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Blow-up Solutions of Quasilinear Hyperbolic Equations With Critical Sobolev Exponent

S. Ibrahim[1] *, A. Lyaghfouri[2]
[1] Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada
[2]Fields Institute, Toronto, ON, Canada

Abstract. In this paper, we show finite time blow-up of solutions of the $p-$wave equation in $\mathbb{R}^N$, with critical Sobolev exponent. Our work extends a result by Galaktionov and Pohozaev [4]

Keywords and phrases: $p-$wave equation, Blow-up, critical Sobolev exponent

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1. Introduction

In this paper, we consider the following $p-$wave equation:

$$\begin{cases} u_{tt} - \Delta_p u = |u|^{p_*-2}u & \text{in } \mathbb{R}^N \times \mathbb{R} \\ u(.,0) = u_0 \in \dot{W}^{1,p}(\mathbb{R}^N) \\ u_t(.,0) = u_1 \in L^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $2 < p < N$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian and $p_* = \frac{pN}{N-p}$ is the critical Sobolev exponent, and where

$$\dot{W}^{1,p}(\mathbb{R}^N) = \{u \in L^{p_*}(\mathbb{R}^N) \ : \ \nabla u \in L^p(\mathbb{R}^N)\ \}.$$

Multiplying equation (1.1) by u_t and integrating by parts, we formally get

$$\int_{\mathbb{R}^N} u_t u_{tt} dx + \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla u_t dx - \int_{\mathbb{R}^N} |u|^{p_*-2} u u_t dx = 0$$

or

$$\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}^N} |u_t|^2 dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx\Big) = 0.$$

Setting $E(u(.,t)) = \frac{1}{2}\int_{\mathbb{R}^N} |u_t|^2 dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx$, we obtain

$$\frac{dE(u(.,t))}{dt} = 0.$$

*Corresponding author. E-mail: ibrahims@uvic.ca

In particular, we have

$$E(u(.,t)) = E(u(.,0)) = E(u_0,u_1) = \frac{1}{2}\int_{\mathbb{R}^N} |u_1|^2 dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_0|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u_0|^{p_*} dx. \tag{1.2}$$

Recall that the function

$$w_p(x) = \Big(1 + \frac{p-1}{(N-p)N^{\frac{1}{p-1}}}|x|^{\frac{p}{p-1}}\Big)^{\frac{p-N}{p}}$$

solves the equation

$$\Delta_p u = -|u|^{p_*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

and is known as the *ground state*. It is related to the best constant of the Sobolev inequality

$$||u||_{p_*} \le C(N,p)||\nabla u||_p \quad \forall u \in \dot{W}^{1,p}(\mathbb{R}^N), \tag{1.4}$$

where $C(N,p)$ is characterized by

$$\begin{cases} \text{if } \ ||u||_{p_*} = C(N,p)||\nabla u||_p \quad \text{and } \ u \neq 0, \ \text{then} \\ u = Cw_p(\sigma(x-x_0)) \ \text{ for some constants } C \neq 0, \sigma > 0, \ \text{and } x_0 \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

For more details, we refer for example to [2], [10] and [1].

Define

$$\mathcal{E}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx$$

and

$$\mathcal{K}(u) = \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} |u|^{p_*} dx.$$

One can easily derive the following identities satisfied by the ground state.

Proposition 1.1.

$$\int_{\mathbb{R}^N} |w_p|^{p_*} dx = \int_{\mathbb{R}^N} |\nabla w_p|^p dx = \frac{1}{C^N(N,p)} \tag{1.6}$$

$$\textit{and} \quad \mathcal{E}_p = \mathcal{E}(w_p) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w_p|^p dx = \frac{1}{NC^N(N,p)}. \tag{1.7}$$

Proof. $i)$ Multiplying equation (1.3) by u and integrating over $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} \Delta_p w_p \cdot w_p dx = -\int_{\mathbb{R}^N} |w_p|^{p_*} dx$$

or

$$\int_{\mathbb{R}^N} |\nabla w_p|^p dx = \int_{\mathbb{R}^N} |w_p|^{p_*} dx. \tag{1.8}$$

It follows that

$$||w_p||_{p_*} = ||\nabla w_p||_p^{\frac{p}{p_*}} = ||\nabla w_p||_p^{\frac{p}{p_*}-1}||\nabla w_p||_p = ||\nabla w_p||_p^{-\frac{p}{N}}||\nabla w_p||_p = ||w_p||_{p_*}^{-\frac{p_*}{N}}||\nabla w_p||_p.$$

We deduce that $C(N,p) = ||w_p||_{p_*}^{-\frac{p_*}{N}}$, and we obtain from (1.8)

$$\int_{\mathbb{R}^N} |\nabla w_p|^p dx = \int_{\mathbb{R}^N} |w_p|^{p_*} dx = ||w_p||_{p_*}^{p_*} = \frac{1}{C^N(N,p)}.$$

$ii)$ Using (1.8), we get

$$\begin{aligned}\mathcal{E}_p &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w_p|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |w_p|^{p_*} dx \\ &= \Big(\frac{1}{p} - \frac{1}{p_*}\Big)\int_{\mathbb{R}^N} |w_p|^{p_*} dx \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |w_p|^{p_*} dx \\ &= \frac{1}{NC^N(N,p)}.\end{aligned}$$

□

Now we prove the following variational characterization of w_p.

Proposition 1.2.

$$\mathcal{E}_p = inf\Big\{\ \mathcal{E}(u)\ :\ u \in \dot{W}^{1,p}(\mathbb{R}^N),\ \mathcal{K}(u) = 0,\ u \neq 0\ \Big\}. \tag{1.9}$$

Proof. First, by (1.6) we have $K(w_p) = 0$, and therefore

$$\inf\Big\{\ \mathcal{E}(u)\ :\ u \in \dot{W}^{1,p}(\mathbb{R}^N),\ \mathcal{K}(u) = 0,\ u \neq 0\ \Big\} \leq \mathcal{E}_p.$$

Second, recall [2], [10] and [1] that

$$||\nabla w_p||_p^p = \inf\Big\{\ ||\nabla u||_p^p\ :\ u \in \dot{W}^{1,p}(\mathbb{R}^N),\ ||u||_{p_*} = ||w_p||_{p_*}\ \Big\}, \tag{1.10}$$

$$||w_p||_{p_*}^{p_*} = \sup\Big\{\ ||u||_{p_*}^{p_*}\ :\ u \in \dot{W}^{1,p}(\mathbb{R}^N),\ ||u||_{p_*} = ||w_p||_{p_*}\ \Big\}. \tag{1.11}$$

Now let $u \in \dot{W}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be such that $\mathcal{K}(u) = 0$, and set $\widetilde{u} = \frac{||w_p||_{p_*}}{||u||_{p_*}} u$. Since $||\widetilde{u}||_{p_*} = ||w_p||_{p_*}$, we have by (1.10)

$$\begin{aligned} ||\nabla w_p||_p^p \leq ||\nabla \widetilde{u}||_p^p &= \frac{||w_p||_{p_*}^p}{||u||_{p_*}^p} ||\nabla u||_p^p \\ \text{or}\quad \frac{||\nabla w_p||_p^p}{||w_p||_{p_*}^p} &\leq \frac{||\nabla u||_p^p}{||u||_{p_*}^p}. \end{aligned} \tag{1.12}$$

Since $\mathcal{K}(u) = \mathcal{K}(w_p) = 0$, we have $||\nabla w_p||_p^p = ||w_p||_{p_*}^{p_*}$ and $||\nabla u||_p^p = ||u||_{p_*}^{p_*}$. We obtain from (1.12)

$$||w_p||_{p_*}^{p_*-p} \leq ||u||_{p_*}^{p_*-p}. \tag{1.13}$$

From Proposition 1.1, we know that

$$\mathcal{E}_p = \frac{1}{N} ||w_p||_{p_*}^{p_*} \tag{1.14}$$

Moreover, since $\mathcal{K}(u) = 0$, we have

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \frac{1}{p}\mathcal{K}(u) + \Big(\frac{1}{p} - \frac{1}{p_*}\Big)\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \Big(\frac{1}{p} - \frac{1}{p_*}\Big)\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \frac{1}{N}||u||_{p_*}^{p_*}. \end{aligned} \tag{1.15}$$

We deduce from (1.13)-(1.15) and since $p_* > p$, that

$$\begin{aligned}&\big(N\mathcal{E}_p\big)^{\frac{p_*-p}{p_*}} \le \big(N\mathcal{E}(u)\big)^{\frac{p_*-p}{p_*}} \\ &\text{or} \quad \mathcal{E}_p \le \mathcal{E}(u).\end{aligned}$$

Hence the proposition holds. □

We have also:

Proposition 1.3. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be such that $\mathcal{E}(u) < \mathcal{E}_p$. Then we have*

$$\mathcal{K}(u) \neq 0, \quad ||u||_{p_*} \neq ||w_p||_{p_*} \ \textit{and} \ ||\nabla u||_p \neq ||\nabla w_p||_p. \tag{1.16}$$

Moreover,

$$\mathcal{K}(u) < 0 \ \Leftrightarrow \ ||u||_{p_*} > ||w_p||_{p_*} \ \Leftrightarrow \ ||\nabla u||_p > ||\nabla w_p||_p. \tag{1.17}$$

$$\mathcal{K}(u) > 0 \ \Leftrightarrow \ ||u||_{p_*} < ||w_p||_{p_*} \ \Leftrightarrow \ ||\nabla u||_p < ||\nabla w_p||_p. \tag{1.18}$$

Proof. $i)$ • If we have $\mathcal{K}(u) = 0$, then from (1.9) we have $\mathcal{E}_p \le \mathcal{E}(u)$, which contradicts the hypothesis.

• Assume that $||u||_{p_*} = ||w_p||_{p_*}$. Then we have by (1.10)

$$\begin{aligned}\mathcal{E}_p &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w_p|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |w_p|^{p_*} dx \\ &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w_p|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &\le \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \mathcal{E}(u).\end{aligned}$$

which again contradicts the hypothesis.

• Similarly, assume that $||\nabla u||_p = ||\nabla w_p||_p$. Then arguing as before and using (1.11), we get a contradiction.

$ii)$ • Assume that $||u||_{p_*}^{p_*} > ||w_p||_{p_*}^{p_*}$. We recall from (1.15) that

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \frac{1}{p}\mathcal{K}(u) + \Big(\frac{1}{p} - \frac{1}{p_*}\Big)\int_{\mathbb{R}^N} |u|^{p_*} dx \\ &= \frac{1}{p}\mathcal{K}(u) + \frac{1}{N}||u||_{p_*}^{p_*}. \end{aligned} \tag{1.19}$$

Using the fact that $\mathcal{E}(u) < \mathcal{E}_p$ and (1.6)-(1.7), we obtain from (1.19)

$$\begin{aligned}\mathcal{K}(u) &= p\mathcal{E}(u) - \frac{p}{N}||u||_{p_*}^{p_*}\\ &< p\mathcal{E}_p - \frac{p}{N}||u||_{p_*}^{p_*}\\ &= \frac{p}{N}(||w_p||_{p_*}^{p_*} - ||u||_{p_*}^{p_*})\\ &< 0.\end{aligned}$$

• Assume that $\mathcal{K}(u) < 0$. Then we have

$$\int_{\mathbb{R}^N} |\nabla u|^p dx < \int_{\mathbb{R}^N} |u|^{p_*} dx. \tag{1.20}$$

Set $\widetilde{u} = \frac{||w_p||_{p*}}{||u||_{p_*}} u$. Then clearly $||\widetilde{u}||_{p_*} = ||w_p||_{p_*}$ and using (1.10), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w_p|^p dx \leq \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^p dx\\ &\text{that is } \int_{\mathbb{R}^N} |\nabla w_p|^p dx \leq \frac{||w_p||_{p_*}^p}{||u||_{p_*}^p}\int_{\mathbb{R}^N} |\nabla u|^p dx\\ &\text{or equivalently } \frac{1}{||w_p||_{p_*}^p}\int_{\mathbb{R}^N} |\nabla w_p|^p dx \leq \frac{1}{||u||_{p_*}^p}\int_{\mathbb{R}^N} |\nabla u|^p dx.\end{aligned} \tag{1.21}$$

Taking into account (1.6) and (1.20), we get from (1.21)

$$\begin{aligned}&\frac{||w_p||_{p_*}^{p_*}}{||w_p||_{p_*}^p} < \frac{||u||_{p_*}^{p_*}}{||u||_{p_*}^p}\\ &\text{or } ||w_p||_{p_*}^{p_*-p} < ||u||_{p_*}^{p_*-p}\\ &\text{or } ||w_p||_{p_*} < ||u||_{p_*}.\end{aligned}$$

Thus $K(u) < 0 \Leftrightarrow \ ||u||_{p_*} > ||w_p||_{p_*}$.

• Assume that $\mathcal{K}(u) < 0$. We would like to prove that $||\nabla u||_p > ||\nabla w_p||_p$. From the previous result and (1.4) we have

$$\begin{aligned}&||w_p||_{p_*} < ||u||_{p_*} \leq C(N,p)||\nabla u||_p\\ &\text{or } \frac{1}{C(N,p)}||w_p||_{p_*} < ||\nabla u||_p\end{aligned}$$

Taking into account (1.6), this leads to

$$\begin{aligned}&\frac{1}{C(N,p)}||w_p||_{p_*} = ||w_p||_{p_*}^{1+\frac{p_*}{N}} < ||\nabla u||_p\\ &\text{that is } ||w_p||_{p_*}^{\frac{p_*}{p}} < ||\nabla u||_p\\ &\text{or equivalently } ||\nabla w_p||_p < ||\nabla u||_p.\end{aligned}$$

• Conversely, assume that $||\nabla u||_p > ||\nabla w_p||_p$ and let us show that necessarily $\mathcal{K}(u) < 0$. Write

$$\mathcal{K}(u) = p_*\big(\frac{1}{p}||\nabla u||_p^p - \frac{1}{p_*}||u||_{p_*}\big) - \frac{p}{N-p}||\nabla u||_p^p.$$

From $\mathcal{E}(u) < \mathcal{E}(w_p) = \frac{1}{N}||\nabla w_p||_p^p$, we have

$$\mathcal{K}(u) < p_*\mathcal{E}_p(w_p) - \frac{p}{N-p}||\nabla u||_p^p = \frac{p}{N-p}\big(||\nabla w_p||_p^p - ||\nabla u||_p^p\big) < 0$$

as desired.

$iii)$ The assertion (1.18) about $\mathcal{K}(u) > 0$ is then a consequence of the previous two statements in Proposition 1.3. □

2. Blow-up in finite time of solutions

The main result of this section is the following theorem.

Theorem 2.1. *(Finite-time blow-up) Let* $(u_0, u_1) \in \big(\dot{W}^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)\big) \times L^p(\mathbb{R}^N)$*. Assume that there exists a solution* $u \in C^0(I, \dot{W}^{1,p}(\mathbb{R}^N)) \cap C^1(I, L^p(\mathbb{R}^N))$ *of (1.1) corresponding to the initial data* (u_0, u_1) *defined on the maximal time interval* I *and satisfying*

$$E((u_0, u_1)) < E((w_p, 0)) \ \text{ and } \ \mathcal{K}(u_0) < 0.$$

Then I *must be a finite interval.*

Remark 2.2. We point out that a blow-up result for the equation (1.1) has been obtained by Galaktinov-Pohozaev [4] under the following assumptions: $\int_{\mathbb{R}^N} u_0 u_1 dx > 0$ and $E((u_0, u_1)) \leq 0$, which are clearly more restrictive than our assumptions.

Remark 2.3. The existence of local solutions of (1.1) is still an interesting open question. The Cauchy problem is not only quasilinear but also with a degenerate or singular p-Laplacian ($p > 2$ or $1 < p < 2$, respectively). If a strong enough dissipation is put into the equation, then there are results about local or global existence of solution. See for example [5], [3] and [11].

In our proof, we follow an idea from [8] where the problem (1.1) was studied when $p = 2$. To pave the way for the proof of Theorem 2.1, we shall need a number of lemmas.

Lemma 2.4. *Let* $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ *be such that we have, for some* $\delta_0 > 0$*,*

$$||\nabla u||_p^p < ||\nabla w_p||_p^p \ \text{ and } \ \mathcal{E}(u) \leq (1-\delta_0)\mathcal{E}_p.$$

Then we have

$$||\nabla u||_p^p \leq (1-\delta_0)||\nabla w_p||_p^p \ \text{ and } \ \mathcal{E}(u) > 0.$$

Proof. We argue as in the proof of [7] Lemma 3.4, and consider the function $f(y) = \frac{1}{p}y - \frac{C^{p_*}(N,p)}{p_*}y^{\frac{p_*}{p}}$ and set $\overline{y} = ||\nabla u||_p^p$. We first observe that by the inequality (1.4)

$$\begin{aligned} f(\overline{y}) &= \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{C^{p_*}(N,p)}{p_*}\Big(\int_{\mathbb{R}^N}|\nabla u|^p dx\Big)^{\frac{p_*}{p}} \\ &\leq \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{C^{p_*}(N,p)}{p_*}\frac{1}{C^{p_*}(N,p)}\int_{\mathbb{R}^N}|u|^{p_*}dx \\ &= \mathcal{E}(u) \\ &\leq (1-\delta_0)\mathcal{E}_p \\ &= \frac{1-\delta_0}{NC^N(N,p)}. \end{aligned} \tag{2.1}$$

Next we have

$$\begin{aligned}
f(0) &= 0\\
f'(y) &= \frac{1}{p} - \frac{C^{p_*}(N,p)}{p_*}\frac{p_*}{p}y^{\frac{p_*}{p}-1} = \frac{1}{p} - \frac{C^{p_*}(N,p)}{p}y^{\frac{p}{N-p}}\\
f''(y) &= -\frac{C^{p_*}(N,p)}{N-p}y^{\frac{2p-N}{N-p}} \le 0
\end{aligned}$$

$$\begin{aligned}
f'(y) = 0 \;\Leftrightarrow\; y = y_C &= \frac{1}{C^N(N,p)} = ||\nabla w_p||_p^p\\
f(y_C) &= \frac{1}{p}y_C - \frac{C^{p_*}(N,p)}{p_*}y_C^{\frac{p_*}{p}}\\
&= \frac{1}{p}\frac{1}{C^N(N,p)} - \frac{C^{p_*}(N,p)}{p_*}\frac{1}{(C(N,p))^{\frac{N^2}{N-p}}}\\
&= \frac{1}{p}\frac{1}{C^N(N,p)} - \frac{1}{p_*}\frac{1}{C^N(N,p)}\\
&= \frac{1}{NC^N(N,p)}.
\end{aligned}$$

It follows that f is nonnegative and strictly increasing between 0 and y_C and concave in $[0,\underline{y}]$, where $\underline{y} = \frac{1}{C^N(N,p)}\left(\frac{N}{N-p}\right)^{\frac{N-p}{p}}$ is such that $f(\underline{y}) = 0$.
Since f is strictly increasing between 0 and y_C and $0 < \overline{y} < y_C$, we obtain from (2.1) $\mathcal{E}(u) \ge f(\overline{y}) > 0$.
Using the concavity of f and (2.1), we obtain

$$f(\overline{y}) \le (1-\delta_0)f(y_C) \le f\big((1-\delta_0)y_C\big). \tag{2.2}$$

Since f is strictly increasing between 0 and y_C, we obtain from (2.2)

$$\overline{y} \le (1-\delta_0)y_C \Leftrightarrow ||\nabla u||_p^p \le (1-\delta_0)||\nabla w_p||_p^p.$$

□

Lemma 2.5. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ be such that we have, for some $\delta_0 > 0$,*

$$\mathcal{K}(u) < 0 \;\; \textit{and} \;\; \mathcal{E}(u) \le (1-\delta_0)\mathcal{E}_p.$$

Then we have for the positive constant $\overline{\delta} = \left(1 + \frac{p\delta_0}{N-p}\right)^{\frac{N-p}{p}} - 1$

$$||\nabla u||_p^p \ge (1+\overline{\delta})||\nabla w_p||_p^p.$$

Proof. From the assumption and Proposition 1.3, we know that $||\nabla u||_p > ||\nabla w_p||_p$. Using the inequality (1.4) and (1.6)-(1.7), we obtain

$$\begin{aligned}
||\nabla u||_p^p > ||\nabla w_p||_p^p = N\mathcal{E}_p &\ge \frac{N}{1-\delta_0}\mathcal{E}(u)\\
&\ge \frac{N}{1-\delta_0}\Big(\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{C^{p_*}(N,p)}{p_*}\Big(\int_{\mathbb{R}^N}|\nabla u|^p dx\Big)^{\frac{p_*}{p}}\Big)\\
&= \frac{N}{1-\delta_0}\Big(\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{C^{p_*}(N,p)}{p_*}\Big(\int_{\mathbb{R}^N}|\nabla u|^p dx\Big)^{\frac{N}{N-p}}\Big).
\end{aligned}$$

This leads to

$$\frac{NC^{p_*}(N,p)}{(1-\delta_0)p_*}\Big(\int_{\mathbb{R}^N}|\nabla u|^p dx\Big)^{\frac{p}{N-p}} \geq \frac{N}{(1-\delta_0)p} - 1 = \frac{N-p+p\delta_0}{(1-\delta_0)p}$$
$$\text{or } \Big(\int_{\mathbb{R}^N}|\nabla u|^p dx\Big)^{\frac{p}{N-p}} \geq \frac{N-p+p\delta_0}{N-p}\frac{1}{C^{p_*}(N,p)}$$
$$\text{or } \int_{\mathbb{R}^N}|\nabla u|^p dx \geq \Big(1+\frac{p\delta_0}{N-p}\Big)^{\frac{N-p}{p}}\frac{1}{C^N(N,p)} = (1+\overline{\delta})||\nabla w_p||_p^p$$

and the lemma follows. □

Lemma 2.6. *Let $(u_0,u_1) \in \dot{W}^{1,p}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Assume that there exists a solution u of (1.1) corresponding to the initial data (u_0,u_1) defined on the maximal interval I and satisfying $E((u_0,u_1)) < E((w_p,0))$. Then we have*

$$\mathcal{K}(u_0) < 0 \;\Rightarrow\; ||\nabla u(t)||_p > ||\nabla w_p||_p \quad \forall t \in I. \tag{2.3}$$

Proof. Let $\delta_0 > 0$ be a fixed positive number such that $E((u_0,u_1)) < (1-\delta_0)E((w_p,0))$. Since $\mathcal{E}(u_0) \leq E((u_0,u_1)) \leq (1-\delta_0)E((w_p,0)) = (1-\delta_0)\mathcal{E}_p$ and $\mathcal{K}(u_0) < 0$, we know from Proposition 1.3 that

$$||\nabla u_0||_p > ||\nabla w_p||_p. \tag{2.4}$$

Now let $t \in I$. By (1.2), we have $E((u(t),u_t(t))) = E((u_0,u_1))$. We deduce that

$$\mathcal{E}(u(t)) \leq E((u(t),u_t(t))) = E((u_0,u_1)) \leq (1-\delta_0)E((w_p,0)) = (1-\delta_0)\mathcal{E}_p. \tag{2.5}$$

If $||\nabla u(t)||_p^p < ||\nabla w_p||_p^p$, we obtain from (2.5) and Lemma 2.4

$$||\nabla u(t)||_p^p \leq (1-\delta_0)||\nabla w_p||_p^p.$$

If $||\nabla u(t)||_p^p > ||\nabla w_p||_p^p$, we have by Lemma 2.5, for $\overline{\delta} = \Big(1+\frac{p\delta_0}{N-p}\Big)^{\frac{N-p}{p}} - 1$

$$||\nabla u(t)||_p^p \geq (1+\overline{\delta})||\nabla w_p||_p^p.$$

Taking into account (2.4) and the continuity of $||\nabla u(t)||_p$, the lemma holds. □

Remark 2.7. Under the assumption of Lemma 2.6, we can similarly show that

$$||\nabla u_0||_p < ||\nabla w_p||_p \;\Rightarrow\; ||\nabla u(t)||_p < ||\nabla w_p||_p \quad \forall t \in I.$$

We are now able to prove the main result of the paper.

Proof. of Theorem 2.1. Let $\delta_0 > 0$ be a fixed positive number such that

$$E((u_0,u_1)) < (1-\delta_0)E((w_p,0)),$$

and let $y(t) = \int_{\mathbb{R}^N} |u(x,t)|^2 dx$. We obtain by using (1.1) and integrating by parts

$$\begin{aligned} y'(t) &= 2\int_{\mathbb{R}^N} u(x,t)u_t(x,t)dx \\ y''(t) &= 2\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + 2\int_{\mathbb{R}^N} u(x,t)u_{tt}(x,t)dx \\ &= 2\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + 2\int_{\mathbb{R}^N} u(\Delta_p u + |u|^{p_*-2}u)dx \\ &= 2\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx - 2\mathcal{K}(u(t)) \\ &= 2\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx - 2\int_{\mathbb{R}^N} |\nabla u(t)|^p dx + 2\int_{\mathbb{R}^N} |u(x,t)|^{p_*} dx. \end{aligned} \tag{2.6}$$

For convenience, we set $\widetilde{\delta}_0 = \delta_0 E((w_p,0))$, and we obtain

$$E((w_p,0)) \geq E((u_0,u_1)) + \widetilde{\delta}_0 = E((u(t),u_t(t))) + \widetilde{\delta}_0,$$

which leads to

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x,t)|^{p_*} dx \geq \frac{p_*}{2}\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + \frac{p_*}{p}\int_{\mathbb{R}^N} |\nabla u(t)|^p dx \\ -p_* E((w_p,0)) + \widetilde{\delta}_0 p_*. \end{aligned} \tag{2.7}$$

We deduce from (2.6)-(2.7) and Lemma 2.6 that

$$\begin{aligned} y''(t) \geq{}& (2+p_*)\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + \frac{2p_*}{p}\int_{\mathbb{R}^N} |\nabla u(t)|^p dx \\ & -2p_* E((w_p,0)) + 2\widetilde{\delta}_0 p_* - 2\int_{\mathbb{R}^N} |\nabla u(t)|^p dx \\ ={}& (2+p_*)\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + 2\Big(\frac{p_*}{p} - 1\Big)\int_{\mathbb{R}^N} |\nabla u(t)|^p dx \\ & -\frac{2p_*}{N}\int_{\mathbb{R}^N} |\nabla w_p|^p dx + 2\widetilde{\delta}_0 p_* \\ ={}& (2+p_*)\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + \frac{2p_*}{N}\Big(\int_{\mathbb{R}^N} |\nabla u(t)|^p dx - \int_{\mathbb{R}^N} |\nabla w_p|^p dx\Big) + 2\widetilde{\delta}_0 p_* \\ \geq{}& (2+p_*)\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + 2\widetilde{\delta}_0 p_*. \end{aligned} \tag{2.8}$$

Assuming that $[0,\infty) \subset I$, and integrating (2.8), we obtain for $t > 0$, $y'(t) \geq 2\widetilde{\delta}_0 p_* t + y'(0)$. We deduce that there exists $t_0 > 0$ such that $y'(t) > 0$ for all $t \geq t_0$. We obtain for $t > t_0$, by using Cauchy-Schwarz inequality

$$\begin{aligned} & y''(t)y(t) \geq (2+p_*)\Big(\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx\Big)\Big(\int_{\mathbb{R}^N} |u(x,t)|^2 dx\Big) \geq \frac{p_*+2}{4} y'^2(t) \\ & \text{or } \frac{y''(t)}{y'(t)} \geq \frac{p_*+2}{4}\frac{y'(t)}{y(t)}. \end{aligned} \tag{2.9}$$

Integrating (2.9) between t_0 and t, we obtain for $C_0 = \frac{y'(t_0)}{(y(t_0))^{\frac{p_*+2}{4}}}$

$$\begin{aligned} & \ln(y'(t)) \geq \frac{p_*+2}{4}\ln(y(t)) + \ln(C_0) \\ & \text{or } y'(t) \geq C_0 (y(t))^{\frac{p_*+2}{4}} \\ & \text{or } \frac{y'(t)}{(y(t))^{\frac{p_*+2}{4}}} \geq C_0. \end{aligned} \tag{2.10}$$

Integrating again (2.10) between t_0 and t, we obtain for $C_1 = \frac{4}{(p_*-2)(y(t_0))^{\frac{p_*-2}{4}}}$

$$\frac{4}{(p_* - 2)(y(t))^{\frac{p_*-2}{4}}} \le C_0(t_0 - t) + C_1.$$

We have reached a contradiction, since the right hand side of the last inequality becomes negative for t large enough, while the left hand side is always nonnegative, since $p_* > 2$.

In the same way we get a contradiction, if we assume that $(-\infty, 0] \subset I$ □

Remark 2.8. From the proof of the Theorem, we see that we just needed an energy inequality which might be satisfied by just weak solutions.

Remark 2.9. Define the sets:

$$R_1 = \{(u, v) \in \dot{W}^{1,p}(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \ : \ E((u, v)) < \mathcal{E}_p \text{ and } \mathcal{K}(u) > 0 \} \cup \{(0, 0)\}$$
$$R_2 = \{(u, v) \in \dot{W}^{1,p}(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \ : \ E((u, v)) < \mathcal{E}_p \text{ and } \mathcal{K}(u) < 0 \}.$$

It is clear from Proposition 1.3, Lemma 2.6 and Remark 2.7, that R_1 and R_2 are invariant sets under the flow of the wave equation (1.1). Moreover, Theorem 2.1 shows that data in R_2 lead to finite time blow up solutions. However, we were not able so far to show that data in R_1 lead to global solutions. Nevertheless, we have the following *à priori* bounds on the solutions: for all $t \in I$ the maximal time of existence

$$\int_{\mathbb{R}^N} |\nabla u(t)|^p dx < N\mathcal{E}_p, \ \ \forall t \in I,$$

$$\frac{1}{2}\int_{\mathbb{R}^N} |u_t(t)|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} |u(t)|^{p_*} dx < \mathcal{E}_p \ \ \forall t \in I.$$

Indeed, by assumption we have $(u_0, u_1) \in R_1$ and therefore we obtain from (1.2) and the continuity of $\mathcal{K}(u(t))$ that $(u(t), u_t(t)) \in R_1$ for all $t \in I$. In particular we have

$$\mathcal{K}(u(t)) > 0 \quad \forall t \in I, \tag{2.11}$$

and

$$E((u(t), u_t(t))) = E((u_0, u_1)) < \mathcal{E}_p, \ \ \forall t \in I,$$

which can be written

$$\frac{1}{2}\int_{\mathbb{R}^N} |u_t(t)|^2 dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u(t)|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^N} |u(t)|^{p_*} dx < \mathcal{E}_p$$
$$\text{or } \frac{1}{2}\int_{\mathbb{R}^N} |u_t(t)|^2 dx + \frac{1}{p}\mathcal{K}(u(t)) + \frac{1}{N}\int_{\mathbb{R}^N} |u(t)|^{p_*} dx < \mathcal{E}_p. \tag{2.12}$$

It follows from (2.11)-(2.12) that we have

$$\frac{1}{2}\int_{\mathbb{R}^N} |u_t(t)|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} |u(t)|^{p_*} dx < \mathcal{E}_p \ \ \forall t \in I. \tag{2.13}$$

Now using (2.13), we obtain also from (2.12) that

$$\int_{\mathbb{R}^N} |\nabla u(t)|^p dx < p\mathcal{E}_p + \frac{p}{p_*}\int_{\mathbb{R}^N} |u(t)|^{p_*} dx < (p + \frac{Np}{p_*})\mathcal{E}_p = N\mathcal{E}_p, \ \ \forall t \in I. \tag{2.14}$$

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