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Solvability Conditions for a Linearized Cahn-Hilliard Equation of Sixth Order

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Abstract. We obtain solvability conditions in $H^6(\mathbb{R}^3)$ for a sixth order partial differential equation which is the linearized Cahn-Hilliard problem using the results derived for a Schrödinger type operator without Fredholm property in our preceding article [18].

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1. Introduction

Let us consider a binary mixture and denote by c its composition, which is the fraction of one of its two components. Then the evolution of such composition is described by the Cahn-Hilliard equation (see, e.g. [1], [11]):

$$\frac{\partial c}{\partial t} = M\Delta^m\left(\frac{d\phi}{dc} - K\Delta c\right), \quad m = 1, 2, \tag{1.1}$$

where M and K are some constants and ϕ stands for the free energy density. The Flory-Huggins solution theory yields

$$\frac{d\phi}{dc} = k_1 + k_2 c + k_3 T(\ln c - \ln(1-c)), \tag{1.2}$$

where $k_i, i = 1, 2, 3$ are certain thermodynamical constants and T stands for the temperature (see e.g. [7]). Here the constants k_1, k_2 and K characterize interaction of components in the binary mixture and can be positive or negative. If the components in the medium are identical, they vanish and problem (1.1) becomes a bi-Laplacian equation.

Let us denote the right side of (1.2) by $F(T, c)$. We assume that the variation of the composition is small enough and linearize it around some constant $c = c_0$, such that

$$F(T, c) \approx k_1 + k_2 c + k_3 T(\alpha + \beta(c - c_0)),$$

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with $\alpha = \ln\left(\dfrac{c_0}{1-c_0}\right)$ and $\beta = \dfrac{1}{c_0(1-c_0)}$. Let us substitute the expression above for $F(T, c)$ into equation (1.1). We arrive at

$$\frac{\partial u}{\partial t} = M\Delta^m(k_1 + k_2 c_0 + \alpha k_3 T + (k_2 + k_3 \beta T)u - K\Delta u), \quad m = 1, 2, \tag{1.3}$$

with $u = c - c_0$. The Cahn-Hilliard equation of sixth order arises, for instance, in the context of epitaxially growing thin films, describes the formation of quantum dots and their faceting (see e.g. [15], [16]). Such issues as the existence, stability and some qualitative properties of solutions of the Cahn-Hilliard equation have been studied extensively in recent years (see e.g. [3], [8], [11]). While the case of $m = 1$ was covered in [22], we investigate in the present work the existence of stationary solutions of problem (1.3), which can be written as

$$\Delta^2(\Delta u + V(x)u + au) = f(x), \tag{1.4}$$

where

$$V(x) = -\frac{k_3\beta T_0(x)}{K}, \quad f(x) = \frac{\alpha k_3}{K}\Delta^2 T_0(x) + g(x), \quad a = -\frac{k_2 + k_3\beta T_\infty}{K}.$$

Here we write $T(x) = T_\infty + T_0(x)$, where T_∞ stands for the value of the temperature at infinity, the limiting value of $T_0(x)$ equals to zero as $|x| \to \infty$ and $g(x)$ is a source term. We obtain a similar equation if we linearize the nonlinear Cahn-Hilliard equation about a solitary wave.

The potential function $V(x)$ is smooth and vanishes at infinity. The exact conditions on it will be stated below, the function $f(x)$ belongs to the appropriate weighted Hölder space, which will yield its square integrability, and a here is a nonnegative constant. We will study equation (1.4) in the space of three dimensions.

Let us recall that Fredholm solvability conditions affirm that an operator problem $Lu = f$ is solvable if and only if its right side is orthogonal to all solutions w of the homogeneous adjoint equation $L^*w = 0$. This fundamental result holds if the operator L satisfies the Fredholm property, namely its kernel has a finite dimension, its image is closed, the codimension of the image is also finite.

The operator

$$Lu = \Delta^2(\Delta u + V(x)u + au),$$

considered as acting from $H^6(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ (or in the corresponding Hölder spaces) does not satisfy the Fredholm property. Indeed, when $V(x)$ along with its derivatives of up to the fourth order vanishes at infinity, the essential spectrum of this operator is the set of all complex λ for which the equation

$$\Delta^2(\Delta u + au) = \lambda u$$

has a nontrivial bounded solution. When applying the Fourier transform, we arrive at

$$\lambda = |\xi|^4(a - |\xi|^2), \quad \xi \in \mathbb{R}^3.$$

Thus the essential spectrum contains the origin. As a consequence, the operator does not satisfy the Fredholm property, and solvability conditions of equation (1.4) are not known. We will derive solvability conditions for this equation using the technique developed in our preceding articles [18], [19], [20], [21], [22] and [23]. This method is based on the spectral decomposition of self-adjoint operators.

Evidently, problem (1.4) can be conveniently rewritten in the equivalent form of the system of the three second order equations

$$\begin{cases} \Delta\theta = f(x), \\ -\Delta v = \theta(x), \\ -\Delta u - V(x)u - au = v(x) \end{cases} \tag{1.5}$$

in which the first two have explicit solutions due to the fast rate of decay of their right sides by means of Assumption 3 below, namely

$$\theta_0(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|}dy, \quad v_0(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\theta_0(y)}{|x-y|}dy \tag{1.6}$$

with their properties established in Lemma 2.1. The $''0''$ subscript in formulas (1.6) indicates that we deal with particular solutions defined up to a harmonic function of the Poisson equations involved in the system above. Note that each of the equations of system (1.5) contains second order differential operators on $L^2(\mathbb{R}^3)$ without Fredholm property. Their essential spectra are $\sigma_{ess}(-\Delta) = [0, \infty)$ and $\sigma_{ess}(-\Delta - V(x) - a) = [-a, \infty)$ for $V(x) \to 0$ at infinity (see e.g. [9]), such that neither of these operators has a finite dimensional kernel and a closed image. Solvability conditions for operators of that kind have been studied extensively in recent articles for a single Schrödinger type operator (see [18]), the sums of second order differential operators (see [19]), the Laplacian operator with the drift term (see [20]). Non Fredholm operators arise as well while studying the existence and stability of stationary and travelling wave solutions of certain reaction-diffusion equations (see e.g. [5], [6], [17]). For the third equation in system (1.5) we introduce the corresponding homogeneous problem

$$-\Delta w - V(x)w - aw = 0. \tag{1.7}$$

We make the following auxiliary assumptions on the potential function of equation (1.4). Note that the first one contains conditions on $V(x)$ analogous to those stated in Assumption 1.1 of [18] (see also [19], [20]).

Assumption 1. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \le \frac{C}{1+|x|^{3.5+\delta}}$$

with some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad \text{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and below C stands for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \le c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and further down the norm of a function $f_1 \in L^p(\mathbb{R}^3),\ 1 \le p \le \infty$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^3)}$.

Assumption 2. $\Delta^2 V \in L^2(\mathbb{R}^3)$ *and* $\dfrac{\partial^3 V}{\partial x_i \partial x_j \partial x_k} \in L^\infty(\mathbb{R}^3),\ i, j, k = 1, 2, 3.$

Remark 1.1. Note that $\dfrac{\partial^2 V}{\partial x_j \partial x_k} \in L^2(\mathbb{R}^3),\ j, k = 1, 2, 3$. Indeed, $\|\Delta V\|^2_{L^2(\mathbb{R}^3)}$ can be easily written as the sum of two integrals, in which the first one is taken over the unit ball centered at the origin in the Fourier space and the second one is over the ball's complement. These terms can be bounded above by the sum of the norms $\|V\|^2_{L^2(\mathbb{R}^3)}$ and $\|\Delta^2 V\|^2_{L^2(\mathbb{R}^3)}$, which are finite by means of Assumptions 1 and 2 respectively.

Let us use the notation

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f_1(x)\bar{f}_2(x)dx,$$

with a slight abuse of notations when these functions are not square integrable, like for instance some of those used in Assumption 3 below. Indeed, if the product of functions $f_1(x)$ and $f_2(x)$ belongs to $L^1(\mathbb{R}^3)$, the integral in the right side of the identity above is well defined. We introduce the auxiliary quantity

$$\rho(x) := (1+|x|^2)^{\frac{1}{2}}, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{1.8}$$

and the space $C_a^\mu(\mathbb{R}^3)$, where a is a real number and $0 < \mu < 1$ which consists of all functions u such that

$$u\rho^a \in C^\mu(\mathbb{R}^3).$$

Here $C^\mu(\mathbb{R}^3)$ denotes the Hölder space and the norm on $C_a^\mu(\mathbb{R}^3)$ is defined as

$$\|u\|_{C_a^\mu(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3}|\rho^a(x)u(x)| + \sup_{x,y\in\mathbb{R}^3}\frac{|\rho^a(x)u(x) - \rho^a(y)u(y)|}{|x-y|^\mu}.$$

Thus the space of all functions for which

$$\partial^\alpha u \in C^\mu_{a+|\alpha|}(\mathbb{R}^3), \ |\alpha| \le l,$$

where l is a nonnegative integer is being designated as $C_a^{\mu+l}(\mathbb{R}^3)$. Let $P(s)$ stand for the set of polynomials of three variables of the order less than or equal to s for $s \ge 0$ and $P(s)$ is empty when $s < 0$. We make the following technical assumption on the right side of the linearized Cahn-Hilliard problem.

Assumption 3. *Let the function $f(x) : \mathbb{R}^3 \to \mathbb{R}$, such that $f(x) \in C^\mu_{8+\varepsilon}(\mathbb{R}^3)$ for some $0 < \varepsilon < 1$, the function $\theta_0(x)$ is given by (1.6) and the orthogonality relations*

$$(f(x), p(x))_{L^2(\mathbb{R}^3)} = 0, \quad (\theta_0(x), q(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.9}$$

hold for any polynomial $p(x) \in P(5)$ satisfying the equation $\Delta p(x) = 0$ and any polynomial $q(x) \in P(3)$ such that $\Delta q(x) = 0$ respectively.

Remark 1.2. The examples of such polynomials of the fifth and the third order are

$$p(x) = x_1^5 + 5x_1x_2^4 - 10x_1^3x_2^2 + x_3, \quad q(x) = x_1^3 - 2x_1x_2^2 - x_1x_3^2.$$

The set of possible $p(x)$ and $q(x)$ includes constants, linear functions of three variables and many other examples as well. Note that the inner products involved in the left sides of (1.9) are well defined via the integrals involving functions $f(x)$ and $\theta_0(x)$, which rates of decay are given by formulas (2.6) and (2.1) respectively.

By means of Lemma 2.3 of [18], under our Assumption 1 above on the potential function, the operator $-\Delta - V(x) - a$ is self-adjoint and unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ via the wave operators (see [10], [14])

$$\Omega^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta-V)}e^{it\Delta}$$

with the limit understood in the strong L^2 sense (see e.g. [13] p.34, [4] p.90). Thus, $-\Delta - V(x) - a$ on $L^2(\mathbb{R}^3)$ possesses only the essential spectrum $\sigma_{ess}(-\Delta - V(x) - a) = [-a, \infty)$. Its functions of the continuous spectrum satisfy

$$[-\Delta - V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3. \tag{1.10}$$

In the integral formulation we have the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [13] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy. \tag{1.11}$$

The orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3 \tag{1.12}$$

hold. By means of the spectral theorem these functions form the complete system in $L^2(\mathbb{R}^3)$. We introduce the following Sobolev type space (see also [20], [21])

$$\tilde{W}^{2,\infty}(\mathbb{R}^3) := \{w(x) : \mathbb{R}^3 \to \mathbb{C} \mid w, \nabla w, \Delta w \in L^\infty(\mathbb{R}^3)\}. \tag{1.13}$$

Our main result is as follows.

Theorem 1.3. *Let Assumptions 1, 2 and 3 hold, $a \geq 0$ and $v_0(x)$ is given by (1.6). Then problem (1.4) admits a unique solution $u_a \in H^6(\mathbb{R}^3)$ if and only if*

$$(v_0(x), w(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.14}$$

for any $w(x) \in \tilde{W}^{2,\infty}(\mathbb{R}^3)$ satisfying equation (1.7).

Remark 1.4. Note that $\varphi_k(x) \in \tilde{W}^{2,\infty}(\mathbb{R}^3),\ k \in \mathbb{R}^3$, which was established in Lemma A3 of [20] under our assumptions on the potential function. Via (1.10) these perturbed plane waves satisfy homogeneous problem (1.7) when the wave vector k belongs to the sphere in three dimensions centered at the origin of radius $\sqrt{a}$. The inner product in the left side of (1.14) is well defined via the integral involving functions $v_0(x)$, which rate of decay is given by (2.2) and $w(x)$, which is bounded.

2. Proof of the main result

We begin with some auxiliary lemmas.

Lemma 2.1. *Let Assumption 3 hold. Then we have*

$$\theta_0(x) \in L^2(\mathbb{R}^3), \quad v_0(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \quad \textit{and} \quad |x|v_0(x) \in L^1(\mathbb{R}^3).$$

Proof. According to the result of [2], for the solution of the Poisson equation $\theta_0(x)$ (see (1.6)) under the condition $f(x) \in C^{\mu}_{8+\varepsilon}(\mathbb{R}^3)$ and the first orthogonality relation in (1.9) given in Assumption 3 we have $\theta_0(x) \in C^{\mu+2}_{6+\varepsilon}(\mathbb{R}^3)$. Thus $\sup_{x\in\mathbb{R}^3}|\rho!l^{6+\varepsilon}\theta_0| \leq C$ and therefore

$$|\theta_0(x)| \leq \frac{C}{(\rho(x))^{6+\varepsilon}},\ x \in \mathbb{R}^3 \quad a.e. \tag{2.1}$$

By means of definition (1.8) the function $\theta_0(x)$ is square integrable. Since $\theta_0(x)$ satisfies the second orthogonality relation in (1.9), via the result of [2] we arrive at $v_0(x) \in C^{\mu+4}_{4+\varepsilon}(\mathbb{R}^3)$. Hence $\sup_{x\in\mathbb{R}^3}|\rho^{4+\varepsilon}v_0| \leq C$, such that

$$|v_0(x)| \leq \frac{C}{(\rho(x))^{4+\varepsilon}},\ x \in \mathbb{R}^3 \quad a.e. \tag{2.2}$$

The statement of the lemma about $v_0(x)$ easily follows from definition (1.8). □

Lemma 2.2. *Let Assumptions 1 and 2 hold. Then* $\dfrac{\partial V}{\partial x_i} \in L^\infty(\mathbb{R}^3), \quad i = 1, 2, 3.$

Proof. By using the standard Fourier transform and estimating from above the sum of two integrals in which the first one is taken over the unit ball centered at the origin in the Fourier space and the second one is over the ball's complement, it can be easily shown that

$$\left\| \frac{\partial V}{\partial x_i} \right\|^2_{L^2(\mathbb{R}^3)} \le \|V\|^2_{L^2(\mathbb{R}^3)} + \|\Delta V\|^2_{L^2(\mathbb{R}^3)} < \infty, \quad i = 1, 2, 3$$

by means of Assumption 1 and Remark 1.1. Similarly

$$\left\| \Delta \frac{\partial V}{\partial x_i} \right\|^2_{L^2(\mathbb{R}^3)} \le \|V\|^2_{L^2(\mathbb{R}^3)} + \|\Delta^2 V\|^2_{L^2(\mathbb{R}^3)} < \infty, \quad i = 1, 2, 3$$

via Assumptions 1 and 2. Therefore, $\dfrac{\partial V}{\partial x_i} \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3), \quad i = 1, 2, 3$ due to the Sobolev embedding theorem. □

Using these auxiliary results we proceed to prove the main statement.

Proof. of Theorem 1.3. The linearized Cahn-Hillard equation (1.4) is equivalent to system (1.5) in which the second equation admits a solution $v_0(x)$ given by (1.6). The function $v_0(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $|x|v_0(x) \in L^1(\mathbb{R}^3)$ by means of Lemma 2.1 and Assumption 3. Then according to Theorem 3 of [21] the third equation in system (1.5) with $v_0(x)$ in its right side admits a unique solution in $H^2(\mathbb{R}^3)$ if and only if orthogonality relation (1.14) holds. This function $u_a(x) \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem, $a \ge 0$ satisfies the equation

$$-\Delta u_a - V(x)u_a - au_a = v_0(x).$$

Our goal is to show that the solution of problem (1.4) $u_a(x) \in H^6(\mathbb{R}^3)$. Let us apply the Laplacian operator to both sides of the formula above and use the second equation in (1.5). This yields

$$\Delta^2 u_a + \Delta(V u_a) + a\Delta u_a = \theta_0(x). \tag{2.3}$$

We use the formula

$$\Delta(V u_a) = V\Delta u_a + 2\nabla V.\nabla u_a + u_a \Delta V, \tag{2.4}$$

where the "dot" denotes the standard scalar product of two vectors in three dimensions. The first term in the right side of (2.4) is square integrable since $V(x) \in L^\infty(\mathbb{R}^3)$ via Assumption 1 and $\Delta u_a(x) \in L^2(\mathbb{R}^3)$. Similarly $u_a\Delta V \in L^2(\mathbb{R}^3)$ since $u_a(x) \in L^\infty(\mathbb{R}^3)$ and $\Delta V \in L^2(\mathbb{R}^3)$ (see Remark 1.1). For the second term in the right side of (2.4) we have $\nabla u_a(x) \in L^2(\mathbb{R}^3)$ and $\nabla V \in L^\infty(\mathbb{R}^3)$ due to Lemma 2.2. Hence $\nabla V.\nabla u_a \in L^2(\mathbb{R}^3)$, which yields $\Delta(V u_a) \in L^2(\mathbb{R}^3)$ and $\theta_0(x)$ is square integrable, which was proven in Lemma 2.1. Thus from equation (2.3) we obtain $\Delta^2 u_a \in L^2(\mathbb{R}^3)$. All partial third derivatives of u_a are square integrable as well. Indeed, for $i, j, k = 1, 2, 3$, $\left\| \dfrac{\partial^3 u_a}{\partial x_i \partial x_j \partial x_k} \right\|^2_{L^2(\mathbb{R}^3)}$ can be estimated by means of the sum of the two integrals in which the first one is taken over the unit ball centered at the origin in the Fourier space and the second one is over the ball's complement. These terms can be bounded above by $\|u_a\|^2_{L^2(\mathbb{R}^3)}$ and $\|\Delta^2 u_a\|^2_{L^2(\mathbb{R}^3)}$ respectively, which are finite. This implies that $u_a \in H^4(\mathbb{R}^3)$. Let us apply the Laplacian operator to both sides of formula (2.3) and use the first equation in system (1.5), which yields

$$\Delta^3 u_a + \Delta(V\Delta u_a) + 2\Delta(\nabla V.\nabla u_a) + \Delta(u_a \Delta V) + a\Delta^2 u_a = f(x). \tag{2.5}$$

Our goal is to analyze the terms of formula (2.5) in order to prove that they are contained in $L^2(\mathbb{R}^3)$. The right side of this equation is square integrable via Assumption 3. Indeed, since $\sup_{x\in\mathbb{R}^3}|\rho^{8+\varepsilon} f| \le C$, we obtain the estimate

$$|f(x)| \le \frac{C}{(\rho(x))^{8+\varepsilon}}, \ x \in \mathbb{R}^3 \quad a.e., \tag{2.6}$$

where $\rho(x)$ is defined explicitly in (1.8). For the second term in the left side of (2.5) we have the formula

$$\Delta(V\Delta u_a) = \Delta V \Delta u_a + 2\nabla V.\nabla(\Delta u_a) + V\Delta^2 u_a \in L^2(\mathbb{R}^3). \tag{2.7}$$

Indeed, the first term in the right side of (2.7) is square integrable since $\Delta V \in L^2(\mathbb{R}^3)$ (see Remark 1.1) and $\Delta u_a \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem. The second term in the right side of (2.7) belongs to $L^2(\mathbb{R}^3)$ because $\nabla V \in L^\infty(\mathbb{R}^3)$ via Lemma 2.2 and $\nabla(\Delta u_a) \in L^2(\mathbb{R}^3)$ for $u_a \in H^4(\mathbb{R}^3)$. The last term in the right side of (2.7) is square integrable since $V(x) \in L^\infty(\mathbb{R}^3)$ via Assumption 1 and $\Delta^2 u_a \in L^2(\mathbb{R}^3)$.

Clearly, the third term in the left side of (2.5) can be easily written as

$$2\sum_{k=1}^{3} \Delta\left(\frac{\partial V}{\partial x_k}\frac{\partial u_a}{\partial x_k}\right).$$

We express each term of the sum above with $k = 1, 2, 3$ as

$$\frac{\partial u_a}{\partial x_k}\left(\Delta\frac{\partial V}{\partial x_k}\right) + 2\left(\nabla\frac{\partial V}{\partial x_k}\right).\left(\nabla\frac{\partial u_a}{\partial x_k}\right) + \frac{\partial V}{\partial x_k}\left(\Delta\frac{\partial u_a}{\partial x_k}\right) \in L^2(\mathbb{R}^3). \tag{2.8}$$

Indeed, the first term in (2.8) belongs to $L^2(\mathbb{R}^3)$ since $\frac{\partial u_a}{\partial x_k}$ is square integrable and $\Delta\frac{\partial V}{\partial x_k} \in L^\infty(\mathbb{R}^3)$ by means of Assumption 2. The second term in (2.8) is square integrable because $\nabla\frac{\partial V}{\partial x_k} \in L^2(\mathbb{R}^3)$ (Remark 1.1) and $\nabla\frac{\partial u_a}{\partial x_k} \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem for $u_a \in H^4(\mathbb{R}^3)$. The last term in (2.8) is contained in $L^2(\mathbb{R}^3)$ due to the fact that $\Delta\frac{\partial u_a}{\partial x_k} \in L^2(\mathbb{R}^3)$ and by means of Lemma 2.2 we have $\frac{\partial V}{\partial x_k} \in L^\infty(\mathbb{R}^3)$.

The fourth term in the left side of (2.5) can be expressed as

$$\Delta(u_a\Delta V) = \Delta u_a \Delta V + 2\nabla u_a.\nabla(\Delta V) + u_a\Delta^2 V \in L^2(\mathbb{R}^3). \tag{2.9}$$

This can be easily seen by analyzing the terms in the right side of (2.9). The first one is present in formula (2.7) above and its square integrability was proven. The second one is contained in $L^2(\mathbb{R}^3)$ since $\nabla u_a \in L^2(\mathbb{R}^3)$ and $\nabla(\Delta V) \in L^\infty(\mathbb{R}^3)$ by means of Assumption 2. The last term in the right side of (2.9) is square integrable because $u_a \in L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem and $\Delta^2 V \in L^2(\mathbb{R}^3)$, which was stated in Assumption 2.

Hence formula (2.5) implies that $\Delta^3 u_a \in L^2(\mathbb{R}^3)$. Note that any fifth partial derivative of u_a is square integrable as well since its $L^2(\mathbb{R}^3)$ norm can be easily estimated above in terms of finite $L^2(\mathbb{R}^3)$ norms of u_a and $\Delta^3 u_a$. Therefore, the solution of problem (1.4) $u_a \in H^6(\mathbb{R}^3)$.

To investigate the issue of uniqueness of such a solution we suppose there exist $u_1, u_2 \in H^6(\mathbb{R}^3)$, which are two solutions of problem (1.4). Then their difference $u(x) = u_1(x) - u_2(x) \in H^6(\mathbb{R}^3)$ satisfies equation (1.4) with vanishing right side, namely

$$\Delta^2(\Delta u + V(x)u + au) = 0. \tag{2.10}$$

Obviously $u, \Delta u \in L^2(\mathbb{R}^3)$ and $Vu \in L^2(\mathbb{R}^3)$ since $V(x)$ satisfies Assumption 1. Hence $v(x) = -\Delta u - V(x)u - au \in L^2(\mathbb{R}^3)$ and solves the second equation in system (1.5), such that

$$\Delta\theta = 0. \tag{2.11}$$

Clearly we have

$$-\Delta v = \Delta^2 u + u\Delta V + 2\nabla V.\nabla u + V\Delta u + a\Delta u = \theta(x) \in L^2(\mathbb{R}^3). \tag{2.12}$$

This can be easily verified since $\Delta^2 u$ is square integrable for $u \in H^6(\mathbb{R}^3)$. The second term in the middle part of (2.12) is contained in $L^2(\mathbb{R}^3)$ because $u \in L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem and $\Delta V \in L^2(\mathbb{R}^3)$ (see Remark 1.1). The third term in the middle part of this formula above is square integrable since $\nabla V \in L^\infty(\mathbb{R}^3)$ due to Lemma 2.2 and $\nabla u \in L^2(\mathbb{R}^3)$. Finally, $V\Delta u \in L^2(\mathbb{R}^3)$ because $V \in L^\infty(\mathbb{R}^3)$ by means of Assumption 1 and $\Delta u \in L^2(\mathbb{R}^3)$.

Due to the fact that the Laplace operator does not possess any nontrivial square integrable zero modes, equations (2.11) and (2.12) yield $\theta(x) = 0$ a.e. in $\mathbb{R}^3$. Then by means of (2.12) with $v(x) \in L^2(\mathbb{R}^3)$ we obtain similarly $v(x) = 0$ a.e. in $\mathbb{R}^3$. Therefore, we arrive at the homogeneous problem $(-\Delta - V(x) - a)u = 0$ a.e. with $u(x) \in L^2(\mathbb{R}^3)$. This Schrödinger type operator in brackets is self-adjoint and unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ as discussed above, which implies that $u(x) = 0$ a.e. in $\mathbb{R}^3$. □

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