-deflexion curve is given in the figure 1-A-b. The table 1, gives the number of steps (branch) to get this curve for different orders and algorithms in the case of series and Padé representations. In this table, we remark, at the first hand, that the four algorithms (Algo1), (Algo2), (Algo3) and (PAL) using series or Padé representations require the same number of ANM steps to get a deflection about $w = 4$. On the other hand, the algorithm (Algo4) requires 50% of steps less than the others algorithms. For example, (Algo4) needs 4 ANM steps to obtain the whole branch solution ($N = 20$ and $\epsilon_{pol} = 10^{-6}$). The same remark is observed for the other orders. The use of Padé representation reduces the number of steps as has been observed in [3].

As a second numerical example, we consider the buckling of a circular arch submitted to a vertical force F applied at the center and to a perturbation force $P = F/100$ placed excentrally (see figure 2-a). The structure is discretized in 200 DKT18 elements. In this test, the algorithm based on pseudo-arclength parameterization require less steps than the three algorithms (Algo1), (Algo2), (Algo3) based on local parameterization for the two representations. We can clearly see, in the table 2, that the (Algo4) reduces the number of ANM steps. For example, with the algorithm



A

Figure 1: a- Bending of a square plate, b- Load-deflection curve at the middle of the plate.



B

Figure 2: a- Buckling of a circular arch, b -Load-deflection at the center of the arch.

	Series representation					Padé representation				
	PAL	Algo1	Algo2	Algo3	Algo4	PAL	Algo1	Algo2	Algo3	Algo4
$N = 10$	16	16	16	16	6	8	9	9	9	4
$N = 20$	9	10	10	10	4	8	8	8	8	4
$N = 30$	8	9	9	9	4	7	8	8	8	3

Table 1: Bending of a square plate, Number of steps to get the curve for different parameterizations at orders $N = 10,\ 20,\ 30$, $\epsilon_{pol} = 10^{-6}$ and $\epsilon_{pad} = 10^{-4}$.

(PAL), using the power series (respectively Padé representation), the whole branch solution has been obtained in 109 ANM steps (respectively 36 ANM steps). If we use the algorithm (Algo4) based on power series (respectively on Padé representation), the number of ANM steps is reduced to 83 (respectively to 30) for $N = 20$, $\epsilon_{pol} = 10^{-14}$ and $\epsilon_{pad} = 10^{-9}$. This remark is valid for the three tested orders. One can also see, that for this example, the Padé representation reduces considerably the number of ANM steps.

	Series representation					Padé representation				
	PAL	Algo1	Algo2	Algo3	Algo4	PAL	Algo1	Algo2	Algo3	Algo4
$N = 10$	618	1231	903	929	435	233	322	305	298	217
$N = 20$	109	196	136	162	83	36	45	38	43	30
$N = 30$	94	106	76	89	50	26	39	31	34	24

Table 2: Buckling of a circular arch, Number of steps to get the curve for different parameterizations at orders $N = 10,\ 20,\ 30$, $\epsilon_{pol} = 10^{-14}$ and $\epsilon_{pad} = 10^{-9}$.

4. Conclusion

In this paper, we have introduced local parameterizations in the ANM algorithm. This concept is based on local parameterization; any component of the unknown can be used as a parameter. Four algorithms have been proposed and tested on two numerical examples. The efficient algorithm seems to be the algorithm (Algo4) based on the auxiliary equation (2.7) where the index i corresponds to the maximum component in modulus of V_{seq} (2.10) instead of the tangent vector T. The local parameterization (Algo4) gives the best results for two numerical tests in comparison with the other tested local parameterizations ((Algo1), (Algo2) and (Algo3)). This algorithm has been tested on others numerical examples from thin elastic shells. The same conclusion holds.

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