*Math. Model. Nat. Phenom.*
**Vol. 8, No. 1, 2013, pp. 132–142**
DOI: 10.1051/mmnp/20138109

# Semiclassical Limits of Heat Kernels of Laplacians on the *h*-Heisenberg Group

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**Abstract.** We construct the heat kernels of the sub-Laplacian and the Laplacian on the $h$-Heisenberg group and compute the limits as $h \to 0$ of the heat kernels.

**Keywords and phrases:** $h$-Heisenberg group, sub-Laplacian, Laplacian, Fourier-Wigner transforms, Hermite functions, heat kernels, semiclassical limits

**Mathematics Subject Classification:** 47F05, 47G30

## 1. The Laplacians on the *h*-Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via the obvious identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C},$$

then for any positive number $h$, the set $\mathbb{C} \times \mathbb{R}$ becomes a noncommutative group $\mathbb{H}_h$ that we call the $h$-Heisenberg group when equipped with the multiplication $\cdot_h$ given by

$$(z, t) \cdot_h (w, s) = \left( z + w, t + s + \frac{h}{4}[z, w] \right), \quad (z, t), (w, s) \in \mathbb{H}_h,$$

where $[z, w]$ is the symplectic form of $z$ and $w$ defined by

$$[z, w] = 2\,\mathrm{Im}(z\overline{w}).$$

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}_h$. Then a basis for $\mathfrak{h}$ is given by $X_h$, $Y_h$ and $T_h$, where

$$X_h = \frac{\partial}{\partial x} + \frac{h}{2} y \frac{\partial}{\partial t},$$

$$Y_h = \frac{\partial}{\partial y} - \frac{h}{2} x \frac{\partial}{\partial t}$$

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and

$$T_h = \frac{\partial}{\partial t}.$$

In fact, if we let $\gamma_j : \mathbb{R} \to \mathbb{H}_h$, $j = 1, 2, 3$, be curves in $\mathbb{H}_h$ such that for all $s$ in $\mathbb{R}$,

$$\gamma_1(s) = (s, 0, 0),$$

$$\gamma_2(s) = (0, s, 0)$$

and

$$\gamma_3(s) = (0, 0, s),$$

then for all smooth functions $f$ on $\mathbb{H}_h$,

$$(X_h f)(z,t) = \left.\frac{d}{ds}\right|_{s=0} f((z,t) \cdot_h \gamma_1(s)),$$

$$(Y_h f)(z,t) = \left.\frac{d}{ds}\right|_{s=0} f((z,t) \cdot_h \gamma_2(s))$$

and

$$(T_h f)(z,t) = \left.\frac{d}{ds}\right|_{s=0} f((z,t) \cdot_h \gamma_3(s))$$

for all $(z, t)$ in $\mathbb{H}_h$.

The sub-Laplacian $\mathcal{L}_h$ and the Laplacian $\Delta_h$ on $\mathbb{H}_h$ are defined, respectively, by

$$\mathcal{L}_h = -(X_h^2 + Y_h^2)$$

and

$$\Delta_h = -(X_h^2 + Y_h^2 + T_h^2).$$

Simple computations give

$$\mathcal{L}_h = -\Delta - \frac{1}{4}(x^2 + y^2)h^2 \frac{\partial^2}{\partial t^2} + h \left( x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \frac{\partial}{\partial t}$$

and

$$\Delta_h = \mathcal{L}_h - \frac{\partial^2}{\partial t^2},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Details on the Heisenberg group $\mathbb{H}_h$, the sub-Laplacians $\mathcal{L}_h$ and $\Delta_h$ are the same as those for $\mathbb{H}_1$, $\mathcal{L}_1$ and $\Delta_1$, and can be found in [1, 2, 5].

We give in this paper a self-contained and detailed construction of the heat kernels $K_\rho^{\mathcal{L}_h}$ and $K_\rho^{\Delta_h}$, $\rho > 0$, of the sub-Laplacian $\mathcal{L}_h$ and the Laplacian $\Delta_h$ respectively on the $h$-Heisenberg group. The new results are the limits as $h \to 0$ of the heat kernels $K_\rho^{\mathcal{L}_h}$ and $K_\rho^{\Delta_h}$, $\rho > 0$. Explicit formulas for the limits are given and dubbed the semiclassical limits in view of the fact that the number $h$ can be thought of as Planck's constant in the $h$-Heisenberg group.

In Section 2, we show how to transform the Laplacians $\mathcal{L}_h$ and $\Delta_h$ to families of twisted Laplacians $L_{h\tau}$ and $\Delta_{h\tau}$ respectively on $\mathbb{C}$ parametrized by $\tau \in \mathbb{R} \setminus \{0\}$. This has the advantage of reducing the number of independent variables of the Laplacians from three to two and can be seen as a method of descent. The $h\tau$-Fourier–Wigner transforms of Hermite functions are developed in Section 3, which can then be used in Section 4 to construct the heat kernels of $L_{h\tau}$ and $\Delta_{h\tau}$ and hence the heat kernels of $\mathcal{L}_h$

and $\Delta_h$. It is worth pointing out that if we scale the variable $t$ by $t = ht'$, then the $h$-Heisenberg group is reduced to the usual Heisenberg group with the corresponding reductions of the sub-Laplacian and the Laplacian. Thus, the results in Section 4 can be obtained by scaling the known results in, for instance, [1,2]. We conclude the paper with the semiclassical limits of the heat kernels on the $h$-Heisenberg group.

To put the results of this paper in perspective, we note that the formula for the heat kernel of $\mathcal{L}_h$ can be traced to the independent works of Gaveau [3] and Hulanicki [4]. To recall, for positive time $\rho$, the heat kernel $K_\rho^{\mathcal{L}_h}$ of $\mathcal{L}_h$ on $\mathbb{H}_h$ is the kernel of the integral operator $e^{-\rho\mathcal{L}_h}$ such that

$$e^{-\rho\mathcal{L}_h} f = f *_{\mathbb{H}_h} K_\rho^{\mathcal{L}_h} \tag{1.1}$$

for all suitable functions $f$ on $\mathbb{H}_h$, where

$$(f *_{\mathbb{H}_h} K_\rho^{\mathcal{L}_h})(z,t) = \int_{\mathbb{H}_h} f((z,t) \cdot_h (w,s)^{-1}) K_\rho^{\mathcal{L}_h}(w,s)\, dw\, ds, \quad (z,t) \in \mathbb{H}_h,$$

provided that the integral exists.

The heat kernel $K_\rho^{\Delta_h}$, $\rho > 0$, of $\Delta_h$ on $\mathbb{H}_h$ can be defined similarly.

## 2. The Twisted Laplacians

Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ be linear partial differential operators on $\mathbb{R}^2$ given by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Let $h > 0$ and $\tau \in \mathbb{R} \setminus \{0\}$. Then we define the vector fields $Z_{h\tau}$ and $\overline{Z_{h\tau}}$ by

$$Z_{h\tau} = \frac{\partial}{\partial z} + \frac{h}{2}\tau\overline{z}, \quad \overline{z} = x - iy,$$

and

$$\overline{Z_{h\tau}} = \frac{\partial}{\partial \overline{z}} - \frac{h}{2}\tau z, \quad z = x + iy.$$

The vector fields $Z_{h\tau}$ and $\overline{Z_{h\tau}}$, and the identity operator $I$ form a basis for a Lie algebra in which the Lie bracket of two elements is their commutator. In fact, $-\overline{Z_{h\tau}}$ is the formal adjoint of $Z_{h\tau}$. Let $L_{h\tau}$ be the linear partial differential operator on $\mathbb{R}^2$ defined by

$$L_{h\tau} = -\frac{1}{2}(Z_{h\tau}\overline{Z_{h\tau}} + \overline{Z_{h\tau}}Z_{h\tau}).$$

Then $L_{h\tau}$ is an elliptic partial differential operator on $\mathbb{R}^2$ given by

$$L_{h\tau} = -\Delta + \frac{1}{4}(x^2+y^2)h^2\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)h\tau.$$

Thus, $L_{h\tau}$ is the ordinary Hermite operator $-\Delta + \frac{1}{4}(x^2+y^2)h^2\tau^2$ perturbed by the partial differential operator $-iNh\tau$, where

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator. As such, $L_{h\tau}$ can be considered as a twisted Laplacian. If $h\tau = 1$, then we recover the twisted Laplacian studied in detail in [7].

To see the connection of the twisted Laplacian $L_{h\tau}$ with the sub-Laplacian, we define for every function $f$ in $L^1(\mathbb{H}_h)$, the function $f^\tau$ on $\mathbb{C}$ by

$$f^\tau(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{it\tau} f(z,t)\, dt, \quad z \in \mathbb{C},$$

provided that the integral exists. $f^\tau(z)$ is in fact the inverse Fourier transform of $f(z,t)$ with respect to $t$ evaluated at $\tau$. It is to be noted that the Fourier transform $\hat{F}$ of a function $F$ in $L^1(\mathbb{R}^n)$ is defined by

$$\hat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

We also need to introduce the twisted convolution $f *_\lambda g$ of two measurable functions $f$ and $g$ on $\mathbb{C}$ given by

$$(f *_\lambda g)(z) = \int_{\mathbb{C}} f(z-w) g(w) e^{i\lambda[z,w]} dw, \quad z \in \mathbb{C},$$

provided that the integral exists.

The following result, which is the $h$-analog of Theorem 3.3 in [8], is useful to us.

**Theorem 2.1.** *Let $f$ and $g$ be functions in $L^1(\mathbb{H}_h)$. Then for $\tau \in \mathbb{R} \setminus \{0\}$,*

$$(f *_{\mathbb{H}_h} g)^\tau = (2\pi)^{1/2} (f^\tau *_{h\tau/4} g^\tau).$$

*Proof.* For all $z$ in $\mathbb{C}$,

$$\begin{aligned} &(f *_{\mathbb{H}_h} g)^\tau(z) \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{it\tau} \left( \int_{\mathbb{H}_h} f((z,t) \cdot_h (w,s)^{-1}) g(w,s)\, dw\, ds \right) dt \\ &= (2\pi)^{-1/2} \int_{\mathbb{H}_h} \left[ \int_{-\infty}^{\infty} e^{it\tau} f\left( z-w, t-s-\frac{h}{4}[z,w] \right) dt \right] g(w,s)\, dw\, ds. \end{aligned} \tag{2.1}$$

Now, for all $z$ and $w$ in $\mathbb{C}$, and all $s$ in $\mathbb{R}$,

$$\begin{aligned} &(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{it\tau} f\left( z-w, t-s-\frac{h}{4}[z,w] \right) dt \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i(t+s+\frac{h}{4}[z,w])\tau} f(z-w,t)\, dt \\ &= e^{i(s+\frac{h}{4}[z,w])\tau} f^\tau(z-w). \end{aligned} \tag{2.2}$$

Then by (2.1) and (2.2), we get for all $z$ in $\mathbb{C}$,

$$\begin{aligned} (f *_{\mathbb{H}_h} g)^\tau(z) &= \int_{\mathbb{H}_h} e^{i(s+\frac{h}{4}[z,w])\tau} f^\tau(z-w) g(w,s)\, dw\, ds \\ &= \int_{\mathbb{C}} \left( \int_{-\infty}^{\infty} e^{is\tau} g(w,s)\, ds \right) f^\tau(z-w) e^{ih\tau[z,w]/4} dw \\ &= (2\pi)^{1/2} \int_{\mathbb{C}} f^\tau(z-w) g^\tau(w) e^{ih\tau[z,w]/4} dw \\ &= (2\pi)^{1/2} (f^\tau *_{h\tau/4} g^\tau). \end{aligned}$$

□

Now, using Theorem 2.1 and the definition of the heat kernel as given by (1.1), we see that for positive time $\rho$ and suitable functions $f$ on $\mathbb{H}_h$,

$$e^{-\rho L_{h\tau}} f^\tau = (2\pi)^{1/2} K_\rho^{\mathcal{L}_{h\tau}} *_{-h\tau/4} f^\tau, \quad \tau \in \mathbb{R} \setminus \{0\}. \tag{2.3}$$

So, if we can compute the heat kernel of $L_{h\tau}$ and hence $K_\rho^{\mathcal{L}_{h\tau}}$, then the heat kernel of $\mathcal{L}_h$ on $\mathbb{H}_h$ can be obtained by taking the Fourier transform of $K_\rho^{\mathcal{L}_{h\tau}}$ with respect to $\tau$.

Similarly, if we take the inverse Fourier transform of the Laplacian $\Delta_h$ with respect to $t$, then we get the twisted Laplacian $\Delta_{h\tau}$ of $\Delta_h$ given by

$$\Delta_{h\tau} = L_{h\tau} + \tau^2.$$

The heat kernel $K_\rho^{\Delta_h}$, $\rho > 0$, of $\Delta_h$ can then be obtained by taking the Fourier transform of $K_\rho^{\Delta_{h\tau}}$ with respect to $\tau$.

## 3. $h\tau$-Fourier–Wigner Transforms of Hermite Functions

Let $f$ and $g$ be functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$. Then for $\tau \in \mathbb{R} \setminus \{0\}$, the $h\tau$-Fourier–Wigner transform $V_{h\tau}(f, g)$ of $f$ and $g$ is defined by

$$V_{h\tau}(f,g)(q,p) = (2\pi)^{-1/2} |h\tau|^{1/2} \int_{-\infty}^{\infty} e^{ih\tau qy} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)}\, dy$$

for all $q$ and $p$ in $\mathbb{R}$.

For $k = 0, 1, 2, \ldots$, the Hermite function $e_k$ of order $k$ is the function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x)$$

for all $x$ in $\mathbb{R}$, where $H_k$ is the Hermite polynomial of degree $k$ given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2})$$

for all $x$ in $\mathbb{R}$.

For $\tau \in \mathbb{R} \setminus \{0\}$ and $k = 0, 1, 2, \ldots$, we define $e_k^{h\tau}$ to be the function on $\mathbb{R}$ by

$$e_k^{h\tau}(x) = |h\tau|^{1/4} e_k(\sqrt{|h\tau|}x), \quad x \in \mathbb{R}.$$

For $\tau \in \mathbb{R} \setminus \{0\}$ and $j, k = 0, 1, 2, \ldots$, we define $e_{j,k}^{h\tau}$ on $\mathbb{R}^2$ by

$$e_{j,k}^{h\tau} = V_{h\tau}(e_j^{h\tau}, e_k^{h\tau}).$$

The connection of $\{e_{j,k}^{h\tau} : j, k = 0, 1, 2, \ldots\}$ with $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ studied in [7] is given by the following formula.

**Theorem 3.1.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and $j, k = 0, 1, 2, \ldots$,*

$$e_{j,k}^{h\tau}(q,p) = |h\tau|^{1/2} e_{j,k}\left(\frac{h\tau}{\sqrt{|h\tau|}} q, \sqrt{|h\tau|} p\right), \quad q, p \in \mathbb{R}.$$

*Proof.* For $\tau \in \mathbb{R} \setminus \{0\}$ and $j, k = 0, 1, 2, \ldots$,

$$\begin{aligned}
&e^{h\tau}_{j,k}(q,p) \\
&= V_{h\tau}(e^{h\tau}_j, e^{h\tau}_k)(q,p) \\
&= (2\pi)^{-1/2}|h\tau|^{1/2}\int_{-\infty}^{\infty} e^{ih\tau q y} e^{h\tau}_j\left(y+\frac{p}{2}\right)\overline{e^{h\tau}_k\left(y-\frac{p}{2}\right)}\,dy \\
&= (2\pi)^{-1/2}|h\tau|\int_{-\infty}^{\infty} e^{ih\tau q y} e_j\left(\sqrt{|h\tau|}\left(y+\frac{p}{2}\right)\right)\overline{e_k\left(\sqrt{|h\tau|}\left(y-\frac{p}{2}\right)\right)}\,dy \\
&= (2\pi)^{-1/2}|h\tau|^{1/2}\int_{-\infty}^{\infty} e^{ih\tau q y/\sqrt{|h\tau|}} e_j\left(y+\sqrt{|h\tau|}\frac{p}{2}\right)\overline{e_k\left(y-\sqrt{|h\tau|}\frac{p}{2}\right)}\,dy \\
&= |h\tau|^{1/2} e_{j,k}\left(\frac{h\tau}{\sqrt{|h\tau|}}q, \sqrt{|h\tau|}p\right)
\end{aligned}$$

for all $q$ and $p$ in $\mathbb{R}$. □

**Theorem 3.2.** *For $\tau \in \mathbb{R} \setminus \{0\}$, $\{e^{h\tau}_{j,k} : j, k = 0, 1, 2, \ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

Theorem 3.2 follows from Theorem 3.1 and Theorem 21.2 in [6] to the effect that $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

**Theorem 3.3.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and $j, k = 0, 1, 2, \ldots$,*

$$L_{h\tau} e^{h\tau}_{j,k} = (2k+1)|h\tau| e^{h\tau}_{j,k}.$$

Theorem 3.3 can be proved using Theorem 3.1 and Theorem 22.2 in [6] telling us that for $j, k = 0, 1, 2, \ldots$, $e_{j,k}$ is an eigenfunction of $L_1$ corresponding to the eigenvalue $2k+1$.

The following theorem is an analog of Theorem 3.3 for the Laplacian $\Delta_h$.

**Theorem 3.4.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and $j, k = 0, 1, 2, \ldots$,*

$$\Delta_{h\tau} e^{h\tau}_{j,k} = \{(2k+1)|h\tau| + \tau^2\} e^{h\tau}_{j,k}.$$

The following formula is the main tool for the construction of the heat kernel of $L_{h\tau}$.

**Theorem 3.5.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and nonnegative integers $\alpha$, $\beta$, $\mu$ and $\nu$,*

$$e^{h\tau}_{\alpha,\beta} *_{h\tau/4} e^{h\tau}_{\mu,\nu} = (2\pi)^{1/2}|h\tau|^{-1/2}\delta_{\beta,\mu} e^{h\tau}_{\alpha,\nu},$$

*where $\delta_{\beta,\mu}$ is the Kronecker delta.*

When $h\tau = 1$, the formula is the same as that in Theorem 4.1 in [7]. Theorem 3.4 can be proved using the formula for $h\tau = 1$ and Theorem 3.1.

## 4. The Heat Kernels of Twisted Laplacians

Using Theorem 3.3 and the spectral theorem, we get for all functions $f$ in $L^2(\mathbb{R}^2)$,

$$e^{-\rho L_{h\tau}} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-(2k+1)|h\tau|\rho} (f, e_{j,k}^{h\tau}) e_{j,k}^{h\tau}, \quad \rho > 0,$$

where ( , ) is the inner product in $L^2(\mathbb{R}^2)$. So, for $\rho > 0$,

$$e^{-\rho L_{h\tau}} f = \sum_{k=0}^{\infty} e^{-(2k+1)|h\tau|\rho} \sum_{j=0}^{\infty} (f, e_{j,k}^{h\tau}) e_{j,k}^{h\tau}$$

and our first task is to compute $\sum_{j=0}^{\infty} (f, e_{j,k}^{h\tau}) e_{j,k}^{h\tau}$. To this end, we note that for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} f *_{h\tau/4} e_{k,k}^{h\tau} &= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (f, e_{j,l}^{\tau}) e_{j,l}^{h\tau} *_{h\tau/4} e_{k,k}^{h\tau} \\ &= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (f, e_{j,l}^{h\tau}) (2\pi)^{1/2} |h\tau|^{-1/2} \delta_{l,k} e_{j,k}^{h\tau} \\ &= (2\pi)^{1/2} |h\tau|^{-1/2} \sum_{j=0}^{\infty} (f, e_{j,k}^{h\tau}) e_{j,k}^{h\tau}. \end{aligned}$$

Hence, for $k = 0, 1, 2, \ldots$,

$$\sum_{j=0}^{\infty} (f, e_{j,k}^{h\tau}) e_{j,k}^{h\tau} = (2\pi)^{-1/2} |h\tau|^{1/2} (f *_{h\tau/4} e_{k,k}^{h\tau}).$$

Therefore

$$e^{-\rho L_{h\tau}} f = (2\pi)^{-1/2} |h\tau|^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1)|h\tau|\rho} e_{k,k}^{h\tau} *_{-h\tau/4} f, \quad \rho > 0.$$

Now, using Theorem 3.1 and Mehler's formula, we get for all $z = (q, p)$ in $\mathbb{C}$ and for $\rho > 0$,

$$\begin{aligned} &(2\pi)^{-1/2} |h\tau|^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1)|h\tau|\rho} e_{k,k}^{h\tau}(q, p) \\ &= (2\pi)^{-1/2} |h\tau| e^{-|h\tau|\rho} \sum_{k=0}^{\infty} e^{-2k|h\tau|\rho} e_{k,k} \left( \frac{h\tau}{\sqrt{|h\tau|}} q, \sqrt{|h\tau|} p \right) \\ &= (2\pi)^{-1} |h\tau| e^{-|h\tau|\rho} \frac{1}{1 - e^{-2|h\tau|\rho}} e^{-|h\tau|\,|z|^2 \frac{1}{4} \frac{1+e^{-2|h\tau|\rho}}{1-e^{-2|h\tau|\rho}}} \\ &= \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|h\tau|\,|z|^2 \coth(h\tau\rho)}. \end{aligned}$$

So, the heat kernel $\kappa_\rho^{L_{h\tau}}$, $\rho > 0$, of $L_{h\tau}$ on $\mathbb{R}^2$ is given by

$$\kappa_\rho^{L_{h\tau}}(z, w) = \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|h\tau|\,|z-w|^2 \coth(h\tau\rho)} e^{-i\frac{h\tau}{4}[z,w]}, \quad z, w \in \mathbb{C}.$$

Hence by (2.3), we have the following result.

**Theorem 4.1.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and $\rho > 0$,*

$$K_\rho^{\mathcal{L}_{h\tau}} = (2\pi)^{-1/2} k_\rho^{\mathcal{L}_{h\tau}}, \tag{4.1}$$

*where*

$$k_\rho^{\mathcal{L}_{h\tau}}(z) = \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}h\tau\,|z|^2 \coth(h\tau\rho)}, \quad z \in \mathbb{C}. \tag{4.2}$$

An immediate application of Theorem 4.1 is the following formula for the heat kernel $K_\rho^h$, $\rho > 0$, of $\mathcal{L}_h$.

**Theorem 4.2.** *For $\rho > 0$, the heat kernel $K_\rho^{\mathcal{L}_h}$ of $\mathcal{L}_h$ is given by*

$$K_\rho^{\mathcal{L}_h}(z,t) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-it\tau} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}h\tau|z|^2 \coth(h\tau\rho)} d\tau, \quad (z,t) \in \mathbb{H}_h.$$

By a change of variable, we can write

$$K_\rho^{\mathcal{L}_h}(z,t) = \frac{1}{8\pi^2 h\rho^2} \int_{-\infty}^{\infty} e^{-\frac{\tau}{\rho}\left(i\frac{t}{h} + \frac{1}{4}|z|^2 \coth\tau\right)} \frac{\tau}{\sinh\tau} d\tau, \quad (z,t) \in \mathbb{H}.$$

If we let $f_h(z,\tau,t) = \tau g_h(z,\tau,t)$, where

$$g_h(z,\tau,t) = i\frac{t}{h} + \frac{1}{4}|z|^2 \coth\tau,$$

and $V(\tau) = \frac{\tau}{\sinh\tau}$, then

$$K_\rho^{\mathcal{L}_h}(z,t) = \frac{1}{8\pi h\rho^2} \int_{-\infty}^{\infty} e^{-f_h(z,\tau,t)/\rho} V(\tau)\, d\tau, \quad (z,t) \in \mathbb{H}_h.$$

Similarly, the heat kernel $\kappa_\rho^{\Delta_{h\tau}}$, $\rho > 0$, of $\Delta_{h\tau}$ on $\mathbb{R}^2$ is given by

$$\kappa_\rho^{\Delta_{h\tau}}(z,w) = \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}h\tau \coth((h\tau\rho) - \tau^2\rho} e^{-i\frac{h\tau}{4}[z,w]}, \quad z,w \in \mathbb{C}.$$

The analogs of Theorems 4.1 and 4.2 are then given by, respectively, the following results.

**Theorem 4.3.** *For $\tau \in \mathbb{R} \setminus \{0\}$ and $\rho > 0$,*

$$K_\rho^{\Delta_{h\tau}} = (2\pi)^{-1/2} k_\rho^{\Delta_{h\tau}}, \tag{4.3}$$

*where*

$$k_\rho^{\Delta_{h\tau}} = \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|h\tau|\,|z|^2 \coth(h\tau\rho)}, \quad z \in \mathbb{C}. \tag{4.4}$$

**Theorem 4.4.** *For $\rho > 0$, the heat kernel $K_\rho^{\Delta_h}$ of $\Delta_h$ on $\mathbb{H}_h$ is given by*

$$K_\rho^{\Delta_h}(z,t) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} e^{-it\tau} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}h\tau|z|^2 \coth(h\tau\rho) - \tau^2\rho} d\tau, \quad (z,t) \in \mathbb{H}_h.$$

## 5. Semiclassical Limits of Heat Kernels

We begin with the heat kernel of the twisted Laplacian $L_{h\tau}$.

**Theorem 5.1.** *For $\tau \in \mathbb{R} \setminus \{0\}$,*

$$K_\rho^{\mathcal{L}_{h\tau}}(z) \to (2\pi)^{-1/2} \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)}$$

*as $h \to 0$ for all $z$ in $\mathbb{C}$ and all positive numbers $\rho$, where $K_\rho^{\mathcal{L}_{h\tau}}$ is the heat kernel of $L_{h\tau}$ given by* (4.1) *and* (4.2).

*Proof.* Since

$$\lim_{x\to 0} \frac{x}{\sinh x} = 1$$

and

$$\lim_{x\to 0} x \coth x = 1,$$

it follows that

$$\begin{aligned} \lim_{h\to 0} K_\rho^{\mathcal{L}_{h\tau}}(z) &= \lim_{h\to 0} (2\pi)^{-1/2} \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau \coth(h\tau\rho)} \\ &= \lim_{h\to 0} (2\pi)^{-1/2} \frac{1}{4\pi\rho} \frac{h\tau\rho}{\sinh(h\tau\rho)} e^{-\frac{1}{4\rho}|z|^2 h\tau\rho \coth(h\tau\rho)} \\ &= (2\pi)^{-1/2} \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)}, \end{aligned}$$

as asserted. □

We give in the following theorem the main result on the semiclassical limit of the heat kernel of $\mathcal{L}_h$ on $\mathbb{H}_h$ as $h \to 0$.

**Theorem 5.2.** *For all positive time $\rho$ and all $(z,t)$ in $\mathbb{R}^3$,*

$$K_\rho^{\mathcal{L}_h}(z,t) \to \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)} \delta(t)$$

*in $\mathcal{S}'(\mathbb{R}^3)$ as $h \to 0$, where $\mathcal{S}'(\mathbb{R}^3)$ is the space of all tempered distributions on $\mathbb{R}^3$.*

*Proof.* Let $\varphi$ be in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ on $\mathbb{R}^3$. Then

$$\begin{aligned} &K_\rho^{\mathcal{L}_h}(\varphi) \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{\mathbb{C}} \left( \int_{-\infty}^{\infty} e^{-it\tau} \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau \coth(h\tau\rho)} d\tau \right) \varphi(z,t) dz\, dt \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{C}} \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau \coth(h\tau\rho)} \hat{\varphi}(z,\tau)\, dz\, d\tau, \end{aligned}$$

where for all $z$ in $\mathbb{C}$ and $\tau$ in $\mathbb{R}$, $\hat{\varphi}(z,\tau)$ is the partial Fourier transform of $\varphi$ with respect to $t$ evaluated at $\tau$. By Theorem 5.1 and Lebesgue's dominated convergence theorem, we conclude that

$$\begin{aligned} \lim_{h\to 0} K_\rho^{\mathcal{L}_h}(\varphi) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{\mathbb{C}} \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)} \hat{\varphi}(z,\tau)\, dz\, d\tau \\ &= \int_{\mathbb{C}} \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)} \varphi(z,0)\, dz \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{C}} \frac{1}{4\pi\rho} e^{-|z|^2/(4\rho)} \delta(t) \varphi(z,t)\, dz\, dt, \end{aligned}$$

and the proof is complete. □

We end this paper with the analogs of Theorems 5.1 and 5.2 for the Laplacian $\Delta_h$.

**Theorem 5.3.** *For* $\tau \in \mathbb{R} \setminus \{0\}$,

$$K_\rho^{\Delta_{h\tau}}(z) \to (2\pi)^{-1/2} \frac{1}{4\pi\rho} e^{-(|z|^2/(4\rho))-\tau^2\rho}$$

*as* $h \to 0$ *for all* $z$ *in* $\mathbb{C}$ *and all positive numbers* $\rho$, *where* $K_\rho^{\Delta_{h\tau}}$ *is the heat kernel of* $\Delta_{h\tau}$ *given by* (4.3) *and* (4.4).

*Proof.* Since

$$\lim_{h\to 0} \frac{h\tau}{\sinh(h\tau\rho)} = \frac{1}{\rho}$$

and

$$\lim_{h\to 0} h\tau \coth(h\tau\rho) = \frac{1}{\rho},$$

it follows that

$$\lim_{h\to 0} K_\rho^{\Delta_{h\tau}}(z) = \lim_{h\to 0} (2\pi)^{-1/2} \frac{1}{4\pi} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau\rho \coth(h\tau\rho)-\tau^2\rho} = (2\pi)^{-1/2} \frac{1}{4\pi\rho} e^{-(|z|^2/(4\rho))-\tau^2\rho}$$

for all $z$ in $\mathbb{C}$ and all positive numbers $\rho$. □

We can now give the semiclassical limit of the heat kernel of $\Delta_h$ on $\mathbb{H}_h$.

**Theorem 5.4.** *For all positive time* $\rho$ *and all* $(z, t)$ *in* $\mathbb{R}^3$,

$$K_\rho^{\Delta_h}(z,t) \to \frac{1}{(4\pi\rho)^{3/2}} e^{-(|z|^2+t^2)/(4\rho)}$$

*as* $h \to 0$.

*Proof.* By Theorem 5.3, we get for all positive time $\rho$ and all $(z, t)$ in $\mathbb{R}^3$,

$$K_\rho^{\Delta_h}(z,t) = \int_{-\infty}^{\infty} e^{-it\tau} \frac{1}{8\pi^2} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau \coth(h\tau\rho)-\tau^2\rho} d\tau.$$

By Theorem 5.3 and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\lim_{h\to 0} K_\rho^{\Delta_h}(z,t) \\ &= \int_{-\infty}^{\infty} \lim_{h\to 0} e^{-it\tau} \frac{1}{8\pi^2} \frac{h\tau}{\sinh(h\tau\rho)} e^{-\frac{1}{4}|z|^2 h\tau \coth(h\tau\rho)-\tau^2\rho} d\tau \\ &= \int_{-\infty}^{\infty} e^{-it\tau} \frac{1}{8\pi^2} \frac{1}{\rho} e^{-|z|^2/(4\rho)} e^{-\tau^2\rho} d\tau \\ &= \frac{1}{8\pi^2\rho} e^{-|z|^2/(4\rho)} \int_{-\infty}^{\infty} e^{-it\tau} e^{-\tau^2\rho} d\tau. \end{aligned} \tag{5.1}$$

Since

$$\int_{-\infty}^{\infty} e^{-it\tau-\tau^2\rho} d\tau = \sqrt{\frac{\pi}{\rho}} e^{-t^2/(4\rho)},$$

we get by (5.1)

$$\lim_{h\to 0} K_\rho^{\Delta_h}(z,t) = \frac{1}{8\pi^2\rho} e^{-|z|^2/(4\rho)} \sqrt{\frac{\pi}{\rho}} e^{-t^2/(4\rho)} = \frac{1}{(4\pi\rho)^{3/2}} e^{-(|z|^2+t^2)/(4\rho)}.$$

□

*Acknowledgements.* We are grateful to the referee for the useful comments for improving the paper.

## References


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