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A Posteriori Error Estimates on Stars for Convection Diffusion Problem

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Abstract. In this paper, a new a posteriori error estimator for nonconforming convection diffusion approximation problem, which relies on the small discrete problems solution in stars, has been established. It is equivalent to the energy error up to data oscillation without any saturation assumption nor comparison with residual estimator.
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1. Introduction

The convection diffusion problem is an intersecting elliptic model problem, nonsymmetric and it is a linearization of some important classes of partial differential equations, like Navier-Stokes equations or Maxwell equations. Convection diffusion problem have a wide applications in many mathematical environment studies to model pollutant transport in the atmosphere, groundwater, surface water and for the analysis and computation of solutions of chemotaxis problem from mathematical biology.

A posteriori error estimators provide the basis for adaptive mesh refinement and quantitative error control [2, 3, 5, 8]. One of the most successful estimators was proposed by Bank and Weiser and extended by many authors [1, 4], it is based on the solution of local Neumann problems on elements, which seem to allow for cancellation and thus lead to better results than the residual estimators. The classical proof of equivalence with the energy error require the *saturation assumption* : this says that the solution can be approximated asymptotically better with quadratic than with linear finite elements. The saturation assumption is shown to be superfluous by Nochetto

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in [7]. However, removing this assumption requires comparison with residual estimators. More recently, a new a posteriori error estimators on stars was proposed in [6], and the proof of the equivalence with energy error was applied directly without reference to residual estimators.

In this paper, we extended the results in [6] to the case of nonconforming finite elements and the convection diffusion case. A new a posteriori error estimator is introduced based on the solution of a small discrete problem in stars. We prove the reliability and the efficiency of the estimator without saturation assumption nor comparison with residual estimator. We consider the simpler case of nonconforming approximations for convection diffusion problem, and we introduce a technique which allowed us to define a new a posteriori error estimator which is equivalent to the energy error.

2. Setting of the problem

In this paper we consider the following convection-diffusion problem:

$$(Pb) \quad \left\{ \begin{array}{rl} -\varepsilon\Delta u + \beta.\nabla u = & f \;\; in \;\; \Omega, \\ u = & 0 \;\; on \;\; \Gamma = \partial\Omega. \end{array} \right.$$

In the following we assume that $0 < \varepsilon << 1$, $\beta \in (W^{1,\infty}(\Omega))^d$, such that $-\frac{1}{2}div\beta \geq 0$ and $f \in L^2(\Omega)$. Given $\Omega \in \mathbb{R}^d$ a simply connected polygon domain ($d = 2$ or 3). Let $\mathcal{T}_h$ be a family of conforming shape-regular triangulation of Ω, we denoted by E_I the set of interior edges and by E_f the set of all edges included in Γ. Let V_h be the lowest order non-conforming Crouzeix-Raviart finite element space defined by:

$$V_h = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h,\ v_{h|T} \in P_1(T), \forall E \in E_I,\ \int_E [v_h]_E d\sigma = 0 \text{ and } \forall E \in E_f,\ \int_E v_h d\sigma = 0\}.$$

Where $[.]_E$ denoted the jump of the function across E.

2.1. Discretization of the problem

For each $T \in \mathcal{T}_h$, we denote by $P_k(T)$ the polynomial space of degree less than or equal to k. For each $T \in \mathcal{T}_h$ and $E \in E_T$ an internal edge of T with vertices $(Q_1, .., Q_d)$ ($(Q_0, Q_1, .., Q_d)$ are the vertices of T), we introduce the d-simplex T_E with vertices $(P, Q_1, .., Q_d)$ where P is the point with barycentric coordinates:

$$\lambda_0(P) = \delta_{E,T} \quad and \quad \lambda_0(P) = \frac{1 - \delta_{E,T}}{d},$$

for $i = 1, .., d$ where $\{\lambda_i\}$ are the standard barycentric coordinates associated to the vertices $\{Q_i\}_{i=0}^d$ and $\delta_{E,T} = \min\{\frac{1}{d+1}, \frac{\sqrt{\varepsilon}}{h_T}\}$. For each $T \in \mathcal{T}_h$ and all $E \in E_T$, we introduce the unique function $b_{E,T} \in C^0(T)$ defined by:

$$b_{E,T} \in P_d(T_E), \quad b_{E,T} = 0 \;\; on \;\; T/T_E, \quad b_{E,T}(a_E) = 1.$$

We denoted by $\{x_i\}_{i\in\mathcal{N}}$ the set of all nodes of the triangulation $\mathcal{T}_h$. For each $i \in \mathcal{N}$, ϕ_i denoted the canonical continuous piecewise linear basis function corresponding to x_i. The star ω_i is the interior relative to Ω of the support of ϕ_i, such that $\sum_{i\in\mathcal{N}} \phi_i = 1$, and h_i is the maximal size of the elements forming ω_i. Finally, Γ_i will denote the union of the sides touching x_i that are contained in Ω, and $\overline{\Gamma}_i$ will denote the union of the sides touching x_i that are contained in $\overline{\Omega}$.
For each star ω_i, $i \in \mathcal{N}$ we introduce the space defined if x_i is an interior node by:

$$V(\omega_i) = \{v \in H^1_{loc}(\omega_i) : \int_{\omega_i} v\phi_i dx = 0\},$$

and

$$W(\omega_i) = \{v \in H^1_{loc}(\omega_i) : v = 0 \text{ on } \partial\omega_i \cap \Gamma\} \quad \text{otherwise .}$$

We define a finite dimensional local spaces $\mathcal{P}^2_0(\omega_i)$ and $\hat{\mathcal{P}}^2_0(\omega_i)$ by:

Definition 1. *For $i \in \mathcal{N}_i$, let $\mathcal{P}^2(\omega_i)$ denote the space of piecewise quadratic functions on the star ω_i that vanish on $\partial\omega_i$. The spaces $\mathcal{P}^2_0(\omega_i)$ and $\hat{\mathcal{P}}^2_0(\omega_i)$ are defined as $\mathcal{P}^2_0(\omega_i) = \mathcal{P}^2(\omega_i) \cap W(\omega_i)$ and $\hat{\mathcal{P}}^2_0(\omega_i) = \mathcal{P}^2(\omega_i) \cap V(\omega_i)$.*

In the following we consider the energy norm:$||u||^2_{\varepsilon,\omega_i} = \varepsilon \|\nabla u\|^2_{0,\omega_i} + \|u\|^2_{0,\omega_i}$. Let $v_h \in V_h$ be fixed and we denoted by $\nabla_h v_h$ the vector belonging to $(L^2(\Omega))^d$ defined by: $\forall T \in \mathcal{T}_h; \quad \nabla_h v_h = \nabla v_h \quad \text{on } T$.

2.2. A posteriori error estimator

Let u_h be a solution of the following stabilized discrete problem:

$$(P_h) \quad \begin{cases} \forall v_h \in V_h \cap H^1_0(\Omega), \\ \displaystyle\sum_{T\in\mathcal{T}_h} \int_T [\varepsilon\nabla u_h.\nabla v_h + \beta.\nabla u_h v_h]dx + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_{\partial T^-} \beta.n[u_h]v_h d\sigma = \int_\Omega f v_h dx. \end{cases}$$

For each $i \in \mathcal{N}$, we consider the local problems:

$$(P_1) \quad \begin{cases} \text{Find } \eta_i \in \widehat{\mathcal{P}}^2_0(\omega_i) \text{ such that } \forall \mu_i \in \widehat{\mathcal{P}}^2_0(\omega_i), \\ \displaystyle\int_{\omega_i} (\varepsilon\nabla\eta_i.\nabla\mu_i)\phi_i dx = \int_{\omega_i} (\varepsilon\nabla_h u_h.\nabla\mu_i)\phi_i dx + \int_{\omega_i} (\beta.\nabla_h u_h \mu_i)\phi_i dx \\ \displaystyle +\frac{1}{2}\sum_{T\in\mathcal{T}_h}\int_{\partial T^-} \beta.n[u_h]\mu_i\phi_i d\sigma - \int_{\omega_i} f\mu_i\phi_i dx, \end{cases}$$

and

$$(P_2) \quad \begin{cases} \text{Find } \alpha_i \in \mathcal{P}^2_0(\omega_i) \text{ such that } \forall \mu_i \in \mathcal{P}^2_0(\omega_i), \\ \displaystyle\int_{\omega_i} (\varepsilon\nabla\alpha_i.\nabla\mu_i)\phi_i dx = \int_{\omega_i} \varepsilon\nabla_h u_h.\ \text{Curl } (\mu_i\phi_i)dx. \end{cases}$$

Using Lax-Milgram theorem,we prove that the discrete problems have unique solution. Finally we set: $\forall i \in \mathcal{N}, \ \ E^2_{1,i}(u_h) = \displaystyle\int_{\omega_i} \varepsilon|\nabla\eta_i|^2\phi_i dx$ and $\forall i \in \mathcal{N}, \ \ E^2_{2,i}(u_h) = \displaystyle\int_{\omega_i} \varepsilon(\nabla\alpha_i)^2\phi_i dx$. We have the following global upper bond an local lower bound of the error:

Theorem 2. *Let $u_h \in V_h$ such that (P_h) holds. There is a positives constants C_1; C_2 and C_3 depending only on the minimum angle of $\mathcal{T}_h$ such that*

$$\begin{aligned} \|u-u_h\|_{\varepsilon,\Omega} &\leq C_1 \Bigg[(\sum_{i\in\mathcal{N}} (E_{1,i}^2(u_h) + E_{2,i}^2(u_h)))^{\frac{1}{2}} + \|\beta\|_{1,\infty} h_i (\sum_{i\in\mathcal{N}} E_{(2,i)}^2(u_h))^{\frac{1}{2}} \\ &+ \sum_{i\in\mathcal{N}} h_i \sum_{T\subset\omega_i} \|[u_h]\|_{L^2(\partial T)} + (\|\beta\|_{0,\infty} + \|\beta\|_{H(div,\omega_i)}) h_i \|u_h\|_{0,\omega_i} + osc(f) \Bigg], \end{aligned} \tag{2.1}$$

where $osc(f)$ is the data oscillation defined by: $osc(f) = (\sum_{i\in\mathcal{N}} \alpha_T^2 \|(f-f_i)\phi_i^{\frac{1}{2}}\|_{0,\omega_i})^{\frac{1}{2}}$, where $f_i = \dfrac{\int_{\omega_i} f\phi_i dx}{\int_{\omega_i}\phi_i dx}$ and $alpha_T = min(1, \frac{h_T}{\sqrt{\varepsilon}})$. For any $i \in \mathcal{N}$, we have:

$$E_{1,i}(u_h) \leq C_2(1 + \|\beta\|_{1,\infty} h_i) ||u-u_h||_{\varepsilon,w_i}, \tag{2.2}$$

and

$$E_{2,i}(u_h) \leq C_3 ||u-u_h||_{\varepsilon,w_i}. \tag{2.3}$$

Proof. Remark that using Helmholtz-decomposition, we have

$$\nabla_h u_h - \nabla u = \nabla w + \text{ Curl } \zeta, \tag{2.4}$$

with $w \in H_0^1(\Omega)$, $\zeta \in H^1(\Omega)$ and $\int_\Omega \nabla w.\text{ Curl } \zeta dx = 0$. Let us remark also that the orthogonality implies the following error decomposition:

$$\varepsilon\|\nabla_h u_h - \nabla u\|_{0,\Omega}^2 = \varepsilon\|\nabla w\|_{0,\Omega}^2 + \varepsilon\|\text{ Curl } \zeta\|_{0,\Omega}^2, \tag{2.5}$$

and the following equalities:

$$\varepsilon\|\nabla w\|_\Omega^2 = \int_\Omega \varepsilon(\nabla_h u_h - \nabla u).\nabla w dx, \tag{2.6}$$

and

$$\varepsilon\|\text{ Curl } \zeta\|_{0,\Omega}^2 = \int_\Omega \varepsilon(\nabla_h u_h - \nabla u).\text{ Curl } \zeta dx. \tag{2.7}$$

The estimates of the expressions in (2.5) will be established in the Lemmas 6 and 7. As a main tool we use the following technical Lemmas:

Lemma 3. *For each node $i \in \mathcal{N}$ there exists an operator $\Pi_i : W(\omega_i) \longrightarrow \mathcal{P}_0^2(\omega_i)$, such that for any $v \in W(\omega_i)$ the following conditions hold:*

1. *For all edge $E \subset \Gamma_i$, $\int_E (v - \Pi_i v)\phi_i d\sigma = 0$, and $\int_{\omega_i} (v-\Pi_i v)\phi_i dx = 0$.*

2. $\varepsilon(\int_{\omega_i} |\nabla \Pi_i v|^2 \phi_i)^{\frac{1}{2}} + (\int_{\omega_i} |\Pi_i v|^2 \phi_i)^{\frac{1}{2}} \leq C \left[\varepsilon(\int_{\omega_i} |\nabla v|^2 \phi_i)^{\frac{1}{2}} + \int_{\omega_i} |v|^2 \phi_i)^{\frac{1}{2}} \right]$,

where the constant C depends only on the minimum angles of $\mathcal{T}_h$.

Lemma 4. *For each node $i \in \mathcal{N}$ there exists an operator $\hat{\Pi}_i : W(\omega_i) \longrightarrow \widehat{\mathcal{P}}_0^2(\omega_i)$, such that for any $v \in W(\omega_i)$ the following conditions hold:*

1. *For all edge $E \subset \overline{\Gamma}_i$, $\int_E (v - \hat{\Pi}_i v)\phi_i d\sigma = 0$, and $\int_{\omega_i} (v - \hat{\Pi}_i v)\phi_i dx = 0$.*
2. $\varepsilon(\int_{\omega_i} |\nabla \hat{\Pi}_i v|^2 \phi_i)^{\frac{1}{2}} + (\int_{\omega_i} |\hat{\Pi}_i v|^2 \phi_i)^{\frac{1}{2}} \leq C \left[\varepsilon(\int_{\omega_i} |\nabla v|^2 \phi_i)^{\frac{1}{2}} + \int_{\omega_i} |v|^2 \phi_i)^{\frac{1}{2}} \right]$,

where the constant C depends only on the minimum angles of $\mathcal{T}_h$.

Lemma 5. *For each node $i \in \mathcal{N}$, functions $v \in W(\omega_i)$, $\zeta \in V(\omega_i)$ and $u_h \in V_h$. We have*

$$\int_{\omega_i} \varepsilon \nabla_h u_h . \nabla((\Pi_i v)\phi_i) dx = \int_{\omega_i} \varepsilon \nabla_h u_h . \nabla(v\phi_i) dx,$$

and

$$\int_{\omega_i} \varepsilon \nabla_h u_h . \textit{ Curl } ((\hat{\Pi}_i \zeta)\phi_i) dx = \int_{\omega_i} \varepsilon \nabla_h u_h . \nabla(\zeta\phi_i) dx.$$

Lemma 6. *There exist a positive constant C, depending only on the minimum angle of $\mathcal{T}_h$ such that*

$$\varepsilon^{\frac{1}{2}} \| \textit{ Curl } \zeta \|_{0,\Omega} \leq C (\sum_{i \in \mathcal{N}} E^2_{(2,i)}(u_h))^{\frac{1}{2}}.$$

And

Lemma 7. *There exists positive constant C^*, depending only on the minimum angle of $\mathcal{T}_h$ such that:*

$$\begin{aligned} \|w\|_{\varepsilon,\Omega} &\leq C^*[(\sum_{i \in \mathcal{N}} E^2_{1,i}(u_h))^{\frac{1}{2}} + \|\beta\|_{1,\infty} h_i (\sum_{i \in \mathcal{N}} E^2_{(2,i)}(u_h))^{\frac{1}{2}} \\ &+ (\sum_{i \in \mathcal{N}} \|\beta\|_{1,\infty} h_i \|u_h\|_{0,\Omega} + h_i \|[u_h]\|_{L^2(\partial T)} + h_i \|[u_h]\|_{L^2(\partial T^-)} + |div\beta| h_i \|u_h\|_{0,\omega_i} + osc(f)]. \end{aligned} \tag{2.8}$$

3. Conclusion

In this work we analyzed an a posteriori error estimator for nonconforming convection diffusion problem, with the Helmholtz-Decomposition technics. These estimators are efficient and robust with respect to the physical parameters of the problem.

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