Math. Model. Nat. Phenom.
Vol. 8, No. 1, 2013, pp. 170–174
DOI: 10.1051/mmnp/20138112

Derivatives of L^p Eigenfunctions of Schrödinger Operators

M. Lukic *

Rice University, 6100 Main Street, Mathematics MS 136, Houston, TX 77005, USA

Abstract. Assuming the negative part of the potential is uniformly locally L^1, we prove a pointwise L^p estimate on derivatives of eigenfunctions of one-dimensional Schrödinger operators. In particular, if an eigenfunction is in L^p, then so is its derivative, for $1 \le p \le \infty$.

Keywords and phrases: Schrödinger operators, derivatives of eigenfunctions, pointwise estimate

Mathematics Subject Classification: 34L40, 35J10

1. Introduction

In this note we study eigenfunctions u of a one-dimensional Schrödinger operator,

$$-u''(x) + V(x)u(x) = zu(x) \tag{1.1}$$

where V is a real-valued function and $z \in \mathbb{C}$. If $V \in L^1_{\mathrm{loc}}$, standard existence and uniqueness results for ODEs (see, e.g., Teschl [10, Theorem 9.1]) state that (1.1) has a two-dimensional space of solutions with $u, u' \in \mathrm{AC}_{\mathrm{loc}}$. Here $\mathrm{AC}_{\mathrm{loc}}$ stands for the space of functions which are absolutely continuous on compact intervals.

We will prove a pointwise L^p estimate on u', generalizing some known inequalities; this pointwise estimate will provide a proof that $u \in L^p$ implies $u' \in L^p$ under a mild condition on the negative part of V. Our estimate will also imply that $u \in L^p$ with $p < \infty$ implies pointwise decay of u and u'.

Throughout the paper, the condition on V will be

$$C_1 = \sup_x \int_x^{x+1} V_-(y)dy < \infty. \tag{1.2}$$

i.e. that the negative part of V is uniformly locally L^1.

Theorem 1.1. *Let $V \in L^1_{\mathrm{loc}}$ obey (1.2), and let $u(x)$ be a solution of (1.1) with $z \in \mathbb{C}$.*

(i) Denoting $C_2 = C_1 + |z|$, there exist constants $C = C_2 + 2\sqrt{C_2}$ and $K = 1/\sqrt{C_2}$ such that

$$|u'(x)| \le C \max_{y \in [x-K, x+K]} |u(y)|. \tag{1.3}$$

*Corresponding author. E-mail: milivoje.lukic@rice.edu

(ii) *Let* $u(x) \neq 0$, $\mathrm{Re}[\overline{u(x)}u'(x)] \geq 0$. *Then*

$$|u(y)| > \frac{|u(x)|}{2} \text{ for } y \in [x, x+\delta), \tag{1.4}$$

where $\delta = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2C_2}}$.

(iii) *For* $1 \leq p < \infty$,

$$|u(x)|^p \leq \frac{2^p}{\delta} \int_{x-\delta}^{x+\delta} |u(y)|^p dy. \tag{1.5}$$

(iv) *For* $1 \leq p < \infty$,

$$|u'(x)|^p \leq \frac{2^p C^p}{\delta} \int_{x-K-\delta}^{x+K+\delta} |u(y)|^p dy. \tag{1.6}$$

(v) *Let* $1 \leq p \leq \infty$ *and let* $w : \mathbb{R} \to (0, \infty)$ *obey*

$$\sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\leq K+\delta}} \frac{w(x)}{w(y)} < \infty. \tag{1.7}$$

Then $u \in L^p(w(x)dx)$ *implies* $u' \in L^p(w(x)dx)$.

(vi) *If* $u \in L^p(dx)$ *for some* $1 \leq p < \infty$, *then*

$$\lim_{x\to\pm\infty} u(x) = \lim_{x\to\pm\infty} u'(x) = 0. \tag{1.8}$$

Results of this type have appeared in the literature as technical lemmas; Stolz proved Theorem 1.1(v) for some weighted L^2 spaces in [9, Proposition 8], and for L^∞ with $z \in \mathbb{R}$ in [8, Lemma 4]. The strongest pointwise estimate of the type (1.6) previously available in the literature is by Simon [7, Lemma 3.1]; it holds for $p = 2$, with the stronger condition that V be uniformly locally L^2. Theorem 1.1 unifies and generalizes these results.

In another direction, the results of Stolz and Simon have been generalized to Sturm–Liouville operators by Schmied–Sims–Teschl [5].

In Section 2, we discuss some applications of Theorem 1.1 to the spectral theory of Schrödinger operators. In Section 3, we present the proof of Theorem 1.1.

It is my pleasure to thank David Damanik, Fritz Gesztesy, Barry Simon and Günter Stolz for useful discussions.

2. Applications to spectral theory

We present some applications of these estimates to spectral theory. These are not new results, but estimates of Theorem 1.1 are relevant to their proofs. These are half-line results, so in this section, $H = -\frac{d^2}{dx^2} + V$ will be the Schrödinger operator on $(0, +\infty)$. We assume 0 is a regular point, i.e. $V \in L^1(0, 1)$, so $u(x)$ and $u'(x)$ have finite limits as $x \to 0$.

Our first application is to an alternative proof that bounded eigenfunctions imply absolutely continuous spectrum. We are referring to the following theorem.

Theorem 2.1. *Let* $V \in L^1_{\mathrm{loc}}$ *be a half-line potential with a regular point at* 0 *which obeys* (1.2) *and let*

$$S = \{E \in \mathbb{R} \mid \textit{solutions of } (1.1) \textit{ are bounded on } [0, \infty)\}.$$

Then the spectral measure μ *of* $H = -\frac{d^2}{dx^2} + V(x)$ *obeys*

(i) $\mu_{\mathrm{sing}}(S) = 0$;

(ii) $\mu_{\mathrm{ac}}(T) > 0$ *for any* $T \subset S$ *with* $|T| > 0$ *(where* $|\cdot|$ *is the Lebesgue measure).*

This theorem was first proved by Behncke [1] and Stolz [8], who proved that (1.2) and boundedness of eigenfunctions for $E \in S$ allows one to use the subordinacy theory of Gilbert–Pearson [2] to imply the conclusions of the above theorem.

A more direct proof was found by Simon [7]. However, the proof in [7] assumes that V is uniformly locally L^2 in order to bound u' locally in terms of u. Replacing that part of the argument by Theorem 1.1(iv), the proof in [7] generalizes to prove Theorem 2.1 in full generality. It should be noted that this method needs the estimate (1.6) for non-real energies z, which Theorem 1.1 provides.

In the remainder of this section, we point out some simple criteria for point spectrum. These criteria use the implication

$$u \in L^2 \implies u' \in L^2. \tag{2.1}$$

This is a special case of Theorem 1.1(v), but we remind the reader that it was previously proved by Stolz [9, Proposition 8].

Simon–Stolz [6] provide a criterion for absence of eigenvalues in terms of transfer matrices. The transfer matrix $T(E,x,y)$ is defined by

$$T(E,x,y)\begin{pmatrix} u(y) \\ u'(y) \end{pmatrix} = \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}.$$

for solutions u of (1.1). The Simon–Stolz criterion uses the condition

$$\int_0^\infty \frac{dx}{\|T(E,x,0)\|^2} = \infty \tag{2.2}$$

to prove that (1.1) has no L^2 solution. Their theorem also assumes V is bounded from below, but their proof, combined with the implication (2.1), gives

Corollary 2.2. *Let $V \in L^1_{\mathrm{loc}}$ be a half-line potential with a regular point at 0 which obeys (1.2) and let $E \in \mathbb{R}$ be such that (2.2) holds. Then $-\Delta + V$, as a Schrödinger operator on $L^2(\mathbb{R}^+)$, doesn't have an eigenvalue at E.*

Proof. The argument of Simon–Stolz [6, Theorem 2.1] goes unchanged to prove $\left\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} \right\| \notin L^2$ for any solution of (1.1) with $z = E$. (2.1) then implies $u \notin L^2$, so E is not an eigenvalue of $-\Delta + V$. □

For a real-valued non-zero solution of (1.1) and $z = E = k^2 > 0$, Prüfer variables are defined by

$$u'(x) = kR_k(x)\cos\theta_k(x)$$
$$u(x) = R_k(x)\sin\theta_k(x)$$

with $R_k(x) > 0$, $\theta_k(x) \in \mathbb{R}$. They were first introduced by Prüfer [4] and have found extensive use in spectral theory, see e.g. Kiselev–Last–Simon [3]. Note that

$$k^2 R_k(x)^2 = u'(x)^2 + k^2 u(x)^2. \tag{2.3}$$

The following corollary is immediate from (2.3) and (2.1).

Corollary 2.3. *Let $V \in L^1_{\mathrm{loc}}$ be a half-line potential with a regular point at 0 which obeys (1.2) and let $E = k^2 > 0$. Then $u \in L^2$ if and only if $R_k \in L^2$.*

3. Proof of Theorem 1.1

The basis of all the estimates will be the following inequality, motivated by work of Stolz [8].

Lemma 3.1. *Let $x < y$ and assume $\omega \in \mathbb{C}$, $u(x) \neq 0$, and $\operatorname{Re}[\bar\omega u(t)] \geq 0$ for $t \in [x,y]$. Then*

$$\operatorname{Re}[\bar\omega u(y)] \geq \operatorname{Re}[\bar\omega u(x)] + (y-x)\operatorname{Re}[\bar\omega u'(x)] - C_2(y-x)(y-x+1)|\omega| \max_{x\leq t\leq y}|u(t)|. \tag{3.1}$$

Proof. Using absolute continuity of u and u',

$$\begin{aligned} u(y) &= u(x) + \int_x^y \left[u'(x) + \int_x^t u''(s)ds\right] dt \\ &= u(x) + (y-x)u'(x) + \int_x^y (y-s)u''(s)ds \end{aligned} \tag{3.2}$$

Denoting $M = \max_{x\leq t\leq y}|u(t)|$, we have $0 \leq \operatorname{Re}[\bar\omega u(s)] \leq |\bar\omega u(s)| \leq |\omega|M$ for $s \in [x,y]$, so by $u'' = Vu - zu$,

$$\begin{aligned} \operatorname{Re}\left[\bar\omega \int_x^y (y-s)u''(s)ds\right] &= \int_x^y (y-s)V(s)\operatorname{Re}[\bar\omega u(s)]\,ds - \int_x^y (y-s)\operatorname{Re}[\bar\omega z u(s)]\,ds \\ &\geq -|\omega|M(y-x)\int_x^y V_-(s)ds - |\omega z|M(y-x)^2 \\ &\geq -|\omega|M(y-x)(y-x+1)(C_1+|z|) \end{aligned}$$

which together with (3.2) proves (3.1). □

Proof of Theorem 1.1. (i) Without loss of generality, assume $\operatorname{Re}[\overline{u(x)}u'(x)] \geq 0$ (the other case follows by considering $u(-x)$).

Let $M = \max_{x-K\leq y\leq x+K}|u(y)|$. Assume that, contrary to (1.3), we have

$$|u'(x)| > C_2(1+2K)M. \tag{3.3}$$

Denote $f(y) = \operatorname{Re}[\overline{u'(x)}u(y)]$. Note that f is continuous, $f(x) \geq 0$ and $f'(x) = \operatorname{Re}[\overline{u'(x)}u'(x)] > 0$, so $f > 0$ in some interval $(x, x+\epsilon)$. We claim that $f > 0$ in $(x, x+K]$; assume to the contrary, that there exists $y \in (x, x+K]$ such that $f(y) = 0$, and pick the smallest such y. Then $f \geq 0$ on $[x,y]$, so applying Lemma 3.1 with $\omega = u'(x)$, we have

$$\begin{aligned} f(y) &\geq f(x) + (y-x)|u'(x)|^2 - C_2(y-x)(y-x+1)|u'(x)|M \\ &\geq M(y-x)|u'(x)|\big(|u'(x)| - C_2(y-x+1)\big) \end{aligned} \tag{3.4}$$

Thus, by (3.3),

$$f(y) > M(y-x)|u'(x)|C_2(2K-(y-x)) > 0 \tag{3.5}$$

contradicting our assumption and proving $f > 0$ on $(x, x+K]$. Taking $y = x+K$ in (3.5), we have

$$\operatorname{Re}[\overline{u'(x)}u(x+K)] > C_2MK^2|u'(x)| = M|u'(x)| \geq |\overline{u'(x)}u(x+K)|$$

which is a contradiction. Thus, the initial assumption (3.3) is wrong.

(ii) Assume the contrary; then there exists $y \in (x, x+\delta)$ such that $|u(y)| = \frac{|u(x)|}{2}$. Let $s \in [x,y)$ be such that

$$|u(s)| = \max_{t\in[x,y]}|u(t)|.$$

Since $\operatorname{Re}[\overline{u(t)}u'(t)] = \frac{1}{2}\frac{d}{dt}|u(t)|^2$, we have $\operatorname{Re}[\overline{u(s)}u'(s)] = 0$ (this is true even if $s = x$ since we know a priori that $\operatorname{Re}[\overline{u(x)}u'(x)] \geq 0$). Note also

$$\operatorname{Re}[\overline{u(s)}u(y)] \leq |\overline{u(s)}u(y)| \leq \frac{|u(s)|^2}{2}, \tag{3.6}$$

so we may pick $t \in (s, y]$ as the smallest $t > s$ with $\mathrm{Re}[\overline{u(s)}u(t)] = \frac{|u(s)|^2}{2}$.

Using (i) with x replaced by s and y replaced by t, and with $\omega = u(s)$ gives

$$\begin{aligned} \mathrm{Re}[\overline{u(s)}u(t)] &\geq |u(s)|^2[1 - C_2(t-s)(t-s+1)] \\ &> |u(s)|^2[1 - C_2\delta(\delta+1)] \\ &= \frac{|u(s)|^2}{2} \end{aligned}$$

where we used $t - s \leq y - x < \delta$. This is a contradiction with (3.6), which completes the proof.

(iii) For $\mathrm{Re}[\overline{u(x)}u'(x)] \geq 0$, the claim follows directly from (ii) by taking the p-th power of (1.4) and integrating from x to $x + \delta$. The case $\mathrm{Re}[\overline{u(x)}u'(x)] < 0$ follows by symmetry, by considering $u(-x)$.

(iv) This follows directly from (i) and (iii).

(v) We start with (1.3) for $p = \infty$ or (1.6) for $p < \infty$, and multiply by $w(x) \leq C_3 w(y)$. For $p < \infty$, integrating in x and using Tonelli's theorem completes the proof.

(vi) If $u \in L^p$ with $p < \infty$, then the right hand sides of (1.5), (1.6) converge to 0 as $x \to \pm\infty$, so the left hand sides also converge to 0. □

References

[1] H. Behncke. *Absolute continuity of Hamiltonians with won Neumann Wigner potentials.* Manuscripta Math. 71 (1991), 163–181.

[2] D. J. Gilbert, D. B. Pearson. *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators.* J. Math. Anal. Appl. 128 (1987), no. 1, 30–56.

[3] A. Kiselev, Y. Last, B. Simon. *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators.* Comm. Math. Phys. 194 (1998), no. 1, 1–45.

[4] H. Prüfer. *Neue Herleitung der Sturm–Liouvilleschen Reihenentwicklung stetiger Funktionen.* Math. Ann. 95 (1926), no. 1, 499–518.

[5] M. Schmied, R. Sims, G. Teschl. *On the absolutely continuous spectrum of Sturm–Liouville operators with applications to radial quantum trees.* Oper. Matrices 2 (2008),417–434.

[6] B. Simon, G. Stolz. *Operators with singular continuous spectrum. V. Sparse potentials.* Proc. Amer. Math. Soc. 124 (1996), no. 7, 2073–2080.

[7] B. Simon. *Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators.* Proc. Amer. Math. Soc. 124 (1996), 3361–3369.

[8] G. Stolz. *Bounded solutions and absolute continuity of Sturm–Liouville operators.* J . Math. Anal. Appl. 169 (1992), 210–228.

[9] G. Stolz. *Localization for random Schrödinger operators with Poisson potential.* Ann. Inst. H. Poincaré Phys. Théor. 63 (1995), no. 3, 297–314.

[10] G. Teschl. *Mathematical methods in quantum mechanics with applications to Schrödinger operators.* Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009.