

Edge-based a Posteriori Error Estimators for Generating Quasi-optimal Simplicial Meshes

A. Agouzal¹, K. Lipnikov² and Yu. Vassilevski³ *

¹ University Lyon1, Institute Camille Jordan, UMR 5208, 69100 Villeurbanne, France

² Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.

³ Institute of Numerical Mathematics, Gubkina str. 8, Moscow 119333, Russia

Abstract. We present a new method for generating a d -dimensional simplicial mesh that minimizes the L^p -norm, $p > 0$, of the interpolation error or its gradient. The method uses edge-based error estimates to build a tensor metric. We describe and analyze the basic steps of our method.

Key words: finite elements, anisotropic meshes, a posteriori error estimates

AMS subject classification: 65D05, 65D15, 65N15, 65N50

1. Introduction

Generation of a mesh adapted to a given function requires a specially designed metric. We develop further a new methodology, proposed originally in [1, 2], for generating a tensor metric \mathfrak{M} from error estimates *prescribed to mesh edges*. The volume and the perimeter of a d -simplex measured in this metric control the norm of error or its gradient. The error equidistribution principle suggests to balance \mathfrak{M} -volumes and \mathfrak{M} -perimeters to produce a \mathfrak{M} -quasi-uniform mesh [3, 4].

The edge-based error estimates come usually from postprocessing or a posteriori error analysis of a discrete solution. Critical advantage of edge-based error estimates over cell-based error estimates is that they provide local directional information about the error. In this paper, we consider the problem of minimizing the P_1 -interpolation error or its gradient, where cell-based and edge-based errors can be easily defined. Most methods that convert the cell-based errors into a metric lose directional information. We describe a new method that uses the edge-based errors to

*Corresponding author. E-mail: yuri.vassilevski@gmail.com

build a metric that preserves directional information. We illustrate the optimal error reduction with numerical experiments. We also summarize the existing error estimates.

2. Edge-based interpolation error estimates

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain and Ω_h be a conformal simplicial mesh with N_h d -simplexes. The volume of a d -simplex Δ and the total length of its edges in a metric \mathfrak{M} are denoted by $|\Delta|_{\mathfrak{M}}$ and $|\partial\Delta|_{\mathfrak{M}}$, respectively. Let $\mathcal{I}_1 u$ be the continuous piecewise linear interpolant of u , and $\mathcal{I}_{1,\Delta} u$ be its restriction to Δ . Our goal is to generate meshes that minimize the L^p -norm, $p \in (0, \infty]$, of the interpolation error $e = u - \mathcal{I}_1 u$ or its gradient ∇e . Let us consider a d -simplex Δ with vertices \mathbf{v}_i , $i = 1, \dots, d+1$, edge vectors $\mathbf{e}_k = \mathbf{v}_i - \mathbf{v}_j$, $1 \leq i < j \leq d+1$, and mid-edge points \mathbf{c}_k , $k = 1, \dots, n_d$, where $n_d = d(d+1)/2$. Let λ_i , $i = 1, \dots, d+1$, be the linear functions on Δ such that $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol. For every edge \mathbf{e}_k , we define the quadratic bubble function $b_k = \lambda_i \lambda_j$. Let u be a continuous function. On each simplex Δ , we consider its quadratic approximation $u_2 = \mathcal{I}_{2,\Delta} u$, where $\mathcal{I}_{2,\Delta} u$ is the Lagrange interpolant of u . The interpolation error for the linear approximation of u_2 is

$$e_2 = u_2 - \mathcal{I}_{1,\Delta} u_2 = 4 \sum_{k=1}^{n_d} (u_2(\mathbf{c}_k) - \mathcal{I}_{1,\Delta} u_2(\mathbf{c}_k)) b_k \equiv \sum_{k=1}^{n_d} \gamma_k b_k.$$

The L^2 -norm of this error is given by

$$\|e_2\|_{L^2(\Delta)}^2 = |\Delta| (\mathbb{B} \boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma}$ is the vector with n_d components γ_k and \mathbb{B} is the $n_d \times n_d$ symmetric positive definite matrix with positive entries $\mathbb{B}_{k,l} = |\Delta|^{-1} \int_{\Delta} b_k b_l dV$. This error is only a number; therefore, it does not provides any directional information. To recover this information, we split this error into n_d edge-based error estimates $\alpha_k \geq 0$ such that

$$\|e_2\|_{L^2(\Delta)} = |\Delta|^{1/2} \sum_{k=1}^{n_d} \alpha_k \quad \text{and} \quad \sum_{k=1}^{n_d} \alpha_k = (\mathbb{B} \boldsymbol{\gamma}, \boldsymbol{\gamma})^{1/2}. \quad (2.1)$$

In the next section, we motivate the following choice of α_k :

$$\alpha_k = |\gamma_k| (\mathbb{B} \boldsymbol{\gamma}, \boldsymbol{\gamma})^{1/2} \left(\sum_{k=1}^{n_d} |\gamma_k| \right)^{-1}. \quad (2.2)$$

Similarly, the L^2 -norm of gradient of e_2 is given by

$$\|\nabla e_2\|_{L^2(\Delta)}^2 = \left\| \sum_{k=1}^{n_d} \gamma_k \nabla b_k \right\|_{L^2(\Delta)}^2 = |\Delta| (\tilde{\mathbb{B}} \boldsymbol{\gamma}, \boldsymbol{\gamma}),$$

where $\tilde{\mathbb{B}}$ is the symmetric positive definite matrix with entries $\tilde{\mathbb{B}}_{k,l} = |\Delta|^{-1} \int_{\Delta} \nabla b_k \cdot \nabla b_l \, dV$. Again, we split the cell-based error (a number) into n_d edge-based error estimates $\tilde{\alpha}_k \geq 0$ such that

$$\|\nabla e_2\|_{L^2(\Delta)}^2 = |\Delta| \sum_{k=1}^{n_d} \tilde{\alpha}_k \quad \text{and} \quad \sum_{k=1}^{n_d} \tilde{\alpha}_k = (\tilde{\mathbb{B}}\boldsymbol{\gamma}, \boldsymbol{\gamma}). \quad (2.3)$$

In the next section, we motivate the following choice of $\tilde{\alpha}_k$:

$$\tilde{\alpha}_k = |\gamma_k| (\tilde{\mathbb{B}}\boldsymbol{\gamma}, \boldsymbol{\gamma}) \left(\sum_{k=1}^{n_d} |\gamma_k| \right)^{-1}. \quad (2.4)$$

3. Metric derivation from edge-based error estimates

The next lemma shows existence of a tensor metric generated by errors associated with mesh edges.

Lemma 1. *Let α_k , $k = 1, \dots, n_d$, be the values prescribed to edges of a d -simplex Δ such that $\alpha_k \geq 0$ and $\sum_{k=1}^{n_d} \alpha_k > 0$. Then, there exists a constant tensor metric \mathfrak{M}_{Δ} such that*

$$\left(\frac{d!}{(d+1)(d+2)} \right)^{1/d} |\Delta|_{\mathfrak{M}_{\Delta}}^{2/d} \leq \sum_{k=1}^{n_d} \alpha_k \leq |\partial\Delta|_{\mathfrak{M}_{\Delta}}^2. \quad (3.1)$$

The proof is sketched below and the detailed proof can be found in [2]. Let us consider the quadratic function $v_2 = -\frac{1}{2} \sum_{k=1}^{n_d} \alpha_k b_k$ and denote its Hessian by $\mathbb{H}(v_2)$. If $\det(\mathbb{H}(v_2)) \neq 0$, we set $\mathfrak{M}_{\Delta} = |\mathbb{H}(v_2)|$ where $|\mathbb{H}(v_2)|$ is the spectral module of $\mathbb{H}(v_2)$. Otherwise, we increase slightly the largest α_k so that the modified function v_2 has a non-singular Hessian. In practice, increase by 1% was sufficient for all numerical experiments.

The derivation of metric \mathfrak{M}_{Δ} suggests a simple motivation for the choices (2.2) and (2.4) when $\mathbb{H}(v_2)$ is definite. Since the bubble function b_k is non-zero only on one edge, we get

$$(\mathfrak{M}_{\Delta} \mathbf{e}_k, \mathbf{e}_k) = \frac{1}{2} \alpha_k (|\mathbb{H}(b_k)| \mathbf{e}_k, \mathbf{e}_k) = 4 \frac{\alpha_k}{|\gamma_k|} \|e_2\|_{L^{\infty}(\mathbf{e}_k)}.$$

When Δ is the \mathfrak{M}_{Δ} -equilateral simplex, we have

$$\frac{\alpha_1}{|\gamma_1|} \|e_2\|_{L^{\infty}(\mathbf{e}_1)} = \dots = \frac{\alpha_{n_d}}{|\gamma_{n_d}|} \|e_2\|_{L^{\infty}(\mathbf{e}_{n_d})}.$$

Thus, the choice (2.2) means that in a mesh consisting on \mathfrak{M}_{Δ} -equilateral simplexes, we equidistribute L^2 -norm of error over cells and L^{∞} -norm of error over edges.

Let $\tilde{\mathfrak{M}}_{\Delta}$ be the metric corresponding to $\tilde{\alpha}_k$. Repeating the above arguments, we obtain (2.4).

Combining (2.1), (2.3) and (3.1), we get the geometric representation of L^2 -norm of error and its gradient:

$$c_d |\Delta|^{1/2} |\Delta|_{\mathfrak{M}_{\Delta}}^{2/d} \leq \|e_2\|_{L^2(\Delta)} \leq |\Delta|^{1/2} |\partial\Delta|_{\mathfrak{M}_{\Delta}}^2, \quad (3.2)$$

$$c_d |\Delta|^{1/2} |\Delta|_{\tilde{\mathfrak{M}}_{\Delta}}^{1/d} \leq \|\nabla e_2\|_{L^2(\Delta)} \leq |\Delta|^{1/2} |\partial\Delta|_{\tilde{\mathfrak{M}}_{\Delta}}. \quad (3.3)$$

In other words, both norms of the error are controlled from above by \mathfrak{M}_{Δ} -perimeter and from below by \mathfrak{M}_{Δ} -volume of simplex Δ . Now we show how to modify the metric so that the controlling quantities will be measured in the same metric.

4. Generalizations to L^p -norms and C^2 -functions

Let $p \in (0; \infty]$. To control various L^p -norms of e_2 and ∇e_2 , we use the scaling result from [2]:

$$\mathfrak{M}_{\Delta,p} = (\det(\mathfrak{M}_{\Delta}))^{-1/(d+2p)} \mathfrak{M}_{\Delta} \quad \text{and} \quad \widetilde{\mathfrak{M}}_{\Delta,p} = (\det(\widetilde{\mathfrak{M}}_{\Delta}))^{-1/(d+p)} \widetilde{\mathfrak{M}}_{\Delta}.$$

Lemma 2. *It holds*

$$c|\Delta|_{\mathfrak{M}_{\Delta,p}}^{2/d+1/p} \leq \|e_2\|_{L^p(\Delta)} \leq C|\Delta|_{\mathfrak{M}_{\Delta,p}}^{1/p} |\partial\Delta|_{\mathfrak{M}_{\Delta,p}}^2, \quad (4.1)$$

$$\tilde{c}|\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}}^{1/d+1/p} \leq \|\nabla e_2\|_{L^p(\Delta)} \leq \tilde{C}|\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}}^{1/p} |\partial\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}}, \quad (4.2)$$

where constants c , C , \tilde{c} , and \tilde{C} depend only on d .

Up to this moment, we derived the geometric representation of various norms of $e_2 = \mathcal{I}_{2,\Delta}u - \mathcal{I}_{1,\Delta}u$. It was shown in [2] that the norm of e_2 provides a good approximation for the corresponding norm of the true error $e = u - \mathcal{I}_{1,\Delta}u$. For completeness, we summarize these important results. Let \mathcal{F} be the space of symmetric $d \times d$ matrices. For a vector \mathbf{e}_k , we define the following norm:

$$\|\mathbf{e}_k\|_{|\mathbb{H}|}^2 = \max_{\mathbf{x} \in \Delta} (|\mathbb{H}(\mathbf{x})| \mathbf{e}_k, \mathbf{e}_k). \quad (4.3)$$

Then, $\|\partial\Delta\|_{|\mathbb{H}|}^2$ means formally the sum of (4.3) over edges \mathbf{e}_k of Δ .

Lemma 3. *Let $u \in C^2(\bar{\Delta})$. Then, there exist positive constant c_o depending only on d such that*

$$\frac{d+1}{2d} \|e_2\|_{L^\infty(\Delta)} \leq \|e_\Delta\|_{L^\infty(\Delta)} \leq \|e_2\|_{L^\infty(\Delta)} + \frac{1}{4} \inf_{\mathbb{F} \in \mathcal{F}} \|\partial\Delta\|_{|\mathbb{H}-\mathbb{F}|}^2, \quad (4.4)$$

$$\|\nabla e_2\|_{L^\infty(\Delta)} - c_o \text{osc}(\mathbb{H}, \Delta) \leq \|\nabla e_\Delta\|_{L^\infty(\Delta)} \leq \|\nabla e_2\|_{L^\infty(\Delta)} + c_o \text{osc}(\mathbb{H}, \Delta), \quad (4.5)$$

where

$$\text{osc}(\mathbb{H}, \Delta) = \frac{|\partial\Delta|^{d-1}}{|\Delta|} \inf_{\mathbb{F} \in \mathcal{F}} \|\partial\Delta\|_{|\mathbb{H}-\mathbb{F}|}^2.$$

The oscillation terms are conventional in contemporary error analysis. Their value depend on the simplex and particular features of the function. For instance, if $u \in C^2(\bar{\Delta})$, and Δ is shape regular, one has $\text{osc}(\mathbb{H}, \Delta) \leq C|\partial\Delta| \inf_{\mathbb{F} \in \mathcal{F}} |\mathbb{H} - \mathbb{F}|_\infty$. Thus, the oscillation terms are smaller than the errors. Similar analysis can be performed for L^p -norms, $p > 0$.

5. Asymptotic error estimates

Let Ω_h and $\tilde{\Omega}_h$ be simplicial meshes with N_h cells that balance the volume and perimeter of cells:

$$N_h^{-1} |\Omega|_{\mathfrak{M}_p} \simeq |\Delta|_{\mathfrak{M}_{\Delta,p}} \simeq |\partial\Delta|_{\mathfrak{M}_{\Delta,p}}^d \quad \forall \Delta \in \Omega_h,$$

$$N_h^{-1} |\Omega|_{\widetilde{\mathfrak{M}}_p} \simeq |\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}} \simeq |\partial\Delta|_{\widetilde{\mathfrak{M}}_{\Delta,p}}^d \quad \forall \Delta \in \tilde{\Omega}_h.$$

On such meshes, the following error estimates are held

$$\|e\|_{L^p(\Omega)} = \left(\sum_{\Delta \in \Omega_h} \|e\|_{L^p(\Delta)}^p \right)^{\frac{1}{p}} \lesssim \left(\sum_{\Delta \in \Omega_h} |\Delta|_{\mathfrak{M}_{\Delta,p}}^{1+\frac{2p}{d}} \right)^{\frac{1}{p}} \lesssim |\Omega|_{\mathfrak{M}_p}^{\frac{1}{p}+\frac{2}{d}} N_h^{-\frac{2}{d}},$$

$$\|\nabla e\|_{L^p(\Omega)} = \left(\sum_{\Delta \in \tilde{\Omega}_h} \|\nabla e\|_{L^p(\Delta)}^p \right)^{\frac{1}{p}} \lesssim \left(\sum_{\Delta \in \tilde{\Omega}_h} |\Delta|_{\tilde{\mathfrak{M}}_{\Delta,p}}^{1+\frac{p}{d}} \right)^{\frac{1}{p}} \lesssim |\Omega|_{\tilde{\mathfrak{M}}_p}^{\frac{1}{p}+\frac{1}{d}} N_h^{-\frac{1}{d}},$$

where $a \lesssim b$ means existence of constants c and C independent of mesh such that $ca \leq b \leq Ca$. In other words, the \mathfrak{M}_p (resp., $\tilde{\mathfrak{M}}_p$)-quasi-uniform meshes provide asymptotically optimal rate for reduction of the L^p -norm of the error (resp., the gradient of the error).

6. Metric-based mesh adaptation

We use Algorithm 1 to build an adaptive mesh minimizing the L^p -norm of error or its gradient. It provides faster convergence and results in smoother meshes when the metric is continuous. To define a continuous metric we suggest a method of shifts. For every node \mathbf{a}_i in Ω_h , we define the superelement σ_i as the union of all d -simplexes sharing \mathbf{a}_i . Then, $\mathfrak{M}(\mathbf{a}_i)$ is defined as one of the metrics in σ_i with the largest determinant. This method always chooses the worst metric in the superelement. To generate a \mathfrak{M} -quasi-uniform mesh, we use local mesh modifications described in [3] and implemented in package `Ani2D` (sourceforge.net/projects/ani2d).

Algorithm 1 Adaptive mesh generation

- 1: Generate an initial mesh Ω_h and compute the metric \mathfrak{M} .
 - 2: **loop**
 - 3: Generate a \mathfrak{M} -quasi-uniform mesh Ω_h with the prescribed number of simplexes.
 - 4: Recompute the metric \mathfrak{M} .
 - 5: If Ω_h is \mathfrak{M} -quasi-uniform, then exit the loop
 - 6: **end loop**
-

For the numerical illustration, we consider the analytical function

$$u = (x^2y + y^3)/16^3 + \tanh(2(\sin(6y) - 3x)(\sin(6x) - 3y)) \quad (6.1)$$

in the Texas-shape domain inscribed in $[-\frac{3}{2}; \frac{3}{2}]$. The spider-like isolines of u (see Fig. 1) show that this function has both isotropic and anisotropic regions. Table 1 verifies the theoretical estimates from Section 5. The L^∞ -norm of the interpolation error on meshes Ω_h built with Algorithm 1 and metric \mathfrak{M}_p is proportional to N_h^{-1} . The L^∞ -norm of its gradient on meshes $\tilde{\Omega}_h$ built with Algorithm 1 and metric $\tilde{\mathfrak{M}}_p$ is proportional to $N_h^{-1/2}$. The last column in Table 1 shows that the L^∞ -norm of the interpolation error on mesh $\tilde{\Omega}_h$, which is not the optimal mesh for this error, exhibits still the optimal convergence rate; albeit, the error is larger than that on mesh Ω_h . We also note that the mesh $\tilde{\Omega}_h$ (Fig.1, right) is denser than the mesh Ω_h (Fig.1, left) in regions where the solution is sharp.

N_h	$\ e\ _{L^\infty(\Omega_h)}$	$\ \nabla e\ _{L^\infty(\tilde{\Omega}_h)}$	$\ e\ _{L^\infty(\tilde{\Omega}_h)}$
2000	3.72e-2	1.23e+0	2.04e-1
4000	1.76e-2	8.00e-1	9.24e-2
8000	8.12e-3	5.36e-1	4.06e-2
16000	4.60e-3	4.00e-1	1.99e-2
32000	2.19e-3	2.72e-1	1.11e-2
64000	1.22e-3	1.92e-1	6.47e-3
rate	0.99	0.53	1.00

Table 1: Convergence of the L^∞ -norm of the interpolation error and its gradient.

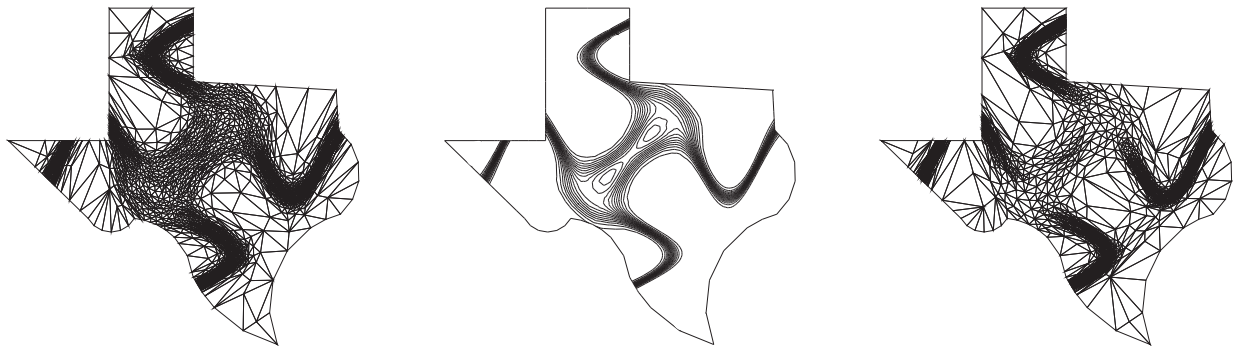


Figure 1: Left: The mesh Ω_h with roughly 2000 triangles minimizing $\|e\|_{L^\infty(\Omega_h)}$. Middle: The iso-lines of function (6.1). Right: The mesh $\tilde{\Omega}_h$ with roughly 2000 triangles minimizing $\|\nabla e\|_{L^\infty(\tilde{\Omega}_h)}$.

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