Math. Model. Nat. Phenom.
Vol. 6, No. 1, 2011, pp. 209-225
DOI: 10.1051/mmnp/20116111

Control of Traveling Solutions in a Loop-Reactor

Y. Smagina[1] and M. Sheintuch

Department of Chemical Engineering, Technion, Haifa, 32000, Israel

Abstract. We consider the stabilization of a rotating temperature pulse traveling in a continuous asymptotic model of many connected chemical reactors organized in a loop with continuously switching the feed point synchronously with the motion of the pulse solution. We use the switch velocity as control parameter and design it to follow the pulse: the switch velocity is updated at every step on-line using the discrepancy between the temperature at the front of the pulse and a set point. The resulting feedback controller, which can be regarded as a dynamic sampled-data controller, is designed using root-locus technique. Convergence conditions of the control law are obtained in terms of the zero structure (finite zeros, infinite zeros) of the related lumped model.

Key words: moving pulses; network of chemical reactors; loop-reactor; distributed systems; control; root-locus method; system zeros.
AMS subject classification: 35C07, 34H05

1. Introduction

Manipulation of pattern-forming processes by applying feedback control is a subject of active research in physics, engineering and physiology. Various types of control have been applied to a wide class of systems with nonlinear dynamics (see, for example, [3] and references therein). Among these studies, the problem of control of moving patterned states that emerge in a reaction-diffusion-advection systems with excitable dynamics has attracted considerable interest: (e.g., [6], [7], [4], [5]). Control of a moving pulse has been rarely addressed because of its mathematically complexity. In the present work we focus on the traveling pulse that emerge in a loop-shaped reactor composed of several reaction-diffusion-advection units with feed switching between them

[1]Corresponding author. E-mail: cermssy@tx.technion.ac.il

(Figure1a). To illustrate this concept consider a loop reactor of overall length L and N units with a feed port and an exit port that are switched at every predetermined time interval, σ. In the first interval the feed flow enters the first reactor and exits from reactor N (denoted as units A, C in Figure1a). At the second interval, the feed flow enters the second reactor (B) and exits from the first reactor (A), and so on. In the desired loop reactor operation a temperature pulse (hot spot) is formed and rotates around the system. Such type reactor is one of suggested technological solutions for low-concentration volatile organic carbons combustion and for exothermic reversible reaction (see [12]).

Each reactor is of length $\Delta = L/N$ and is described by the dimensionless PDEs (see details in work [18])

$$Le\frac{\partial y}{\partial \tau} + V\frac{\partial y}{\partial \xi} - Pe_y^{-1}\frac{\partial^2 y}{\partial \xi^2} = Bf_1(x,y), \tag{1.1}$$

$$\frac{\partial x}{\partial \tau} + V\frac{\partial x}{\partial \xi} - Pe_x^{-1}\frac{\partial^2 x}{\partial \xi^2} = f_1(x,y) \tag{1.2}$$

where $\xi \in [0, L]$,

$$f_1(x,y) = Da(1-x)exp(\frac{\gamma y}{\gamma + y}) \tag{1.3}$$

The two variables are usually the temperature (y) and conversion (x), V is the fluid velocity, $Le >> 1$ is the ratio of solid to fluid heat capacity, the numbers Pe (Pe_x, Pe_y) denote inverse heat conductivity and mass dispersion, B is the reaction exothermicity while γ is the dimensionless activation energy and Da (Damkohler number) is the dimensionless rate constant.

The boundary conditions are

$$Pe_z^{-1}\frac{\partial z}{\partial \xi}\mid_{\xi=\xi_{feed}} = V(z\mid_{\xi=\xi_{feed}} - z_{in}), \quad \frac{\partial z}{\partial \xi}\mid_{\xi=\xi_{feed}+\Delta} = 0 \tag{1.4}$$

where $z = x, y$ and $z_{in} \neq 0$ for every feed port and $z_{in} = z$ of previous unit for other units.

The switching feed loop reactor (Figure1b) can be viewed as a counter-clockwise-rotating loop-shaped reactor, with reactant flow in opposite direction while the feed and exit ports are fixed in space. In the limit of many small reactors, with which we are interested in this contribution, we can view the rotation as a steady process, while the valve separating the feed and exit ports are fixed. Rotating pulses in a loop reactor may emerge provided that the switching velocity ($V_{sw} = \Delta/\sigma$ i.e., unit length/switching time) and the pulse velocity V_{fr} are matched in a certain way, $V_{sw} \cong V_{fr}$ (see [18]) and the pattern is "frozen" in moving coordinates. The "frozen" pulse solutions exist only in a narrow domain of switch velocities (see [14]) and are very sensitive to fluctuations in operating conditions and uncertainties in the model parameters resulting to reaction extinction. To overcome this instability we consider the problem of stabilization of a rotating pulse by automatically tuning the parameter V_{sw} to the value that corresponds to the stable moving the pulse solution. Unlike our previous study [20] the present research is based on the asymptotic continuous model of the loop reactor (limiting case of the loop reactor with $N \to \infty$ [18]). This model on the one hand successfully approximates the dynamics of multi-units

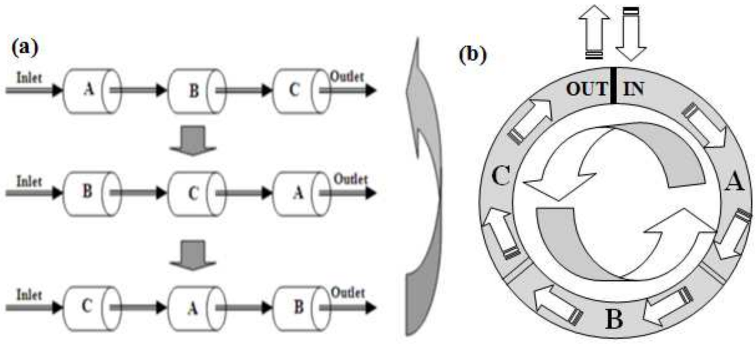

Figure 1: Scheme of the three-reactor three-positions loop system (a) and its presentation as a rotating loop-reactor with flow and rotation in opposite directions (b) (see text for details).

network reactor, while its structure is simpler than that of the original multi-units network model. This simplifies the design of the controller.

This reactor was demonstrated experimentally in our group [11], using ethylene oxidation as a model reaction, and using a model-free control scheme in which feed switching from one unit to the following occurs when the temperature at a certain position in the following units exceeds a threshold value (T_{SW}) which was determined empirically. The difference between model-free and model-based control was discussed elsewhere [20].

The proposed model-based control structure is similar to the well-known no-reset discrete- time iterative learning control [19]. Using root-locus technique and concept of finite and infinite zeros we find admissible sensor positions and obtain solvability conditions for control existence and for assurance of stable pulse motion. Unlike numerical control solutions proposed in the literature [2] our approach gives analytical solution of the problem.

2. Statement

To simplify mathematical manipulations we shall study here the control strategy on a simpler formal continuous one-variable model of the loop reactor [2]

$$\frac{\partial y}{\partial \tau} + V\frac{\partial y}{\partial \xi} - Pe_y^{-1}\frac{\partial^2 y}{\partial \xi^2} = f(y), \tag{2.1}$$

where $\xi \in [0, L]$, L is a length of the ring, y is the state variable, parameters V, Pe_y were described above (Eqns. 1.1-1.4) and $f(.)$ is some nonlinear function of ensuring the existence of the moving pulse solution. At the feed and exit ports the conditions are

[2]To simplify the full model (Eqn.1.1-1.3) to a single-variable problem we use the realistic assumptions of $Le >> 1$ and $Pe_x > Pey >> 1$. Thus, in the full model we may capitalize in the short time scale of the mass balance the absence of mass-mixing, which allow to find an approximate $x(y)$ relation and reduce model (1.1)-(1.3). Also we redefine τ. The detailed description of this simplification is in ([20], Appendix A).

$$Pe_y^{-1}\frac{\partial y}{\partial \xi}\mid_{\xi=\xi_{feed}}= V(y\mid_{\xi=\xi_{feed}} - y_{in}), \quad \frac{\partial y}{\partial \xi}\mid_{\xi=\xi_{feed}+L}= 0 \tag{2.2}$$

and they describe the limiting case of the network loop reactor ($N \to \infty, \Delta \to 0$) when the feed point is changed continuously at a speed of V_{sw} so that $\xi_{feed} = \xi_{feed}(\tau) = [\tau V_{sw}]mod(L)$ [3]. The feed should follow synchronously the motion of the pulse so that $V_{sw} = V_{fr}$ where V_{fr} is the pulse velocity [18]. The corresponding boundary-conditions at the feed- and exit-ports coincide with Eqn.2.2.

We can see that the behaviour of propagating pulses here is similar to one emerging in reaction-diffusion systems with excitable media like heart. For such moving-pulse solutions we state the general control problem as follows: Stabilize rotating pulses in excitable media by using a pulse velocity (V_{fr}) as control parameter.

Redefining $\zeta = \xi - V_{fr}\tau, \quad \tau' = \tau$ we reduce the original Eqn.(2.1-2.2) to the following system in a moving coordinate with $\zeta_{feed} = 0$ [4]

$$\frac{\partial y}{\partial \tau} + (V - V_{fr})\frac{\partial y}{\partial \zeta} - Pe_y^{-1}\frac{\partial^2 y}{\partial \zeta^2} = f(y), \qquad \zeta \in [0, L] \tag{2.3}$$

subject to

$$Pe_y^{-1}\frac{\partial y}{\partial \zeta}\mid_{\zeta=0}= V(y\mid_{\zeta=0} - y_{in}), \quad \frac{\partial y}{\partial \zeta}\mid_{\zeta=L}= 0 \tag{2.4}$$

Let's note that $\zeta = (\xi - V_{fr}\tau)mod(L)$ for the ring domain.

We should note that any solution of Eqn.2.1,2.2 that moves with a specified velocity V_{fr} is transformed into the frozen solution in moving coordinate of Eqn.2.3,2.4.

Let V_{fr}^* be a velocity of a pulse solution of system (2.3),(2.4) with a corresponding $y_o(\zeta)$ that sustains a pulse as $\tau \to \infty$. Proposing that the control variable V_{fr} may be updated at certain time instants, we formulate our purpose as follows: To design a variable V_{fr} that should automatically be tuned to the value V_{fr}^* i.e. $V_{fr} \to V_{fr}^*$ as $\tau \to \infty$. Such control prevents extinction of any pulse solution of Eqn.(2.3-2.4) for large times even when all parameters are not known exactly i.e. control assures $y(\zeta, \tau) \to y_o(\zeta)$ as $\tau \to \infty$.

3. Structure of control

Let's assume that the variable $y(\zeta, \tau)$ is accessible at every point of the domain $[0, L]$. We will use the discrete-time control policy updating the control variable V_{fr} at equal time intervals $\tau_i = i\sigma, \quad i = 1, 2, \ldots$ according to the rule: At a moment $\tau_i = \sigma i$ the present state $y(\zeta^*, \tau_i)$ at a position ζ^* is compared with the set point y^* and V_{fr} is corrected. As a set point we use $y^* = y_o(\zeta^*)$. The resulting control law takes the form of the following dynamic feedback discrete-time controller

[3] $\alpha = [\beta]mod(L)$ denotes the remainder of division of β by L.

[4] Further for simplicity we will use τ.

$$V_{fr}(\tau_{i+1}) = V_{fr}(\tau_i) - k[y(\zeta^*, \tau_i) - y^*], \quad k > 0, \quad i = 1, 2, \dots \tag{3.1}$$

and it is activated at time instants τ_i at the sensor position ζ^*. For sake of definiteness we will study the case $k > 0$[5]. This control is the sampled-data control [9], [10] having the structure of the non-reset iterative learning control [19]. Note that the sensor is located at a position moving with the pulse (i.e. ζ^* is fixed in a moving coordinate); we should address this problem of sensor location in real systems.

To choose the parameters ζ^*, k, σ of controller (3.1), we will study the behaviour of the nonlinear model (2.3, 2.4, 3.1) in proximity of the steady solution y_o. For small deviations $\bar{y} = y - y_o$ and $\bar{V}_{fr} = V_{fr} - V_{fr}^*$ we obtain the following linearized PDE equation

$$\frac{\partial \bar{y}}{\partial \tau} + \mathcal{L}(\bar{y}) = \frac{\partial f}{\partial y} |_{y_o} \bar{y} + V_{fr}^* \frac{\partial \bar{y}}{\partial \zeta} + \frac{\partial y_o}{\partial \zeta} \bar{V}_{fr}(\tau_i), \quad i = 1, \dots \tag{3.2}$$

where

$$\mathcal{L}(\bar{y}) = V \frac{\partial \bar{y}}{\partial \zeta} - Pe_y^{-1} \frac{\partial^2 \bar{y}}{\partial \zeta^2} \tag{3.3}$$

The boundary conditions become

$$Pe_y^{-1} \frac{\partial \bar{y}}{\partial \zeta} |_{\zeta=0} = V \bar{y} |_{\zeta=0}; \quad \frac{\partial \bar{y}}{\partial \zeta} |_{\zeta=L} = 0 \tag{3.4}$$

Expressing $\bar{V}_{fr}$ as the dynamic equation for the deviation of control

$$\bar{V}_{fr}(\tau_{i+1}) = \bar{V}_{fr}(\tau_i) - k\bar{y}(\zeta^*, \tau_i), \quad i = 1, 2, \dots \tag{3.5}$$

where $\bar{y}(\zeta^*, \tau_i) = y(\zeta^*, \tau_i) - y^*, \; y^* = y_o(\zeta^*)$ and inserting (3.5) in (3.2) we obtain the closed-loop system in deviations.

Let us note that in the original fixed coordinate system Eqn.3.2 becomes $\frac{\partial \bar{y}}{\partial \tau} + \mathcal{L}(\bar{y}) = \frac{\partial f}{\partial y} |_{y_o} \bar{y} + \frac{\partial y_o}{\partial \xi} \bar{V}_{fr}(\tau_i)$. Hence the term $\frac{\partial y_o}{\partial \xi} \bar{V}_{fr}(\tau_i)$ that is approximated for 'frozen solution' as $\frac{\partial y}{\partial \xi} \bar{V}_{fr}(\tau_i)$ can change the dynamic behaviour of system (3.2) and its stability. The same applies to the original nonlinear model (Eqn.2.1) obtained by adding the above equation and the steady equation: $\frac{\partial y_o}{\partial \tau} + \mathcal{L}(y_o) = f(y_o)$

$$\frac{\partial y}{\partial \tau} + \mathcal{L}(y) = f(y) + \frac{\partial y_o}{\partial \xi} \bar{V}_{fr}(\tau_i), \quad y = y(\xi, \tau), \quad i = 1, \dots \tag{3.6}$$

[5]Similar as in work [20].

4. Stability conditions

To find stability conditions we convert linearized PDE (3.2) into an infinite set of linear ODEs by Galerkin's method and approximate the infinite dimensional model by the following M - dimensional ODE system [6]

$$\dot{\boldsymbol{a}}(\tau) = A\boldsymbol{a}(\tau) + P\bar{V}_{fr}(\tau_i), \quad \tau_i = \sigma(i-1), \quad i = 1, 2, \ldots \tag{4.1}$$

where $\boldsymbol{a}$ is an M-vector of continuous-time state variable, A is an $M \times M$ matrix and P is an $M \times 1$ column vector. Matrix A takes the form

$$A = -\Lambda + J_y + V_{fr}^* F_y \tag{4.2}$$

where $\Lambda = diag(\rho_1, \rho_2, \ldots, \rho_M)$ and elements of $M \times M$ matrices J_y, F_y and vector P are calculated from the following expressions

$$\{\{J\}\}_{nj} = \int_0^L \frac{\partial f}{\partial y}(y_o(\zeta))\phi_n \phi_j^a d\zeta, \quad \{\{F\}\}_{nj} = \int_0^L \frac{\partial \phi_n}{\partial \zeta}\phi_j^a d\zeta, \quad \{\{P\}\}_j = \int_0^L \frac{\partial y_o}{\partial \zeta}\phi_j^a d\zeta \tag{4.3}$$

where ρ_n, $\phi_n = \phi_n(\zeta)$ and $\phi_n^a = \phi_n^a(\zeta)$, $n = 1, 2, \ldots$ are eigenvalues, eigenfunctions and adjoint eigenfunctions of the linear operator (3.3) with b.c. (3.4).

Now forming the scalar output at instants τ_i

$$w(\tau_i) = H\boldsymbol{a}(\tau_i) \tag{4.4}$$

where elements of the $1 \times M$ row vector H are calculated as

$$\{\{H\}\}_m = \phi_m(\zeta^*), \quad m = 1, 2, \ldots, M \tag{4.5}$$

we present the discrete-time control (3.5) in the form

$$\bar{V}_{fr}(\tau_{i+1}) = \bar{V}_{fr}(\tau_i) - kH\boldsymbol{a}(\tau_i), \quad k > 0, \quad \tau_i = \sigma(i-1), \quad i = 1, 2, \ldots \tag{4.6}$$

It should be noted that equation (4.1) is a hybrid one: It is a connection of continuous-time dynamics and discrete-time control that is fed into the continuous system by means of an ideal zeroth-order hold, i.e. the value $\bar{V}_{fr}(\tau)$ is sampled and held constant between any two consecutive sampling instants. This control action is called a single-rate digital control because it synchronously turns on with a common sample period (σ). For such control the following result is well known (see e.g. [9], [10]).

Assertion 1. *Control (4.6) with appropriate gain k assures that the zero solution of closed-loop hybrid system (4.1, 4.4, 4.6) is uniformly asymptotically stable if and only if the matrix*

$$\begin{bmatrix} G & D \\ -kH & 1 \end{bmatrix}$$

[6] M is the truncated order, an integer satisfying $max(eig(A_M)) \cong max(eig(A_{M+1}))$ where A_m is $M \times M$ matrix.

is Schur stable, i.e. it has no eigenvalues z_i outside of the unit circle: $|z_i| < 1$. Here

$$G = e^{A\sigma}, \quad D = (\int_o^\sigma e^{A\tau} d\tau)P \tag{4.7}$$

and $\sigma = \tau_{i+1} - \tau_i$ *is the period of discretization.*

A more constructive result may be obtained in terms of open-loop system (Eqn. 4.1, 4.4). To obtain this point we sample the continuous-time part of Eqn.4.1 with a period σ coinciding with the sampling interval of the piecewise input signal $\bar{V}_{fr}(\tau_i)$, $\tau_i = \sigma(i-1)$. Letting $\boldsymbol{a}(i) = \boldsymbol{a}(\tau_i)$, $v(i) = \bar{V}_{fr}(\tau_i)$ we obtain the following discrete time equation

$$\boldsymbol{a}(i+1) = G\boldsymbol{a}(i) + Dv(i), \quad i = 1, 2, \ldots \tag{4.8}$$

where $\boldsymbol{a}(i)$ is an M - column vector and the $M \times M$ matrix G and the M-column vector D are calculated by formulas (4.7). The output of system (4.8) is

$$w(i) = H\boldsymbol{a}(i) \quad i = 1, 2, \ldots \tag{4.9}$$

where $w(i) = w(\tau_i)$. In the terms of Eqn.(4.8), (4.9) discrete-time control (4.6) becomes the form of the following dynamic feedback

$$v(i+1) = v(i) - kw(i) = v(i) - kH\boldsymbol{a}(i), \quad k > 0, \quad i = 1, 2, \ldots \tag{4.10}$$

Theorem 2. *Control (4.10) with an appropriate gain k stabilizes the closed-loop discrete-time system (4.8-4.10) by the root-locus technique if the following conditions are satisfied:*

(a) $HD \neq 0 \quad (HP \neq 0)$

(b) the roots s_j of the equation

$$\psi(s) = det \begin{bmatrix} sI - G & -D \\ H & 0 \end{bmatrix} = 0 \tag{4.11}$$

lie within the unit circle: $|s_j| < 1$

(c) $|\alpha| < 1$

where α, the point of intersection of asymptotes, is calculated from the relation ([15], [13])

$$\alpha = 0.5(1 + \sum_{j=1}^{M} z_j - \sum_{j=1}^{M-1} s_j) \tag{4.12}$$

Here s_j are finite zeros of system (4.8), (4.9) and z_j are eigenvalues of the matrix G.

Proof. Let us write Eqns (4.8-4.10) in the standard state space form of a system of order $M+1$ with a single input $u(i)$

$$\begin{bmatrix} \boldsymbol{a}(i+1) \\ v(i+1) \end{bmatrix} = \begin{bmatrix} G & D \\ O & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}(i) \\ v(i) \end{bmatrix} + \begin{bmatrix} O \\ 1 \end{bmatrix} u(i) \tag{4.13}$$

and a single output

$$w(i) = [H, 0] \begin{bmatrix} \boldsymbol{a}(i) \\ v(i) \end{bmatrix} \tag{4.14}$$

where $\boldsymbol{a}(i)$ is a M - column vector and $v(i)$ and $u(i)$ are scalar variables. Here O is the zero matrix of the appropriate dimension, 0 is the zero element.

Control here takes the form of the following feedback linear control

$$u(i) = -kw(i), \quad k > 0 \tag{4.15}$$

At first we recall that as the gain coefficient of the closed-loop system grows indefinitely ($k \to \infty$) than controllable and observable eigenvalues of the $(M+1) \times (M+1)$ dynamics matrix of the closed-loop system, $E_c = \begin{bmatrix} G & D \\ O & 1 \end{bmatrix} - k \begin{bmatrix} O \\ 1 \end{bmatrix} [H, 0]$ approach the location of transmission zeros [16] of the open-loop system (Eqn.4.13, 4.14). At the same time, uncontrollable (unobservable) eigenvalues of this matrix coincide with decoupling [16] zeros of this system. All these zeros (transmission and decoupling ones) form the set of finite zeros [16] that are calculated as the set of complex $s = s_i$ for which the normal rank [8] of the system matrix $P(s) = \begin{bmatrix} sI_M - G & -D & O \\ O & s-1 & 1 \\ H & 0 & 0 \end{bmatrix}$ is reduced [16]. Using the series of the rank equalities we may evaluate rank of as follows

$$rankP(s) = rank \begin{bmatrix} sI_M - G & -D \\ H & 0 \end{bmatrix} \tag{4.16}$$

Thus, the normal rank of matrix $P(s)$ is reduced if and only if the complex values s coincide with one calculated from Eqn.4.11 that are, in fact, finite zeros[7] of the system

$$\boldsymbol{a}(i+1) = G\boldsymbol{a}(i) + Dv(i), \quad w(i) = H\boldsymbol{a}(i) \tag{4.17}$$

Then if finite zeros of any system lie within the unit circle then transmission zeros of this system also lie within the unit circle because the set of transmission zeros is included in the set of finite zeros [16]. Hence, if condition (b) of Theorem 2 (zeros of lie within the unit circle) is satisfied then transmission zeros of the system (4.13),(4.14) also lie within the unit circle and control (4.15) assures that a number of eigenvalues of closed-loop tends to the values of finite zeros that are located within the unit circle. Let us examine the behavior of all $(M+1)$ eigenvalues of the dynamics matrix E_c of the closed-loop system. Denoting the number of finite zeros of a system by M_z we recall that system (4.17) with $HD \neq 0$ has $M_z = M - 1$ finite zeros [21]. From above

[7]Finite zeros coincide with the set of complex for which the normal rank of the system matrix $\begin{bmatrix} sI_M - G & -D \\ H & 0 \end{bmatrix}$ is reduced (see[16]).

reasoning it follows that system (4.13), (4.14) has the same number $(M-1)$ of finite zeros. Thus, the $M-1$ eigenvalues (from $M+1$) of the dynamics matrix E_c of the closed-loop system tend to the finite values (finite zeros) with increasing gain (k) and the remaining two eigenvalues terminate at infinity along asymptotes. To assure stability of the closed-loop system we need to provide that these infinitely increasing eigenvalues (infinite zeros of system (4.13),(4.14)) are located within the unit circle.

To assure this point we use the known properties of root-locus of a system with a scalar input /output ([15], [13]. The angles of asymptotes of these two infinite zeros are given by the formula

$$\eta_{1,2} = \pm 180^o/2 = \pm 90^o$$

These asymptotes intersect on a point α of the real axis. Since the considered system has $M_z = M-1$ finite zeros $(s_1, \ldots, s_M)$ [8] and $M+1$ poles $(z_1, \ldots, z_M, 1)$ [9] where $z_j,\ j = 1, \ldots, M$ are eigenvalues of the matrix G then the point α calculated from relation (4.12) should satisfy condition (c) of Theorem 2. The proof is complete.

Then we need to guarantee that the sampling is realized without loss of controllability and observability and without introducing additional zeros. It is well-known that the controllability (observability) property can be preserved under almost any sampling rate, except countable isolated points. To keep controllability(observability) of the sampled system we need to fulfil the following condition for complex eigenvalues of the matrix A ([22])

$$exp(\lambda_l \sigma) \neq exp(\lambda_j \sigma) \quad \text{for any complex} \quad \lambda_l \neq \lambda_j{}^{10}. \tag{4.18}$$

where σ is the period of discretization

Also, any pole ρ_c of the continuous-time system is related to a certain pole ρ_d of the sampled system as $\rho_d = exp(\rho_c \sigma)$ and vice versa. However, no simple relation exists for the zeros of continuous-time and discrete-time systems [1]. These zeros may lie outside the unit circle and create difficulties for control. On the other hand the condition $HC \neq 0$ guarantees that the continuous-time system $\dot{\boldsymbol{a}} = A\boldsymbol{a} + Cv,\ \ w = H\boldsymbol{a}$ has $(M-1)$ has $M-1$ finite zeros ([21]) and so does the associated discrete system. Thus, if the condition (a) of Theorem 2

$$HP \neq 0 \tag{4.19}$$

is satisfied then the 'pole-zero' behaviour of system (4.8-4.9) is similar to the 'pole-zero' behaviour of system (4.1 - 4.4) with continuous-time output and control. The system (4.13-4.14) possesses the same property because of its special structure.

Let us note that above conditions may be satisfied only by appropriately choosing elements of the matrix H that are, in turn, dependent on the sensor position ζ^*.

---

[8] See the condition $HD \neq 0$.

[9] We propose that system (4.13), (4.14) is controllable and observable. Thus, its poles are eigenvalues of the dynamics matrix of (4.13).

[10] Condition (4.18) is equivalent to the following one: $\sigma \neq m\pi/(\omega_j - \omega_l)$ for any $\lambda_{l.j} = \delta_{l,j} + \omega_{l,j}\sqrt{(}-1)$ with $\delta_l = \delta_j,\ \ \omega_j \neq \omega_l$ and $m = \pm 1, \pm 2, \ldots$ [22].

Remark 3. *The magnitude of the gain coefficient k in the discrete-time control (4.15) should be restricted to keep infinite zeros within the unit circle. Upper bound on the gain may be evaluated from the root-locus plots or using inequality obtained by the authors in [20]: $k \leq \frac{\partial V_{fr}}{\partial y}$.*

Remark 4. *According to the above theory the sensor positions in the original fixed coordinate system should be activated by the following law $\xi_i^* = [\zeta_i^* + \sigma \sum_i V_{fr}^i] mod(L)$ where $V_{fr}^i = V_{fr}(\tau_i)$. Because difference $V_{fr}(\tau_i) - V_{fr}^*(\tau_i)$ is vanishing with growing i in the stable closed-loop system, then we may roughly approximate $\xi_i^* = [\zeta_i^* + \sigma i V_{fr}^*] mod(L)$.*

Remark 5. *We can recommend the following strategy for control design: Let us assign σ (interval of control updating) and some initial value $V_{fr}(0)$. Then we measure the variable y at the moment $\tau_1 = \sigma$ at the position $\xi = \zeta^* + V_{fr}(0)\sigma$ i.e. we evaluate $y(\zeta^* + V_{fr}(0)\sigma, \tau_1)$ and correct V_{fr} according Eqn.3.1 to obtain $V_{fr}(\tau_1)$. Furthermore, at the moment $\tau_2 = 2\sigma$ we measure $y(\zeta^* + (V_{fr}(0) + V_{fr}(\tau_1))\sigma, \tau_2)$ and do new correction of V_{fr} according Eqn.3.1 to obtain $V_{fr}(\tau_2)$ and so on.*

5. Application

As an application we consider a network of N identical fixed-bed reactors (loop reactor) with gradual switching of the inlet/outlet ports at each time interval σ (the original full model, Eqns.1.1-1.4).

In the limiting case of a loop reactor with infinite number of ports ($N \rightarrow \infty$) the stepwise function ξ_{feed} can be replaced by a continuous description of the feed position $\xi_{feed} = \xi_{feed}(\tau) = [\tau V_{sw}(\tau)] mod(L)$ where $V_{sw}(\tau)$ is the switch velocity. We consider the case when a "frozen" front or pulse solution rotates in the network system at a velocity V_{fr} that is kept to coincide with the switch velocity ($V_{sw} \cong V_{fr}$). The operation of such network is approximated by the asymptotic continuous model that in a moving coordinate system ($\zeta = \xi - V_{sw}\tau, \quad \tau' = \tau$) with the fixed position of the inlet and outlet takes the following continuous form

$$Le\frac{\partial y}{\partial \tau} + (V - LeV_{sw})\frac{\partial y}{\partial \zeta} - Pe_y^{-1}\frac{\partial^2 y}{\partial \zeta^2} = Bf_1(x, y), \tag{5.1}$$

$$\frac{\partial x}{\partial \tau} + V\frac{\partial x}{\partial \zeta} - Pe_x^{-1}\frac{\partial^2 x}{\partial \zeta^2} = f_1(x, y) \tag{5.2}$$

with boundary conditions

$$Pe_z^{-1}(\frac{\partial z}{\partial \zeta}\mid_{\zeta=0} - \frac{\partial z}{\partial \zeta}\mid_{\zeta=L}) = V(z\mid_{\zeta=0} - z_{in}), \quad z(0) = z(L) \tag{5.3}$$

where $z = y, x$.

This asymptote (for infinite number of units $N \rightarrow \infty$) was derived and also demonstrated in work of M.Sheintuch and O.Nekhamkina [18]. They also showed that the steady state pulse solution $y_o(\zeta), x_o(\zeta)$ of system (5.1-5.3), the loop reactor with periodic switching, exists if $V_{sw} = V^*_{sw} \in [V^{min}_{sw}, V^{max}_{sw}]$. Outside this domain the solution is extinguished (cold) and the reactor is inactive (see [18]). Our purpose is to find a rule to update the switch velocity V_{sw} to prevent extinction of this solution when initially $V_{sw} \notin [V^{min}_{sw},\ V^{max}_{sw}]$, i.e. V_{sw} -control that assures $y(\zeta,\tau) \to y_o(\zeta),\ x(\zeta,\tau) \to x_o(\zeta)$ as $\tau \to \infty$, We update V_{sw} at instants τ_i according to the rule

$$V_{sw}(\tau_{i+1}) = V_{sw}(\tau_i) - k[y(\zeta^*,\tau_i) - y^*] \tag{5.4}$$

To study linear stability we linearize Eqn. (5.1), (5.2) about $y_o,\ \ x_o,\ \ V^*_{sw}$ and using Galerkin's method approximate the linearized PDEs by the following ODE system

$$\dot{\boldsymbol{a}}(\tau) = A\boldsymbol{a}(\tau) + P\bar{V}_{sw}(\tau_i),\ \tau_i = \sigma(i-1),\ \ i = 1,2,\dots \tag{5.5}$$

where $\boldsymbol{a}$ is the $2M$ -vector[11] of the continuous-time state variable, the $2M \times 2M$ matrix A and $2M \times 1$ column vector P have the following structure

$$A = \begin{bmatrix} Le^{-1}(-\Lambda_1 + BJ_y + LeV^*_{sw}F_y) & Le^{-1}BJ_x \\ J_y & -\Lambda_2 + J_x + V^*_{sw}F_x \end{bmatrix};\quad P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{5.6}$$

Here $\Lambda_{1(2)} = diag(\rho_1,\dots,\rho_M)$, $M \times M$ matrices J_y, J_x, F_x, F_y and $M \times 1$ column vectors P_1, P_2 have the elements respectively

$$\{\{J_y\}\}_{nj} = \int_0^L \frac{\partial f_1}{\partial y}|_{y_o,x_o}\phi_n\phi^a_j d\zeta,\quad \{\{J_x\}\}_{nj} = \int_0^L \frac{\partial f_1}{\partial x}|_{y_o,x_o}\psi_n\psi^a_j d\zeta, \tag{5.7}$$

$$\{\{F_y\}\}_{nj} = \int_0^L \frac{\partial \phi_n}{\partial \zeta}\phi^a_j d\zeta,\quad \{\{F_x\}\}_{nj} = \int_0^L \frac{\partial \psi_n}{\partial \zeta}\psi^a_j d\zeta, \tag{5.8}$$

$$\{\{P_1\}\}_j = \int_0^L \frac{\partial y_o}{\partial \zeta}\phi^a_j d\zeta,\quad \{\{P_2\}\}_j = \int_0^L \frac{\partial x_o}{\partial \zeta}_{\zeta}\psi^a_j d\zeta \tag{5.9}$$

where

$$\frac{\partial f_1}{\partial y}|_{y_o,x_o} = Da(1-x_o)\frac{\gamma^2}{(\gamma+y_o)^2}\exp\frac{\gamma y_0}{\gamma+y_o},\qquad \frac{\partial f_1}{\partial x}|_{y_o,x_o} = -Da\exp\frac{\gamma y_0}{\gamma+y_o} \tag{5.10}$$

In (5.7-5.9) ρ_n, ϕ_n ,ψ_n, $n = 1,2,\dots$ are eigenvalues and eigenfunctions of the linear operator

$$\mathcal{L}(z) = V\frac{\partial z}{\partial \zeta} - Pe_z^{-1}\frac{\partial^2 z}{\partial \zeta^2};\ \ z = y,x \tag{5.11}$$

with boundary condition 5.3. Profiles $x_o(\zeta), y_o(\zeta), \zeta \in [0,L]$ are obtained by numerical simulation of asymptotic model of the network.

[11] M is the truncated order.

Control variable $\bar{V}_{sw}(\tau_i)$ in Eqn. 5.5 satisfies the dynamical equation

$$\bar{V}_{sw}(\tau_{i+1}) = \bar{V}_{sw}(\tau_i) - k\tilde{H}\boldsymbol{a}(\tau_i) \tag{5.12}$$

where $\tilde{H} = [H_1, O]$ is the $1 \times 2M$ row vector , O is zero $1 \times M$ row vector and elements of the $1 \times M$ row vector H_1 are follows

$$\{\{H_1\}\}_m = \phi_m(\zeta^*), \quad m = 1, 2, \ldots, M \tag{5.13}$$

Now solvability conditions that guarantee effectiveness of control (5.12) are assured by Theorem 2 that applies to the discrete system with appropriate $2M \times 2M$ and $2M \times 1$ matrices G and D obtained by sampling continuous part of Eqn. 5.5 and with $1 \times 2M$ row vector $H = [H_1, 0]$. Condition (4.19) here is reduced to $H_1P_1 \neq 0$ where P_1 is $M \times 1$ column vector.

Now we demonstrate the effectiveness of control (5.4) to stabilize the rotation of a pulse solution for initial V_{sw} that is outside of the stable domain. Analysis of the continuous model (Eqn.5.1-5.2 with parameters listed in Figure 2 caption) reveals that rotating pulses in the open-loop system (i.e. without control) are stable when V_{sw} belongs to the range $[1.182 * 10^{-3},\ 1.265 * 10^{-3}]$ (see Figure 2, left, $V_{sw} = 1.25 * 10^{-3}$) and the pulse solution is eventually extinguished outside this range (see Figure 2, right, $V_{sw} = 1.35 * 10^{-3}$).

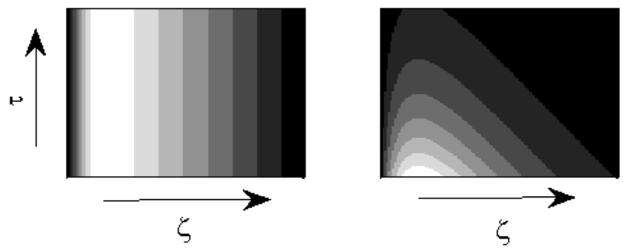

Figure 2: Analysis of the continuous model (Eqn.(1.1,1.2), $Le = 800$; $V = 1$; $L = 1$; $Pe_y = 194.17$; $Pe_x = 200$; $Da = 3.1148$; $B = 0.5233$; $\gamma = 9.1638$): Sustained stable propagation (left, $V^*_{sw} = 1.25 * 10^{-3}$) and failed propagation (right, $V_{sw} = 1.35 * 10^{-3}$) leading to extinction of a loop reactor in the absence of control. Figure shows spatiotemporal gray-scale patterns of the dimensionless temperature.

The typical steady pulse solution of a loop network is presented in Figure 3.

Study of the qualitative properties of the approximate lumped model [12] shows that the sensor position (with $k > 0$) should fall in the range $0 < \zeta^* < 0.05$, i.e., along the ascending section of the pulse because in this range the conditions of Theorem 2 (see Figure 4) and conditions (4.18,4.19) are satisfied. (Elsewhere we show that with $k < 0$ the sensor position can placed along the descending branch.) To evaluate the gain coefficient k we use root-locus plots. Figure 5

[12]The truncated order of this model, $M = 50$, is evaluated by study of convergence of leading eigenvalue of the dynamics matrix A of Eqn.5.5.

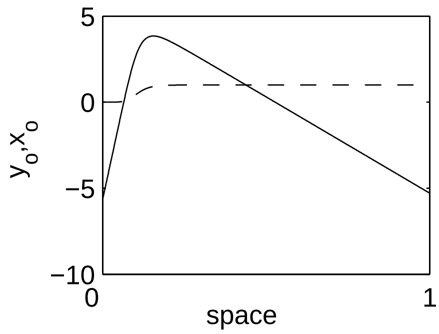

Figure 3: Typical steady pulse solution (y_o (solid line), x_o (dotted line)) of the loop reactor (Eqn. 5.1, 5.2, $V^*_{sw} = 1.25 * 10^{-3}$).

demonstrates root-loci for discrete-time system with sampling time $\sigma = 100, 400, 800$ respectively. We conclude that $k = 0.1 * 10^{-4}$ is enough to stabilize system for $\sigma < 800$. This sampling time should be compared with one rotation time of the pulse, $T = L/V^*_{sw} \cong 800$ for $L = 1$, $V^*_{sw} \cong 1.25 * 10^{-3}$; i.e. we sample once a cycle.

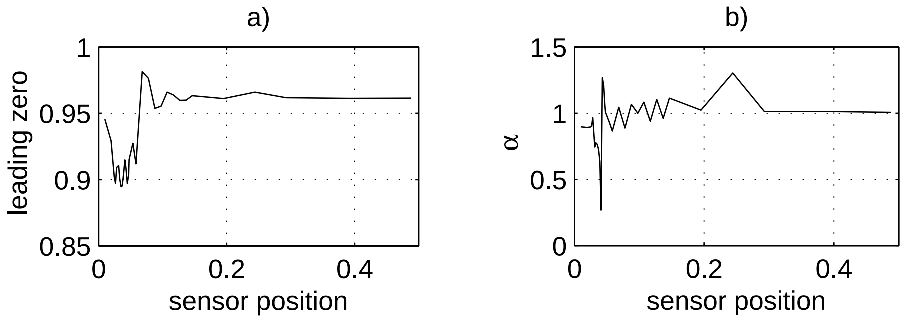

Figure 4: The dependence of the leading zero (a) and value α (Eqn.4.12) (b) on sensor position (ζ^*) for the linearized sampled (discrete-time) model obtained from Eqn.5.5, with matrices A, P from Eqn.5.6-5.9. Discrete-time control (5.12) is updated via $\sigma = 10$. Linearization is performed around the steady solution y_o, x_o (Figure3).

Finally we approve effectiveness of the proposed control law in the actual nonlinear system: we simulate the closed-loop system (Eqn. 5.1-5.4), initially $V_{sw} = 1.36 * 10^{-3} \notin [1.182,\ 1.265] * 10^{-3}$. We use the sensor situated within the acceptable position range ($\zeta^* = 0.0195$) and different switching intervals (sampling time): $\sigma = 100, 400, 800$ (see Figures 6-8): The V_{sw} value converges to a constant value that belongs to the stable range of rotating pulses when $\sigma < 800$.

We can observe that increasing the time interval of control updating considerably increases the amplitude of oscillations of state variable y (compare temperature dynamics in Figures 6 and 7).

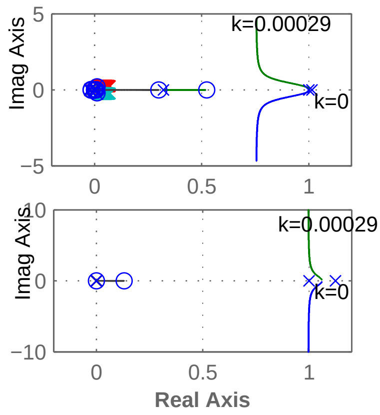

Figure 5: The root-locus for the linearized lumped closed-loop model obtained from Eqn.5.5 with discrete-time control (5.12). First row(left): $\sigma = 100$, leading unstable eigenvalue 1.0147, $\alpha = 0.7539$; first row (right): $\sigma = 400$, leading unstable eigenvalue 1.0602, $\alpha = 0.9181$; second row: $\sigma = 800$, leading unstable eigenvalue 1.240, $\alpha = 0.9968$. Sensor is situated at $\zeta^* = 0.0195$, $V_{sw}^* = 1.1241 * 10^{-3}$.

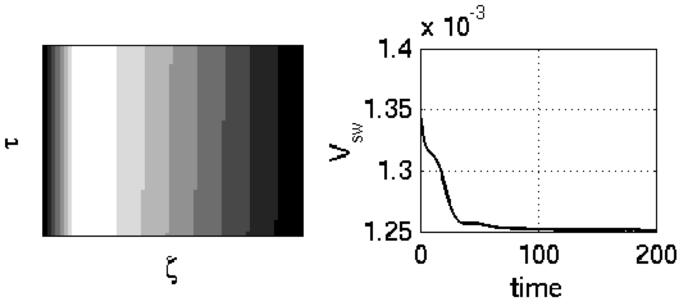

Figure 6: Testing the effectiveness of control law (5.4) in the asymptotic continuous model of loop reactor (Eqn.1.1,1.2), $V_{fr} = V_{sw}$, $k = 0.1 * 10^{-4}$. Control is applied via $\sigma = 100$; $\zeta^* = 0.0195$, initial $V_{sw} = 1.36 * 10^{-3}$, other parameters are as in Figure 2. Figure shows spatiotemporal gray-scale patterns of the dimensionless temperature (left), dynamics of the V_{sw} response (right).

6. Conclusion

We are currently studying the control of a one-dimensional model of pulse propagation in a cardiac system, which takes quite a similar form to Eqns.(2.1). The similarity suggests a mode of control and similarity of tools in control design. In both problems one may design a model-independent feedback control, as opposed to the model-based control employed here. Our analysis suggests several advantages of the latter approach because control in fact, is a distributed feedback control that stabilizes the spatiotemporal evolution of waves by using a finite number of sensor/ actuators. Such type control is effective at large length of space domain unlike model-independed approach that suppress instability in a small portion of tissue (see, for example, works [4, 5].

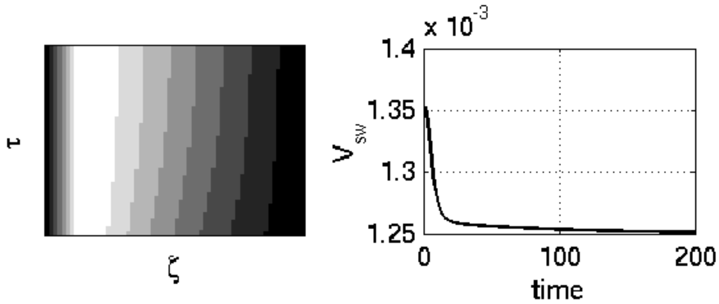

Figure 7: Testing the effectiveness of control law (5.4) in the asymptotic continuous model of loop reactor (Eqn.1.1,1.2), $V_{fr} = V_{sw}$, $k = 0.1 * 10^{-4}$. Control is applied via $\sigma = 400$; $\zeta^* = 0.0195$, initial $V_{sw} = 1.36 * 10^{-3}$, other parameters are as in Figure 2. Figure shows spatiotemporal gray-scale patterns of the dimensionless temperature (left), dynamics of the V_{sw} response (right).

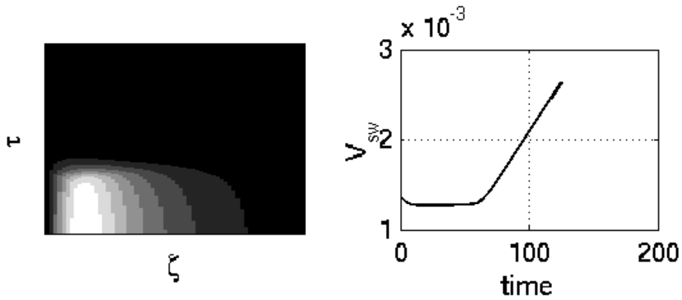

Figure 8: Testing the effectiveness of control law (5.4) in the asymptotic continuous model of loop reactor (Eqn.1.1,1.2), $V_{fr} = V_{sw}$, $k = 0.1 * 10^{-4}$. Control is applied via $\sigma = 800$; $\zeta^* = 0.0195$, initial $V_{sw} = 1.36 * 10^{-3}$, other parameters are as in Figure 2. Figure shows spatiotemporal gray-scale patterns of the dimensionless temperature (left), dynamics of the V_{sw} response (right).

Acknowledgements

Work supported by the US-Israel Binational Science Foundation. MS is a member of the Minerva Center of Nonlinear Dynamics. YS is partially supported by the Center for Absorption in Science, Ministry of Immigrant Absorption, State of Israel.

References

[1] K. J. Astrom, P. Hagander, J. Sternby. *Zeros of sampled system.* Automatica, 20 (1984), No. 1, 31–38.

[2] A. A. Barresi, M. Vanni, M. Brinkmann, G. Baldi. *Control of an autothemal network of nonstationary catalytic reactors.* AIChE Journal, 45 (1999), 1597–1602.

[3] L. G. Stanton, A. A. Golovin. *Global feedback control for pattern-forming systems.* PRE, 76 (2007), 036210.

[4] B. Echebarria, A. Karma. *Spatiotemporal control of cardiac alternans.* Chaos, 12 (2002), 923–930.

[5] B. Echebarria, A. Karma. *Amplitude equation approach to spatiotemporal dynamics of cardiac alternans.* Physical Review E, 76 (2007), 051911.

[6] M. Courtemanche, L. Glass, J. Keener *Instabilities of a propagating pulse in a ring of excitable media.* PRL, 70 (1993), 2182–2185.

[7] M. Courtemanche, J. Keener, L. Glass. *A delay equation representation of pulse circulation on a ring in excitable media.* SIAM.J. Appl. Math., 56 (1996) ,No.1, 119–142.

[8] F. R. Gantmacher. The theory of matrices. New York, Chelsea Pull. Comp, 1977.

[9] L. Hou, A. N. Michel. *Some qualitative properties of sampled-date control systems.* IEEE Transaction on Automatic control, 42 (1997), No.12, 1721–1725.

[10] L. Hou, A. N. Michel.*Some qualitative properties of multirate digital systems.* IEEE Transaction on Automatic control, 44 (1999), No. 4, 765–770.

[11] A. Madai, M. Sheintuch. *Demonstration of loop reactor operation.* AIChE Journal., 54 (2008), 2413–2422.

[12] Yu. Sh. Matros. Catalytic process under unsteady-state conditions. Amsterdam, Elsevier,1989.

[13] M. Mori. *Root-locus method of pulse transfer function for sampled-date control systems.* IRE Transaction on Automatic. Control, 3 (1957), No.1, 13–20.

[14] O. Nekhamkina, M. Sheintuch.*Structure of operating domains of loop reactors.* AIChE Journal, 54 (2008), 1292–1302.

[15] K. Ogata. Modern Control Engineering. Prentice-Hall, London, Englewood Cliffs, 1990.

[16] H. H. Rosenbrock. State-space and multivariable theory, London, Nelson, 1970.

[17] M. Sheintuch, S. Shvartsman. *Spatiotemporal patterns in catalytic reactors.* AIChE Journal, 42 (1996), 1041–1067.

[18] M. Sheintuch, O. Nekhamkina. *The asymptotes of loop reactors.* AIChE Journal, 51 (2005), 224–234.

[19] L. G. Sison, E. K. P. Chong. *No-reset iterative learning control.* Proceedings of 35st IEEE Conference on Decision and Control, Kobe, Japan, 1996, 3062–3063.

[20] Ye. Smagina, M. Sheintuch, M. *Control of rotating pulses in a loop reactor.* Journal of Process Control, 19 (2009), 954–963.

[21] Ye. Smagina. System Zeros, http://arxiv.org/pdf/math.DG/0605092, 2006.

[22] Z. Sun. (2004). *Sampling and control of switched linear systems.* Journal of Franklin Institute, 341 (2004), 657–674.