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# Spectral Properties of Schrödinger-type Operators and Large-time Behavior of the Solutions to the Corresponding Wave Equation

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**Abstract.** Let $L$ be a linear, closed, densely defined in a Hilbert space operator, not necessarily selfadjoint. Consider the corresponding wave equations

$$(1) \quad \ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f, \quad \dot{w} = \frac{dw}{dt}, \quad f \in H.$$

$$(2) \quad \ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0,$$

where $k > 0$ is a constant. Necessary and sufficient conditions are given for the operator $L$ not to have eigenvalues in the half-plane $\mathrm{Re}z < 0$ and not to have a positive eigenvalue at a given point $k_d^2 > 0$. These conditions are given in terms of the large-time behavior of the solutions to problem (1) for generic $f$.

Sufficient conditions are given for the validity of a version of the limiting amplitude principle for the operator $L$.

A relation between the limiting amplitude principle and the limiting absorption principle is established.

**Keywords and phrases:** elliptic operators, wave equation, limiting amplitude principle, limiting absorption principle

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## 1. Introduction

Let $L$ be a linear, densely defined, closed operator in a Hilbert space $H$. Our results and techniques are valid in a Banach space also, but we wish to think about $L$ as of a Schrödinger-type operator in a Hilbert space and, at times, think that $L$ is selfadjoint. For a Schrödinger operator $L = -\nabla^2 + q(x)$ the resolvent $(L - k^2)^{-1}, Imk > 0$, is an integral operator with a kernel $G(x, y, k)$ , its resolvent kernel. If $q$ is a real-valued function, sufficiently rapidly decaying then $L$ is selfadjoint, $G(x, y, k)$ is analytic with respect to $k$ in the half-plane $\mathrm{Im}k > 0$, except, possibly, for a finitely many simple poles $ik_j$, $k_j > 0$, the

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semiaxis $k \geq 0$ is filled with the points of absolutely continuous spectrum of $L$, and there exists a limit

$$\lim_{\epsilon \to 0} G(x, y, k + i\epsilon) = G(x, y, k)$$

for all $k > 0$.

Sufficient conditions for $k^2 = 0$ not to be an eigenvalue of $L$ are found in papers [5], [6]. Spectral analysis of the Schrödinger operators is presented in many books (see, for example, [2] and [11]). In papers [3], [4], such an analysis was given in a class of domains with infinite boundaries apparently for the first time, see also [8]. In [7] an eigenfunctions expansion theorem was proved for non-selfadjoint Schrödinger operators with exponentially decaying complex-valued potential $q$. The operator $L$ in this paper is not necessarily assumed to be selfadjoint.

In [1] the validity of the limiting amplitude principle for some class of selfadjoint operators $L$ has been established.

This principle says that, as $t \to \infty$, the solution to problem

$$\ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0, \quad \dot{u} = \frac{du}{dt}, \tag{1.1}$$

has the following asymptotics

$$u = e^{-ikt}v + o(1), \quad t \to \infty, \tag{1.2}$$

where $k$ is a real number and $v \in H$ solves the equation

$$Lv - k^2 v = f. \tag{1.3}$$

The $v$ is called *the limiting amplitude*. It turns out that a more natural definition of the limiting amplitude is:

$$v = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(s) e^{iks} ds, \tag{1.4}$$

if this limit exists and solves equation (1.3).

Why is this definition more natural than (1.2)? There are good reasons for this. One of the reasons is: if (1.2) and (1.3) hold, then the limit (1.4) exists and solves equation (1.3). The other reason is: the limit (1.4) may exist and solve equation (1.3) although the limit (1.2) does not exist.

**Example.** *If $u = e^{ikt}v + e^{ik_1 t}v_1$, then the limit* (1.2) *does not exist, while the limit* (1.4) *does exist and is equal to $v$.*

To describe our assumptions and results, some preparation is needed.

Consider the problem

$$\ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f. \tag{1.5}$$

Assuming that $||u(t)|| \leq ce^{at}$, where $c > 0$ stands throughout the paper for various generic constants, and $a \geq 0$ is a constant, one can define the Laplace transform of $u(t)$,

$$\mathcal{U} := \mathcal{U}(p) := \int_0^\infty e^{-pt} u(t) dt, \quad \sigma > a,$$

where $p = \sigma + i\tau$, $\mathrm{Re}p = \sigma$.

Let us take the Laplace transform of (1.1) and of (1.5) to get

$$L\mathcal{U} + p^2 \mathcal{U} = \frac{f}{p + ik}, \tag{1.6}$$

and

$$L\mathcal{W} + p^2 \mathcal{W} = f, \tag{1.7}$$

where

$$\mathcal{W} = \mathcal{W}(p) = \int_0^\infty w(t)e^{-pt}dt.$$

We also denote $\mathcal{W}(p) := \bar{w}(t)$.

The complex plane $p$ is related to the complex plane $k$ by the formula

$$p = -ik, \quad k = k_1 + ik_2, \quad k_2 \geq 0, \quad \sigma = k_2 \geq 0. \tag{1.8}$$

We assume throughout that $f$ is generic in the following sense:

*If $I$ is the identity operator and a point $p$ is a pole of the kernel of the operator $(L + p^2 I)^{-1}$, then it is a pole of the same order of the element $(L + p^2 I)^{-1} f = \mathcal{W}$.*

If $k^2$ is an eigenvalue of $L$ and $\operatorname{Re} k^2 < 0$, then $\operatorname{Im} k > 0$, where $k = |k|e^{\frac{i \arg k^2}{2}}$, $p = -ik$, so $\sigma = \operatorname{Re} p > 0$. Let $k > 0$ and assume that $-k^2 < 0$ is an eigenvalue of $L$. Then $ik$ is a pole of the resolvent kernel $G(x, y, k)$, and $p = -i(ik) = k$ is a pole of the kernel of the operator $(L + p^2 I)^{-1}$. If $k^2 > 0$ is an eigenvalue of $L$, then $p = -ik$ is a pole of the operator $(L + p^2 I)^{-1}$.

The following known facts from the theory of Laplace transform will be used.

**Proposition 1.1.** *An analytic in the half-plane $\sigma > \sigma_0 \geq 0$ function $F(p)$ is the Laplace transform of a function $f(t)$, such that $f(t) = 0$ for $t < 0$ and*

$$\int_0^\infty |f(t)|^2 e^{-2\sigma_0 t} dt < \infty \tag{1.9}$$

*if and only if*

$$\sup_{\sigma > \sigma_0} \int_{-\infty}^\infty |F(\sigma + i\tau)|^2 d\tau < \infty. \tag{1.10}$$

**Proposition 1.2.** *If $F(p) = \overline{f(t)}$, then*

$$\frac{F(p)}{p} = \overline{\int_0^t f(s)ds}. \tag{1.11}$$

Let us now formulate the main Assumptions A and B standing throughout this paper.

**Assumption A.** For a generic $f$ the $\mathcal{W}(p) = (L + p^2)^{-1} f$ is analytic in the half-plane $\sigma > 0$, except, possibly, at a finitely many simple poles at the points $-ik_j$, $1 \leq j \leq J$, $k_j$ are real numbers, and at the points $\kappa_m$, $\operatorname{Re} \kappa_m > 0$,

$$\mathcal{W}(p) = \sum_{j=1}^J \frac{v_j}{p + ik_j} + \mathcal{W}_1(p) + \sum_{m=1}^M \frac{b_m}{p - \kappa_m}, \tag{1.12}$$

where $v_j$ and $b_m$ are some elements of $H$, $\mathcal{W}_1(p)$ is an analytic function in the half-plane $\operatorname{Re} p = \sigma > 0$, continuous up to the imaginary axis $\sigma = 0$, and satisfying the following estimate

$$||\mathcal{W}_1(p)|| \leq \frac{c}{1 + |p|^\gamma}, \quad \gamma > \frac{1}{2}. \tag{1.13}$$

**Assumption B.** There exists the limit

$$\lim_{\sigma \to 0} ||\mathcal{W}_1(\sigma - ik) - \mathcal{W}_1(-ik)|| = 0 \tag{1.14}$$

for all real numbers $k$.

**Theorem 1.3.** *Let the Assumption A hold. Then a necessary and sufficient condition for the operator $L$ to have no eigenvalues in the half-plane* $\operatorname{Re} k^2 < 0$ *is the validity of the estimate*

$$\left|\left|\int_0^t w(s)ds\right|\right| = O(e^{\epsilon t}), \quad t \to \infty, \tag{1.15}$$

*for an arbitrary small $\epsilon > 0$.*

*A necessary and sufficient condition for the operator $L$ not to have any positive eigenvalues $k^2 > 0$ is the validity of the estimate*

$$\left|\left|\frac{1}{t}\int_0^t e^{iks} w(s)ds\right|\right| = o(1), \quad t \to \infty, \quad \forall k \in \mathbb{R}. \tag{1.16}$$

*A point $ik_0 > 0, k_0 > 0$, is not a pole of the resolvent kernel of the operator* $(L - k^2 - i0)^{-1}$ *if and only if estimate* (1.16) *holds with $k = k_0 > 0$.*

**Remark.** If condition (1.16) holds for $k = 0$, then $|| \int_0^t w(s)ds|| = o(t)$, so condition (1.15) holds, and the operator $L$ has no eigenvalues in the half-plane $\operatorname{Re} k^2 < 0$.

**Theorem 1.4.** *Let the Assumptions A and B hold. Suppose that estimates* (1.14) *and* (1.15) *hold. Then the limiting amplitude principle* (1.4) *holds for every $k \in \mathbb{R}$, $k \neq k_j, 1 \leq j \leq J$.*

In section 2, proofs are given.

## 2. Proofs

### 2.1. Proof of Theorem 1.3

From the Assumption A and Proposition 1.1, it follows that $\mathcal{W}(p)$ is a Laplace transform of a function $w(t)$ such that

$$w(t) = \sum_{j=1}^{J} v_j e^{-ik_j t} + \sum_{m=1}^{M} b_m e^{\kappa_m t} + w_1(t), \tag{2.1}$$

where

$$w_1(t) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{pt}\mathcal{W}_1(p)dp, \tag{2.2}$$

and the integral in (2.2) converges in $L^2$-sense due to the assumption (1.13). It is clear from formula (2.1) that all $b_m = 0$ if and only if estimate (1.15) holds with $0 < \epsilon < \min_{1\leq m\leq M} \operatorname{Re} \kappa_m$. This proves the first conclusion of Theorem 1.3.

Let us calculate the expression on the left side of formula (1.16) and show that this expression is $o(1)$ unless $k = k_j$ for some $1 \leq j \leq J$. In this calculation it is assumed that $L$ does not have any eigenvalues in the half-plane $\operatorname{Re} k^2 < 0$, in other words, that all $b_m = 0$. Otherwise the expression on the left of formula (1.16) tends to infinity as $t \to \infty$ at an exponential rate.

If all $b_m = 0$ in (2.1), then

$$\sum_{j=1}^{J} v_j \frac{1}{t}\int_0^t e^{i(k-k_j)t}dt + \frac{1}{t}\int_0^t w_1(t)e^{ikt}dt := I_1 + I_2. \tag{2.3}$$

If $k$ and $k_j$ are real numbers, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{i(k-k_j)t}dt = \begin{cases} 1, & k = k_j, \\ 0, & k \neq k_j. \end{cases} \tag{2.4}$$

Thus, $I_1 = 0$ if and only if $k$ does not coincide with any of $k_j$, $1 \le j \le J$.

Let us prove that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t w_1(t)e^{ikt}dt = 0. \tag{2.5}$$

By proposition (1.2) and the Mellin inversion formula, one has

$$I := \frac{1}{t}\int_0^t w_1(t)e^{ikt}dt = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{W}_1(p-ik)\frac{e^{pt}}{pt}dp, \tag{2.6}$$

where $\text{Re}p = \sigma > 0$ can be chosen arbitrarily small.

Let $pt = q$, take $\sigma = \frac{1}{t}$, write $q = 1 + is$, and write the integral on the right side of (2.6) as:

$$I = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \mathcal{W}_1(\frac{q}{t} - ik)\frac{q}{t}\frac{e^q}{q^2}dq. \tag{2.7}$$

If one uses estimate (1.13) and formula $|q| = (1+s^2)^{1/2}$, then one obtains the following inequality

$$||I|| \le \frac{1}{2\pi t}\int_{-\infty}^{\infty} \frac{1}{(1+s^2)^{1/2}}\frac{cds}{[1+|\frac{1+is}{t} - ik|^\gamma]} = \frac{c}{2\pi t^{1-\gamma}}\int_{-\infty}^{\infty}\frac{1}{(1+s^2)^{1/2}}\frac{ds}{(t^\gamma + [1+(s-kt)^2]^{\gamma/2})}. \tag{2.8}$$

Let $s = ty$. Then the integral on the right side of (2.8) can be written as

$$\begin{aligned} &\frac{ct}{2\pi t^{1-\gamma}}\int_{-\infty}^{\infty}\frac{dy}{(1+t^2y^2)^{1/2}}\frac{1}{(t^\gamma + [1+t^2(y-k)^2]^{\gamma/2})} \\ &= \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{(1+t^2y^2)^{1/2}}\frac{1}{(1+[t^{-2}+(y-k)^2]^{\gamma/2})} \\ &\le \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{(1+t^2y^2)^{1/2}}\frac{1}{[1+(y-k)^\gamma]} \to 0, \text{ as } t\to\infty, \end{aligned} \tag{2.9}$$

and the convergence of the last integral to zero is uniform with respect to $k \in \mathbb{R}$.

Thus

$$\lim_{t\to\infty} ||I|| = 0. \tag{2.10}$$

From (2.3)-(2.5) the last two conclusions of Theorem 1.3 follow. Theorem 1.3 is proved. □

## 2.2. Proof of Theorem 1.4

Using Proposition 1.2, equation (1.6), and the Mellin formula, one gets

$$\frac{1}{t}\int_0^t u(t)e^{ikt}dt = \frac{1}{t}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\mathcal{U}(p-ik)}{p}e^{pt}dp, \tag{2.11}$$

where, according to (1.6),

$$\mathcal{U}(p-ik) = \frac{\mathcal{W}(p-ik)}{p}. \tag{2.12}$$

Let $\sigma = \frac{1}{t}$ and $pt = q$. Then

$$\frac{1}{t}\int_0^t u(t)e^{ikt}dt = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\mathcal{W}\left(\frac{q}{t} - ik\right)\frac{e^q}{q^2}dq. \tag{2.13}$$

Estimate (1.15) and Theorem 1.3 imply that all $b_m = 0$ in formula (2.1). Therefore, using formula (2.1) with $b_m = 0$, one gets

$$\mathcal{W} = \sum_{j=1}^{J} v_j\frac{1}{p+ik_j} + \mathcal{W}_1,$$

and

$$\mathcal{W}\left(\frac{q}{t}-ik\right)=\mathcal{W}_1\left(\frac{q}{t}-ik\right)+\sum_{j=1}^{J}v_j\frac{1}{\frac{q}{t}-i(k-k_j)}. \tag{2.14}$$

One has $\overline{t^n}=\frac{n!}{p^{n+1}}$. Therefore

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^q}{q^2}dq=1,$$

and

$$\lim_{t\to\infty}\frac{v_j}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{1}{\frac{q}{t}-i(k-k_j)}\frac{e^q}{q^2}dq=\begin{cases}\frac{iv_j}{k-k_j}, & k\neq k_j,\\ \infty, & k=k_j.\end{cases} \tag{2.15}$$

Furthermore,

$$\lim_{t\to\infty}\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\mathcal{W}_1\left(\frac{q}{t}-ik\right)\frac{e^q}{q^2}dq=\mathcal{W}_1(-ik), \tag{2.16}$$

as follows from assumption (1.14) and the Lebesgue's dominated convergence theorem if one passes to the limit $t\to\infty$ under the sign of the integral (2.16). Let us check that this $v$ solves equation (1.3). This would conclude the proof of Theorem 1.4. We need a lemma.

**Lemma 2.1.** *If $h\in L^1_{loc}(0,\infty)$ and the limit $\lim_{t\to\infty}t^{-1}\int_0^t h(s)ds$ exists, then the limit $\lim_{p\to 0}p\int_0^\infty e^{-pt}h(t)dt$ exists, and*

$$\lim_{t\to\infty}t^{-1}\int_0^t h(s)ds=\lim_{p\to 0}p\int_0^\infty e^{-pt}h(t)dt. \tag{2.17}$$

**Proof of Lemma 1**. One has

$$p\int_0^\infty e^{-pt}h(t)dt=pe^{-pt}\int_0^t h(s)ds|_0^\infty+p^2\int_0^\infty te^{-pt}t^{-1}\int_0^t h(s)dsdt.$$

For any $p>0$ one has

$$pe^{-pt}\int_0^t h(s)ds|_0^\infty=0.$$

Let $q=pt$ and denote $H(t):=t^{-1}\int_0^t h(s)ds$, $J:=\lim_{t\to\infty}H(t)$. Then

$$\lim_{p\to 0}p^2\int_0^\infty te^{-pt}t^{-1}\int_0^t h(s)dsdt=\lim_{p\to 0}\int_0^\infty qe^{-q}H(qp^{-1})dq.$$

Passing in the last integral to the limit $p\to 0$ one obtains (2.17). Lemma 1 is proved. □

Using equation (2.17), one writes $v=\lim_{p\to 0}p\mathcal{U}(p-ik)$, where $\mathcal{U}$ solves equation (1.6). Thus,

$$L\mathcal{U}(p-ik)+(p-ik)^2\mathcal{U}(p-ik)=p^{-1}f.$$

Multiplying both sides of this equation by $p$ and passing to the limit $p\to 0$, one obtains equation (1.3). In the passage to the limit under the sign of the unbounded operator $L$ the assumption that $L$ is closed was used.

Thus, the conclusion of Theorem 1.4 follows. □

If the limit (1.14) exists at a point $p=i\tau$ then one says that the limiting absorption principle holds for the operator $L$ at the point $k=ip=i(-ik)=k, k>0$.

Thus, Assumption B means that the limiting absorption principle holds for $L$ at the point $k>0$, that is, $\lim_{\epsilon\to 0}(L-k^2-i\epsilon)^{-1}f$ exists.

## 3. Applications

Let $L = -\nabla^2 + q(x)$, where $q(x)$ is a real-valued function, $|q(x)| \leq c(1+|x|)^{-2-\epsilon}, \epsilon > 0, x \in \mathbb{R}^3$. Then $L$ is selfadjoint on the domain $H^2(\mathbb{R}^3)$. Its resolvent $(L-k^2-i0)^{-1}$ satisfies Assumptions A and B if one keeps in mind the following.

Let $G(x,y,k)$ be the resolvent kernel of $L$, that is, the kernel of the operator $(L-k^2-i0)^{-1}$,

$$LG(x,y,k) = -\delta(x-y) \quad in \quad \mathbb{R}^3,$$

$G \in L^2(\mathbb{R}^3)$ for $\operatorname{Im} k > 0$. If $f \in L^2(\mathbb{R}^3)$ is compactly supported, then for $k > 0$ the function

$$v(x) := (L-k^2-i0)^{-1} f = \int_{\mathbb{R}^3} G(x,y,k) f(y) dy$$

does not necessarily belong to $L^2(\mathbb{R}^3)$.

For example, if $q(x) = 0$, then $G(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, and the function

$$v(x,k) = \int_{|y|\leq 1} g(x,y,k) dy = O\left(\frac{1}{|x|}\right) \tag{3.1}$$

does not belong to $L^2(\mathbb{R}^3)$ (except for those $k >$ for which $x(x,k) = 0$ in the region $|y| \geq 1$. These numbers $k > 0$ are the zeros of the Fourier transform of the characteristic function of the ball $|y| \leq 1$, see [10], Chapter 11.

By this reason the abstract results of theorem (1.3) and (1.4) can be used in applications if one defines some subspace of $H$, for example, a subspace of functions with compact support, denote by $\mathcal{P}$, a projection operator on this subspace, and replaces $\mathcal{W}$ and $\mathcal{W}_1$ by $\mathcal{PW}$ and $\mathcal{PW}_1$ in equations (1.12) and (1.14). For example, the function (3.1) one replaces by $\eta(x)v(x,k)$, where $\eta(x)$ is a characteristic function of a compact subset of $\mathbb{R}^3$.

The analytic properties of $\eta(x)v(x,k)$ and of $v(x,k)$ as functions of $k$ are the same. A similar suggestion is used in [1].

With the above in mind, one knows (for example, from [2] or [11]) that Assumptions A and B hold for $L = -\nabla^2 + q(x)$.

Consequently, the conclusions of Theorems 1.3 and 1.4 hold.

In addition, the assumptions

$$|q(x)| \leq c(1+|x|)^{-2-\epsilon}, \quad \epsilon > 0, \quad \operatorname{Im} q = 0,$$

imply that $L$ does not have positive eigenvalues, so all $v_j = 0$, and zero is not an eigenvalue of $L \geq 0$ if $\epsilon > 0$ (see [5], [6]).

A new method for estimating of large time behavior of solutions to abstract evolution problems is developed in [9], where some applications of this method are given.

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