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Analysis and Numerical Approximation of an Electro-elastic Frictional Contact Problem

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Abstract. We consider the problem of frictional contact between an piezoelectric body and a conductive foundation. The electro-elastic constitutive law is assumed to be nonlinear and the contact is modelled with the Signorini condition, nonlocal Coulomb friction law and a regularized electrical conductivity condition. The existence of a unique weak solution of the model is established. The finite elements approximation for the problem is presented, and error estimates on the solutions are derived.

Key words: piezoelectric, Coulomb's law, Signorini condition, fixed point process, finite element approximation, error estimates
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1. Strong and weak formulation of the mathematical model

In this section, we state the model of equilibrium process of the elastic piezoelectric body in frictional contact with a conductive deformable foundation and give his weak formulation. We consider an elastic piezoelectric body which initially occupies open bounded subset $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently regular boundary $\partial\Omega = \Gamma$. We decompose Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas\ \Gamma_1 > 0$ and $meas\ \Gamma_a > 0$. The summation convention over repeated indices is used, all indices take values in $\{1, ..., d\}$. Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u,\ v \in \mathbb{R}^d$, $\forall \sigma,\ \tau \in \mathbb{S}^d$ $u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{1/2}$ and $\sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{1/2}$. Moreover, we denote by $u : \Omega \to \mathbb{R}^d$ the

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displacement field, $\varepsilon(u) = (\frac{1}{2}(u_{i,j} + u_{j,i}))$ the small strain tensor, $\sigma : \Omega \to \mathbb{S}^d$ the stress tensor, $D : \Omega \to \mathbb{R}^d$ the electric displacement field, $E(\varphi) = -\nabla\varphi$ the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electric potential. We also use the notations for normal and tangential components of displacement vector and stress: $v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu, \quad \sigma_\nu = (\sigma\nu)\cdot\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$, where ν denote the outward normal vector on Γ. The static equilibrium of the elastic piezoelectric body is described by the following strong equations

$$\operatorname{Div}\sigma + f_0 = 0, \qquad \operatorname{div} D = q_0, \quad \text{in } \Omega, \tag{1.1}$$

where $\operatorname{Div}\sigma = (\sigma_{ij,j})$, and $\operatorname{div} D = (D_{j,j})$, f_0 and q_0 the density of volume forces and volume electric charges, respectively. The constitutive laws of material is assumed of the form (see [3])

$$\sigma = \mathcal{F}\varepsilon(u) - \mathcal{E}^* E(\varphi), \qquad D = \mathcal{E}\varepsilon(u) + \beta E(\varphi) \quad \text{in } \Omega, \tag{1.2}$$

in which $\mathcal{F} : \Omega\times \mathbb{S}^d \to \mathbb{S}^d$ is a nonlinear elasticity operator, $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is a linear piezoelectric operator, $\mathcal{E}^* : \Omega \times \mathbb{R}^d \to \mathbb{S}^d$ is its transpose given by $\mathcal{E}\sigma v = \sigma\mathcal{E}^* v$, $\forall\sigma \in \mathbb{S}^d, v \in \mathbb{R}^d$ and β: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear electric permittivity operator.

To complete the model, we have to prescribe the mechanic and electric boundary conditions. According to the physical setting, we use

$$u = 0 \quad \text{on } \Gamma_1, \quad \sigma\nu = f_2 \quad \text{on } \Gamma_2, \quad \varphi = 0 \quad \text{on } \Gamma_a, \quad D\cdot\nu = q_2 \quad \text{on } \Gamma_b, \tag{1.3}$$

where f_2 and q_2 the density of tractions and surface electric charges, respectively. On the contact surface Γ_3, we model the contact with the Signorini condition, the regularized Coulomb law and the regularized electrical conductivity condition, that is

$$\left.\begin{array}{l} \sigma_\nu(u) \le 0, u_\nu \le 0, \sigma_\nu(u)u_\nu = 0, \\ |\sigma_\tau| \le \mu(\|u_\tau\|)|R\sigma_\nu(u)|, \quad \left\{ \begin{array}{l} |\sigma_\tau| < \mu(\|u_\tau\|)|R\sigma_\nu(u)| \Rightarrow u_\tau = 0 \\ |\sigma_\tau| = -\mu(\|u_\tau\|)|R\sigma_\nu(u)|\frac{u_\tau}{\|u_\tau\|} \Rightarrow u_\tau \ne 0 \end{array}\right. , \\ D\cdot\nu = \psi(u_\nu)\phi_L(\varphi - \varphi_0). \end{array}\right\} \quad \text{on } \Gamma_3. \tag{1.4}$$

Here μ is the coefficient of friction and R is a regularization operator. Finally, ϕ_L is the truncation function, used to control the roundedness of $(\varphi - \varphi_0)$, where φ_0 represents the electric potential of the foundation (see [4]). The formulation of the problem is as follows

Problem P**.** *Find a displacement field* $u : \Omega \to \mathbb{R}^d$ *and an electric potential* $\varphi : \Omega \to \mathbb{R}$ *such that* (1.1)-(1.4) *hold.*

Next, we derive a weak formulation of the problem P. To this end, we use the following functional spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \{\tau = (\tau_{ij});\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}.$$

We also introduce the spaces for the displacement and the electric potential:

$$V = \{v \in H_1;\, v = 0 \text{ on } \Gamma_1\}, \qquad W = \{\xi \in H^1(\Omega)/\, \xi = 0 \text{ on } \Gamma_a\}.$$

On V we consider the inner product given by $(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}$, and let $\|\cdot\|_V$ be the associated norm. It follows from Korns inequality that $\|\cdot\|_V$ and $\|\cdot\|_{H_1}$ are equivalent norms on V . Therefore $(V,\|\cdot\|_V)$ is a real Hilbert space. On W we consider the inner product given by $(\varphi,\xi)_W = (\varphi,\xi)_{H^1}$ for $\varphi,\xi \in W$. Then $(W,\|\cdot\|_W)$ is also Hilbert space. Moreover, let K be the set of admissible displacements given by

$$K = \{v \in V;\ v_\nu \leq 0 \quad \text{on } \Gamma_3\}.$$

The following assumptions are made on the given data:

$(\mathbf{h_1})$ - The elasticity operator $\mathfrak{F}$ is strongly monotone and Lipschitz continuous such that $\forall\xi \in \mathbb{S}^d,\ x \to \mathfrak{F}(x,\xi)$ is Lebesgue measurable on Ω, and $x \to \mathfrak{F}(x,0)$ belongs to $\mathcal{H}$;

$(\mathbf{h_2})$ - The piezoelectric tensor $\mathcal{E} = (e_{ijk})$ is satisfies $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$;

$(\mathbf{h_3})$ - The electric permittivity tensor $\beta = (\beta_{ij}) \in L^\infty(\Omega)$ is assumed to be symmetric and W-elliptic;

$(\mathbf{h_4})$ - The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz, bounded and measurable function on Γ_3 for all $u \in \mathbb{R}$, which satisfies $\psi(x,u) = 0$ for all $u \leq 0$;

$(\mathbf{h_5})$ - The coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Lipschitz and measurable function on Γ_3 for all $u \in \mathbb{R}$, which satisfies $\exists\ \mu^* > 0$ such that $\mu(x,u) \leq \mu^*, \forall u \in \mathbb{R}_+,$ a.e. $x \in \Gamma_3$;

$(\mathbf{h_6})$ - The density of volume forces, tractions, volume electric charges, surface electric charges and given potential have the regularity $f_0 \in L^2(\Omega)^d,\ f_2 \in L^2(\Gamma_3)^d,\ q_0 \in L^2(\Omega),\ q_2 \in L^2(\Gamma_b)$ and $\varphi_0 \in L^2(\Gamma_3)$;

$(\mathbf{h_7})$ - The mapping $R : H'_{\Gamma_3} \to L^\infty(\Gamma_3)$ is a linear and continuous function (see [5]).

Next, we define $f \in V$ and $q \in W$ as for all $v \in V$ and $\xi \in W$

$$(f,v)_V = \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_2} f_2 \cdot v\, da, \text{ and } (q,\xi)_W = \int_\Omega q_0 \xi\, dx - \int_{\Gamma_b} q_2 \xi\, da. \tag{1.5}$$

Finally, we define the mappings $\ell : V \times W \times W \to \mathbb{R}$ and $j : V \times V \to \mathbb{R}$, respectively, by

$$\ell(u,\varphi,\xi) = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\xi\, da, \ \ \forall u \in V,\ \forall\, \varphi,\xi \in W, \tag{1.6}$$

$$j(u,v) = \int_{\Gamma_3} \mu(\|u_\tau\|)|R\sigma_\nu(u)|\ \|v_\tau\|\, da, \ \ \forall\, u,v \in V. \tag{1.7}$$

Using Greens formula and the previous boundary conditions, the variational formulation for the mechanical problem P is as follows:

Problem PV**.** *Find a displacement field* $u \in K$ *and an electric potential* $\varphi \in W$ such that: $\forall\, v \in K$ and $\forall\, \xi \in W$ we have

$$(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + (\mathcal{E}^{*}\nabla\varphi, \varepsilon(v) - \varepsilon(u))_{L^2(\Omega)^d} + j(u, v) - j(u, u) \geq (f, v - u)_V, \quad (1.8)$$

$$(\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d} + \ell(u, \varphi, \xi) = (q, \xi)_W. \quad (1.9)$$

2. Existence and uniqueness result

The main existence and uniqueness result, which we establish in this section, is the following.

Theorem 1. *Under the assumption* $(\mathbf{h_1})$-$(\mathbf{h_7})$*, the problem* PV *has at least one solution. Moreover, there exists* L^**, which depends only on* $\Omega, \Gamma_1, \Gamma_2, \Gamma_3, \mathfrak{F}, \mathcal{E}, \beta$*, such that if* $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$ *the solution is unique.*

Proof. The proof of Theorem 1 will be carried out in several steps. Let consider the product spaces $X = V \times W$ and $Y = L^2(\Gamma_3) \times L^2(\Gamma_3)$ together with the inner products

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W, \;\; \forall x = (u, \varphi), \; y = (v, \xi) \in X, \quad (2.1)$$

$$(\eta, \theta)_Y = (g, \lambda)_{L^2(\Gamma_3)} + (z, \zeta)_{L^2(\Gamma_3)}, \;\; \forall \eta = (g, z), \; \theta = (\lambda, \zeta) \in Y, \quad (2.2)$$

and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $U = K \times W$ be non-empty closed convex subset of X. We define the operator $A : X \to X$, the functions $\widetilde{j}$, $\widetilde{\ell}$ on $X \times X$ and the element $f_3 \in X$ by:

$$(Ax, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} + (\mathcal{E}^{*}\nabla\varphi, \varepsilon(v))_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d}, \quad (2.3)$$

$$\widetilde{j}(x, y) = j(u, v), \quad \widetilde{\ell}(x, y) = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\xi \, da \quad \text{and} \quad f_3 = (f, q) \in X, \quad (2.4)$$

for all $x = (u, \varphi)$ and $y = (v, \xi)$ in X. We have the following equivalence result

Lemma 2. *The couple* $x = (u, \varphi)$ *is a solution to problem* PV *if and only if:*

$$(Ax, y - x)_X + \widetilde{j}(x, y) - \widetilde{j}(x, x) + \widetilde{\ell}(x, y - x) \geq (f_3, y - x)_X, , \forall y = (v, \xi) \in K \times W. \quad (2.5)$$

The proof of this Lemma can be found in [3]. Now to prove the existence and uniqueness result of (2.5), we define two closed convex set $\mathcal{K}_1 = \{g \in L^2(\Gamma_3) \;/\; g \geq 0 \text{ and } \|g\|_{L^2(\Gamma_3)} \leq k_1\}$ and $\mathcal{K}_2 = \{z \in L^2(\Gamma_3) \;/\; \|z\|_{L^2(\Gamma_3)} \leq k_2\}$ with k_1 and k_2 to be specified. Let $\eta = (g, z) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given and consider the elements $f_\eta \in X$ given by $(f_\eta, y)_X = (f, v)_v + (q, \xi)_W - \int_{\Gamma_3} z\, \xi \, da, \;\; \forall\, y =$

$(v,\xi) \in U$, define the functional $\widetilde{j}_g(x) = \int_{\Gamma_3} g\, \|v_\tau\|\, da, \quad \forall\, x = (u,\varphi) \in U$ and consider the following intermediate Problem.

$$(Ax_\eta, y - x_\eta)_X + \widetilde{j}_g(y) - \widetilde{j}_g(x_\eta) \geq (f_\eta, y - x_\eta)_X, \quad \forall\, y = (v,\xi) \in U. \tag{2.6}$$

Keeping in mind $(\mathbf{h_1})$-$(\mathbf{h_3})$ we see that A is a strongly monotone and Lipschitz continuous operator. Moreover, it is easy to see that $\widetilde{j}_g$ is a proper, convex and continuous function on U. The existence and uniqueness result for (2.6) follow from standard arguments of elliptic variational inequalities (see [1]). That is, for any $\eta \in \mathcal{K}_1 \times \mathcal{K}_2$. The problem (2.6) has a unique solution $x_\eta = (u_\eta, \varphi_\eta) \in K \times W$.

We now consider the operator $\Lambda : L^2(\Gamma_3) \times L^2(\Gamma_3) \to L^2(\Gamma_3) \times L^2(\Gamma_3)$ such that for all $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, we have

$$\Lambda\eta = (\mu(\|u_{\eta\tau}\|)|R\sigma_\nu(u_\eta)|, \psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0)), \;\; \forall\, \eta \in L^2(\Gamma_3) \times L^2(\Gamma_3). \tag{2.7}$$

It follows from assumptions $(\mathbf{h_4})$-$(\mathbf{h_5})$, $(\mathbf{h_7})$ that the operator Λ is well-defined. We prove the following result

Proposition 3. *If $k_1 = c_2\, \mu^*\, c_0\, c_*(\|f\|_V + \|q\|_W + M_\psi\, L\, mes(\Gamma_3)^{\frac{1}{2}})$ and $k_2 = M_\psi L\; mes(\Gamma_3)^{\frac{1}{2}}$, then the operator Λ has at least one fixed point.*

Proof. We prove that

1. The mapping $\eta \to x_\eta$, where x_η is the solution to (2.6), is weakly continuous from $L^2(\Gamma_3) \times L^2(\Gamma_3)$ to X and the operator Λ is weakly continuous of $\mathcal{K}_1 \times \mathcal{K}_2$ into itself.

2. $\mathcal{K}_1 \times \mathcal{K}_2$ is a nonempty, convex and closed subset of $L^2(\Gamma_3) \times L^2(\Gamma_3)$. Since $L^2(\Gamma_3) \times L^2(\Gamma_3)$ is a reflexive space, $\mathcal{K}_1 \times \mathcal{K}_2$ is weakly compact.

By Schauder's fixed point theorem the operator Λ has at least one fixed point. Now, the existence part follows from the existence of the fixed point of the operator Λ.
For the uniqueness part. Let $x_1 = (u_1, \varphi_1)$, $x_2 = (u_2, \varphi_2) \in X$ be two solutions of problem (2.5) we establish after some algebra the following estimate

$$\|x_1 - x_2\|_X^2 \leq c(L_\mu + \mu^* + L_\psi\, L + M_\psi)\, \|x_1 - x_2\|_X^2.$$

Let $L^* = 1/c$, then if $L_\mu + \mu^* + L_\psi\, L + M_\psi < L^*$ therefore $x_1 = x_2$.

3. Error estimates for the numerical approximation

In this section, we study the finite element approximation of the variational problem PV. Assume Ω is a polygonal domain, let τ^h be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the partition of the boundary decompositions $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma = \Gamma_a \cup \Gamma_b \cup \Gamma_3$,that is, any point when the boundary condition type changes is a vertex of the partitions, then the side lies entirely in $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, and $\overline{\Gamma}_a \cup \overline{\Gamma}_b \cup \overline{\Gamma}_3$. Corresponding to each partition τ^h. We denote by $\mathbb{P}_1(\Omega^e)$ the space of polynomials of global degree less or equal to one in Ω^e. Let us consider two finite-dimensional spaces $V^h \subset V$ and $W^h \subset W$, approximating the spaces V and W, respectively, that is

$$V^h = \{v^h \in C(\overline{\Omega})^d,\ v^h_{/\Omega^e} \in \mathbb{P}_1(\Omega^e)^d,\ \Omega^e \in \tau^h,\ v^h = 0 \ on\ \overline{\Gamma}_1\},$$
$$W^h = \{\psi^h \in C(\overline{\Omega}),\ \psi^h_{/\Omega^e} \in \mathbb{P}_1(\Omega^e),\ \Omega^e \in \tau^h,\ \psi^h = 0 \ on\ \overline{\Gamma}_a\}.$$

Here $h > 0$ is a discretization parameter. Moreover let us consider the nonempty finite-dimensional closed convex sets of admissible displacements with V^h, defined by $K^h = K \cap V^h$. i.e. $K^h = \{v^h \in V^h,\ v^h_\nu \leq 0 \ on\ \overline{\Gamma}_3\}$. We consider the following discrete approximation of problem PV:

Problem PV^h. *Find a discrete displacement field* $u^h \in K^h$ *and a discrete electric potential* $\varphi^h \in W^h$ *such that*

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u^h), \varepsilon(v^h) - \varepsilon(u^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi^h, \varepsilon(v^h) - \varepsilon(u^h))_{L^2(\Omega)^d} + j(u^h, v^h) - j(u^h, u^h)\\ &\geq (f, v^h - u^h)_V, \quad \forall v^h \in K^h.\end{aligned} \tag{3.1}$$

$$(\beta\nabla\varphi^h, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u^h), \nabla\xi^h)_{L^2(\Omega)^d} + \ell(u^h, \varphi^h, \xi^h) = (q, \xi^h)_W,\ \forall\xi^h \in W^h. \tag{3.2}$$

Using the assumptions of Theorem 1, it can be shown that Problem PV^h has a unique solution $(u^h, \varphi^h) \in K^h \times W^h$. Our interest lies in estimating the numerical errors. We first derive a Céa's type inequality.

Theorem 4. *Let us denote by* (u, φ) *and* (u^h, φ^h) *the respective solutions to problem* PV *and* PV^h*. Under the assumptions of Theorem 1 with the same value of* L^**, the following error estimates are obtained for all* $v^h \in K^h$ *and* $\psi^h \in W^h$*,*

$$\begin{aligned}&\|u - u^h\|_V + \|\varphi - \varphi^h\|_W \leq c \inf_{(v^h,\xi^h)\in K^h\times W^h} \Big\{\|u - v^h\|_V + \|\varphi - \xi^h\|_W\\ &+ \|u - v^h\|_{L^2(\Gamma_3)^d} + \|\varphi - \xi^h\|_{L^2(\Gamma_3)} + \big(\|\mathfrak{F}\varepsilon(u)\|^{\frac{1}{2}}_{\mathcal{H}} + \|\mathcal{E}^*\nabla\varphi^h\|^{\frac{1}{2}}_{\mathcal{H}} + \|f\|^{\frac{1}{2}}_V\big)\|u - v^h\|^{\frac{1}{2}}_V\\ &+ \big(\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\big)^{\frac{1}{2}}\|u - v^h\|^{\frac{1}{2}}_{L^2(\Gamma_3)^d}\Big\},\end{aligned} \tag{3.3}$$

where $c > 0$ *is independent of* h*.*

The proof of Theorem 4 is done by using a properties $(\mathbf{h_1})$-$(\mathbf{h_7})$ and after some tedious algebraic manipulations. The inequality (3.3) is a basis for deriving error estimation and convergence analysis. In an analogous way, if we also suppose that $\sigma_\tau \in L^2(\Gamma_3)^d$ and using the classical results of interpolation (cf. [2]), we have the following result

Theorem 5. *Under the assumptions of Theorem 1 with the same value of* L^**, assume additionally* $\sigma_\tau \in L^2(\Gamma_3)^d$*. Then for some constant* $c > 0$*, we have*

$$\begin{aligned}\|u - u^h\|_V + \|\varphi - \varphi^h\|_W \leq\ & c\, h \Big\{ |u|_{H^2(\Omega)^d} + |\varphi|_{H^2(\Omega)} + h\,|u|_{H^2(\Gamma_3)^d} + h\,|\varphi|_{H^2(\Gamma_3)}\\ &+ (\|\sigma_\tau\|_{L^2(\Gamma_3)^d} + \|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)})^{\frac{1}{2}}|u|^{\frac{1}{2}}_{H^2(\Gamma_3)^d}\Big\}.\end{aligned}$$

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