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A Remark on the Hull of a Multi-Dimensional Limit-Periodic Potential

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Abstract. We discuss the hull of a multi-dimensional limit-periodic potential and show that such a hull is an inverse limit of product cyclic groups. We present the result in an explicit way, which will be useful for a future study of multi-dimensional limit-periodic Schrödinger operators.

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1. Introduction

When investigating spectral properties of almost-periodic Schrödinger operators, mathematicians would like to write them as the following:

$$(H_\omega u)(n) = \sum_{|m-n|=1} u(m) + V_\omega(n)u(n), \tag{1.1}$$

where

$$V_\omega(n) = f(T^n(\omega)), \quad \omega \in \Omega,\ n \in \mathbb{Z}^d \tag{1.2}$$

with a $\mathbb{Z}^d$ action by translations T and a continuous sampling function $f : \Omega \to \mathbb{R}$. If $\Omega = \mathcal{T}^d$ (the multiplicative group of all d-dimensional complex vectors with entries of norm 1), then $\{V_\omega(n)\}_{n\in\mathbb{Z}^d}$ is quasi-periodic and H_ω is called a quasi-periodic Schrödinger operator. If Ω is a Cantor group with minimal translations, $\{V_\omega(n)\}_{n\in\mathbb{Z}^d}$ is limit-periodic and H_ω is called a limit-periodic Schrödinger operator. Conversely, given an almost-periodic Schrödinger operator first, we can just take Ω as the hull of $\{V(n)\}_{n\in\mathbb{Z}^d}$. For a limit-periodic potential V [1], the hull is a Cantor group with minimal translations. By this way, one can separate base dynamics and sampling function, so that it becomes easy and natural to answer questions of the type *how often does phenomenon X occur?* [1, 3–6] presented many spectral properties of limit-periodic Schrödinger operators under this framework. Because of usefulness of the framework, it becomes necessary to describe group structure of the hull of a limit-periodic potential in detail for a future study of the limit-periodic Schrödinger operators (even though we already know that it is a Cantor group). [9] studied the hull of a one-dimensional limit-periodic potential, showing that

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[1] Though V is simply an element of $\ell^\infty(\mathbb{Z}^d)$, we call V a potential throughout the paper.

the hull is isomorphic to the inverse limit of a sequence of cyclic groups, that is, a procyclic group. This paper will generalize to the multi-dimensional case and the main result in this paper is Theorem 2.8, stating that the hull of a multi-dimensional limit-periodic potential is isomorphic to the inverse limit of a sequence of product cyclic groups.

2. Preliminaries

Before stating the main result, let us introduce some preliminary facts. It is well known that there is a close connection between the hulls of limit-periodic potentials in $\ell^\infty(\mathbb{Z}^d)$ and Cantor groups which admit a minimal $\mathbb{Z}^d$ action by translations. For $d = 1$, this was worked out in detail in [1, Section 2]; for $d > 1$, this was worked out in [6, Section 2]. We rewrite some old definitions and results for the reader's convenience.

Definition 2.1. (a) We say that Ω is a *Cantor group* if it is an infinite, totally disconnected, metrizable, compact Abelian group. We fix a metric dist on Ω that is compatible with the topology.

(b) Consider a Cantor group Ω and a $\mathbb{Z}^d$ action by translations, $\{T^n\}_{n\in\mathbb{Z}^d}$. That is, there are $\alpha_1, \dots, \alpha_d \in \Omega$ such that for $\omega \in \Omega$, we have

$$T^n\omega = \omega + \sum_{j=1}^{d} n_j\alpha_j, \tag{2.1}$$

where we write the group operation as +.[2] We say that the action is *minimal* if all orbits are dense, that is, for each $\omega \in \Omega$, we have $\overline{\{T^n\omega : n \in \mathbb{Z}^d\}} = \Omega$.

Definition 2.2. Let $d \in \mathbb{Z}_+$. The group $\mathbb{Z}^d$ acts on $\ell^\infty(\mathbb{Z}^d)$ as $(S_mV)(n) = V(n-m)$ for $n, m \in \mathbb{Z}^d$ and $V \in \ell^\infty(\mathbb{Z}^d)$. The set $\mathrm{orb}(V) = \{S_mV : m \in \mathbb{Z}^d\}$ is called the *orbit* of V and the closure of its orbit is called its *hull*, that is, $\mathrm{hull}(V) = \overline{\mathrm{orb}(V)}$. An element V of $\ell^\infty(\mathbb{Z}^d)$ is called *periodic* if its orbit is finite. It is called *limit-periodic* if it is not periodic and belongs to the closure of the set of periodic elements of $\ell^\infty(\mathbb{Z}^d)$.

V is periodic in the sense of Definition 2.2 if and only if it is periodic in each direction, that is, there are $p_1, \dots, p_d \in \mathbb{Z}_+$ such that for all $n = (n_1, \dots, n_d)$, $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, we have $V(n_1 + k_1p_1, \dots, n_d + k_dp_d) = V(n_1, \dots, n_d)$. We will call $p = (p_1, \dots, p_d) \in (\mathbb{Z}_+)^d$ a *periodicity vector* of V. There exists a smallest periodicity vector for a periodic V. The following proposition describes how limit-periodic potentials in $\ell^\infty(\mathbb{Z}^d)$ may be generated.

Proposition 2.3. *Suppose Ω is a Cantor group that admits a minimal $\mathbb{Z}^d$ action by translations, $\{T^n\}_{n\in\mathbb{Z}^d}$. Then, for every $f \in C(\Omega, \mathbb{R})$ and every $\omega \in \Omega$, the potential V_ω of $\ell^\infty(\mathbb{Z}^d)$ defined by $V_\omega(n) = f(T^n\omega)$ is limit-periodic. Moreover, for each $\omega \in \Omega$, we have* $\mathrm{hull}(V_\omega) = \{V_{\tilde{\omega}} : \tilde{\omega} \in \Omega\}$.

We first prove the following simple lemma:

Lemma 2.4. *Suppose that $\{T^n\}_{n\in\mathbb{Z}^d}$ is an action by translations as in* (2.1) *on the compact Abelian group Ω. Then, for each $j \in \{1, \dots, d\}$, there is a sequence $\{n_k^{(j)}\}_{k\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ such that $\lim_{k\to\infty} n_k^{(j)}\alpha_j = \omega_e$, the identity element of Ω.*

Proof. Let us fix j and explain how to find $\{n_k^{(j)}\}_{k\in\mathbb{Z}_+} \subset \mathbb{Z}_+$. Since Ω is compact, there exists an increasing sequence of positive integers $m_k \to \infty$ such that $m_k\alpha_j$ converges to some $\omega \in \Omega$ as $k \to \infty$. For each k, choose $\tilde{m}_k \in \{m_{k+\ell} : \ell \geq 1\}$ such that $n_k^{(j)} := \tilde{m}_k - m_k \geq k$. Then, $n_k^{(j)} \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} n_k^{(j)}\alpha_j = \omega - \omega = \omega_e$, as desired. ☐

[2]While + is a natural way to denote the group operation in the abstract setting, for the concrete groups that arise as hulls of limit-periodic elements of $\ell^\infty(\mathbb{Z}^d)$, this is ambiguous. Thus, in the concrete setting, we will prefer to use $\cdot$ to denote the group operation.

Proof of Proposition 2.3. For a given $\varepsilon > 0$, we may choose a compact open neighborhood U of the identity $\omega_e \in \Omega$ that is small enough so that $|f(\omega + \omega_U) - f(\omega)| < \varepsilon$ for every $\omega_U \in U$ and every $\omega \in \Omega$.

Since U is compact and open, we can choose $\delta > 0$ such that $\mathrm{dist}(\omega_U, \omega_{\Omega \setminus U}) > \delta$ for every $\omega_U \in U$ and every $\omega_{\Omega\setminus U} \in \Omega \setminus U$.

Lemma 2.4 shows that we can choose $p_1, \ldots, p_d \in \mathbb{Z}_+$ such that $d(\omega_e, p_j\alpha_j) < \delta$ for $j = 1, \ldots, d$. By the defining property of δ, it follows that the closure of

$$\left\{ \sum_{j=1}^{d} n_j p_j \alpha_j : n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \right\}$$

is a compact subgroup of Ω that is contained in U. Its index is bounded by $\prod p_j$.

Now, given $f \in C(\Omega, \mathbb{R})$ and $\omega \in \Omega$, we consider the potential V_ω of $\ell^\infty(\mathbb{Z}^d)$ defined by $V_\omega(n) = f(T^n\omega)$. With the arbitrary choice of $\varepsilon > 0$ above and the resulting U and $\delta > 0$, we consider the following potential V_ω^p of $\ell^\infty(\mathbb{Z}^d)$, $V_\omega^p(n) = f(T^{\tilde{n}}\omega)$, where $\tilde{n} = (\tilde{n}_1, \ldots, \tilde{n}_d)$ is defined by $\tilde{n}_j \in \{0, \ldots, p_j - 1\}$ and $\tilde{n}_j \equiv n_j \mod p_j$. Thus, V_ω^p is periodic. We have

$$\begin{aligned}
\|V_\omega - V_\omega^p\|_\infty &= \sup_{n\in\mathbb{Z}^d} |V_\omega(n) - V_\omega^p(n)| \\
&= \sup_{n\in\mathbb{Z}^d} |f(T^n\omega) - f(T^{\tilde{n}}\omega)| \\
&= \sup_{n\in\mathbb{Z}^d} |f(T^{\tilde{n}}\omega + (T^n\omega - T^{\tilde{n}}\omega)) - f(T^{\tilde{n}}\omega)| \\
&< \varepsilon.
\end{aligned}$$

The first three steps follow by simple rewriting, and the final step follows from the choice of U and the fact that, by construction, $T^n\omega - T^{\tilde{n}}\omega$ belongs to U. This shows that V_ω is limit-periodic since $\varepsilon > 0$ is arbitrary and V_ω^p is periodic.

The statement $\mathrm{hull}(V_\omega) = \{V_{\tilde{\omega}} : \tilde{\omega} \in \Omega\}$ follows since both sides are compact and contain $\mathrm{orb}(V_\omega)$ as a dense subset (for the right-hand side, this is a consequence of the minimality of the action). This completes the proof of the proposition. □

Thus, we have seen that a Cantor group that admits a minimal $\mathbb{Z}^d$ action by translations and a continuous sampling function give rise to limit-periodic potentials of $\ell^2(\mathbb{Z}^d)$. Let us now turn to the converse. That is, given a limit-periodic potential of $\ell^2(\mathbb{Z}^d)$, we want to show that it arises in this way.

Proposition 2.5. *Suppose $V \in \ell^\infty(\mathbb{Z}^d)$ is limit-periodic. Then,* $\mathrm{hull}(V)$ *is compact and it has a unique topological group structure so that V is the identity element and $\mathbb{Z}^d \to \mathrm{hull}(V)$, $m \mapsto S_mV$ is a homomorphism. Moreover, the group structure is Abelian and there exist arbitrarily small compact open neighborhoods of V in* $\mathrm{hull}(V)$ *that are finite index subgroups, and* $\mathrm{hull}(V)$ *admits a minimal $\mathbb{Z}^d$ action by translations.*

Proof. Since V is limit-periodic, we can find for each $\varepsilon > 0$, a periodic V_p with $\|V - V_p\|_\infty < \varepsilon$. Since $\mathrm{orb}(V_p)$ is finite, it follows that $\mathrm{orb}(V)$ is contained in the ε-neighborhood of a finite set. That is, $\mathrm{orb}(V)$ is totally bounded and hence its closure $\mathrm{hull}(V)$ is compact.

Obviously, there is a unique group structure on $\mathrm{orb}(V)$ such that $\mathbb{Z}^d \to \mathrm{orb}(V)$, $m \mapsto S_mV$ is a homomorphism. Our goal is to show that it extends uniquely to a group structure on $\mathrm{hull}(V)$. It suffices to show uniform continuity of the group structure on $\mathrm{orb}(V)$. This will then also show that the resulting extension of the group structure to $\mathrm{hull}(V)$ is Abelian. We have

$$\begin{aligned}
\|S_{m_1+k_1}V - S_{m_2+k_2}V\|_\infty &= \|S_{m_1-m_2}V - S_{k_2-k_1}V\|_\infty \\
&\le \|S_{m_1-m_2}V - V\|_\infty + \|V - S_{k_2-k_1}V\|_\infty \\
&= \|S_{m_1}V - S_{m_2}V\|_\infty + \|S_{k_1}V - S_{k_2}V\|_\infty.
\end{aligned}$$

Here, the first and the third step follow since translations are isometries and the second step follows from the triangle inequality. Put differently, if $a, b, c, d \in \mathrm{orb}(V)$ and we denote the group operation by $\cdot$, then $\|a \cdot b - c \cdot d\|_\infty \leq \|a - c\|_\infty + \|b - d\|_\infty$, which shows the desired uniform continuity.

To prove the last statement about finite index subgroups in small neighborhoods of the identity, let $\varepsilon > 0$ be given. Choose a periodic $V_p \in \ell^\infty(\mathbb{Z}^d)$ with $\|V - V_p\|_\infty < \frac{\varepsilon}{2}$. Also, there are $p_1, \dots, p_d \in \mathbb{Z}_+$ such that for all $n = (n_1, \dots, n_d), k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, we have $V_p(n_1 + k_1 p_1, \dots, n_d + k_d p_d) = V_p(n_1, \dots, n_d)$. In other words, V_p is invariant under S_m for every $m \in (p_1\mathbb{Z}) \times \cdots \times (p_d\mathbb{Z})$. Clearly, the closure of $\{S_m V : m \in (p_1\mathbb{Z}) \times \cdots \times (p_d\mathbb{Z})\}$, which we denote by $\mathrm{hull}_p(V)$, is a compact subgroup of $\mathrm{hull}(V)$ of index at most $\prod p_j$. Since $\mathrm{hull}(V)$ is the union of finitely many disjoint translates of $\mathrm{hull}_p(V)$, it follows that $\mathrm{hull}_p(V)$ is also open. By the invariance property of V_p, $\mathrm{hull}_p(V)$ is contained in the $\frac{\varepsilon}{2}$-ball around V_p, and hence it is contained in the ε-ball around V.

Last, the minimal $\mathbb{Z}^d$ action is given by $T^n = S_n$ with the translations S_n introduced above. Note that this action is indeed an action by translations in the sense of Definition 2.1, simply choosing $\alpha_j = T^{(0,\cdots,1,\cdots,0)}(V)$ with the j-th component being 1. Let us show that this action is minimal. It suffices to show that for $\omega_1, \omega_2 \in \mathrm{hull}(V)$ and $\varepsilon > 0$, there is $n \in \mathbb{Z}^d$ such that $\mathrm{dist}(T^n\omega_1, \omega_2) = \|T^n\omega_1 - \omega_2\|_\infty < \varepsilon$. We can choose $n_1, n_2 \in \mathbb{Z}^d$ such that $\|\omega_j - T^{n_j}V\|_\infty < \frac{\varepsilon}{2}$, $j = 1, 2$. Now set $n := n_2 - n_1$. Putting everything together and using that T is an isometry, we find

$$\begin{aligned}\|T^n\omega_1 - \omega_2\|_\infty &\leq \|T^n\omega_1 - T^{n+n_1}V\|_\infty + \|T^{n+n_1}V - \omega_2\|_\infty \\ &= \|\omega_1 - T^{n_1}V\|_\infty + \|T^{n_2}V - \omega_2\|_\infty \\ &< \varepsilon.\end{aligned}$$

□

Next we will rewrite some results from [2, Appendix 1] in the d-dimensional context, introducing the frequency module for $\mathrm{hull}(V)$. Denote $\mathrm{hull}(V)$ by Ω_V. $\hat{\Omega}_V$, of characters on Ω_V, is naturally a topological subgroup of $\mathcal{T}^d$. By taking inverse image from $[0,1]^d$ to $\mathcal{T}^d$ under the map $(\alpha_1, \cdots, \alpha_d) \to (e^{2i\pi\alpha_1}, \cdots, e^{2i\pi\alpha_d})$, we obtain $\hat{\Omega}_V$ as a subgroup of $[0,1]^d$, called the *frequency module* of V. $\hat{\Omega}_V$ is countable since Ω_V has a countable dense set. Since $\hat{\Omega}_V$ is a subgroup of $[0,1]^d$, given $\alpha, \beta \in \hat{\Omega}_V$, and integers n_1, n_2, we have that $n_1\alpha + n_2\beta \mod \mathbf{1} \in \hat{\Omega}_V$, i.e. $\hat{\Omega}_V$ is a module over $\mathbb{Z}$ (one should automatically consider $\mod \mathbf{1}$ when discussing $\hat{\Omega}_V$ so that we don't have to write $\mod \mathbf{1}$ every time).

The Peter-Weyl theorem assures us that any V is a uniform limit of finite sums of the form $\sum_{j=1}^k c_j e^{i2\pi\alpha^{(j)}\cdot n}$ with $\alpha^{(j)} \in \hat{\Omega}_V$. From this it follows that

Proposition 2.6. *The frequency module, $\hat{\Omega}_V$, is the module generated by*

$$\left\{\alpha : \lim_{k\to\infty} \frac{1}{(2k)^d} \sum_{n\in[-k,k]^d} V(n)e^{-i2\pi n\cdot\alpha} \neq 0, \alpha \in [0,1]^d\right\}.$$

For $\alpha, \beta \in [0,1]^d$, we say that they have a *common divisor* $\gamma \in [0,1]^d$ if there exist $n, m \in \mathbb{Z}^d$ such that for $1 \leq j \leq d$, $n_j\gamma_j = \alpha_j$ and $m_j\gamma_j = \beta_j$ respectively. Like [2, Theorem A.1.3], we then have

Proposition 2.7. *V is limit-periodic if and only if $\hat{\Omega}_V$ has the property that any $\alpha, \beta \in \hat{\Omega}_V$ have a common divisor in $\hat{\Omega}_V$.*

Proof. For $\alpha, \beta \in \hat{\Omega}_V$, by Proposition 2.6

$$a = \lim_{k\to\infty} \frac{1}{(2k)^d} \sum_{n\in[-k,k]^d} V(n)e^{-i2\pi n\cdot\alpha}$$

and

$$b = \lim_{k\to\infty} \frac{1}{(2k)^d} \sum_{n\in[-k,k]^d} V(n)e^{-i2\pi n\cdot\beta}$$

are both non-zero. Choose a periodic potential $P \in \ell^\infty\mathbb{Z}$ with

$$\|P - V\|_\infty \leq \frac{1}{2}\min(|a|, |b|).$$

It follows that

$$\lim_{k\to\infty} \frac{1}{(2k)^d} \sum_{n\in[-k,k]^d} P(n)e^{-i2\pi n\cdot\gamma} \neq 0$$

for $\gamma = \alpha, \beta$. Let $p = (p_1, \cdots, p_2)$ be the smallest periodicity vector of P. The frequency module of P can be generated by $\{(0, \cdots, 0, 1/p_j, 0, \cdots, 0) : 1 \leq j \leq d\}$ as a $\mathbb{Z}$ module. Since α, β both belong to the frequency module of P, $1/p_j$ must divide both α_j and β_j. So $(1/p_1, \cdots, 1/p_d)$ is a common divisor of α and β. Similarly any finite subset of the generating set $\left\{\alpha : \lim_{k\to\infty} \frac{1}{(2k)^d} \sum_{n\in[-k,k]^d} V(k)e^{-i2\pi n\cdot\alpha} \neq 0\right\}$ have a common divisor. Since we can select the greatest common divisor, the property follows.

Conversely, if $\hat{\Omega}_V$ has the property, any finite sum $\sum_{j=1}^k c_j e^{i2\pi\alpha^{(j)}\cdot n}$ with $\alpha_j \in \hat{\Omega}_V$ is periodic since the $\alpha^{(j)}$ have a common divisor. By the Peter-Weyl theorem, V is limit-periodic. □

Since any finite collection of $\alpha^{(j)}$ have a common divisor in $\hat{\Omega}_V$ and furthermore V is a uniform limit of finite sums of the form $\sum_{j=1}^k c_j e^{i2\pi\alpha^{(j)}\cdot n}$, we can find a sequence of periodic $P^{(k)} \in \ell^\infty(\mathbb{Z}^d)$ which satisfy the following: (i) $\lim_{k\to\infty} P^{(k)} = V$ in ℓ^∞-norm. (ii) Write $p^{(k)}$ as the smallest periodicity vector of $P^{(k)}$. We have $p_i^{(k)} | p_i^{(k+1)}$ for every $1 \leq i \leq d$. (iii) $\left(\frac{1}{p_1^{(k)}}, \cdots, \frac{1}{p_d^{(k)}}\right) \in \hat{\Omega}_V$ for all k.

Write $G_V = \left\{\left(\frac{1}{p_1^{(k)}}, \cdots, \frac{1}{p_d^{(k)}}\right) : k \in \mathbb{Z}_+\right\}$. Clearly, $G_V \subset \hat{\Omega}_V$. From the proof of Proposition 2.7, we can see that for any vector $n \in \mathbb{Z}^d$ and $\alpha \in G_V$, $(n_1\alpha_1, n_2\alpha_2, \cdots, n_d\alpha_d) \in \hat{\Omega}_V$. Moreover, with the common divisor property, $\hat{\Omega}_V$ can be generated by G_V by such a $\mathbb{Z}^d$ action, i.e., entry by entry multiplication. Write $F_V = \left\{\left(p_1^{(k)}, \cdots, p_d^{(k)}\right) : k \in \mathbb{Z}_+\right\}$. We call F_V a *frequency integer vector set* of V. Write $X^{(k)} = \mathbb{Z}_{p_1^{(k)}} \times \cdots \times \mathbb{Z}_{p_d^{(k)}}$, a product cyclic group. Let X be the inverse limit of $\{X^{(k)}\}_{k\in\mathbb{Z}_+}$ (we will introduce the inverse limit concept in the next section). Our main result is the following.

Theorem 2.8. *Ω_V is isomorphic to X.*

3. Inverse limits

A *directed* set is a partially ordered set I such that for all $i_1, i_2 \in I$ there is an element $j \in I$ for which $i_1 \leq j$ and $i_2 \leq j$.

Definition 3.1. An inverse system (X_i, ϕ_{ij}) of topological groups indexed by a directed set I consists of a family $(X_i \mid i \in I)$ of topological groups and a family $(\phi_{ij} : X_j \to X_i \mid i, j \in I, i \leq j)$ of continuous homomorphisms such that ϕ_{ii} is the identity map id_{X_i} for each i and $\phi_{ij}\phi_{jk} = \phi_{ik}$ whenever $i \leq j \leq k$.

Definition 3.2. An inverse limit (X, ϕ_i) of an inverse system (X_i, ϕ_{ij}) of topological groups is a topological group together with a compatible family $(\phi_i : X \to X_i)$ of continuous homomorphisms with the following universal property: whenever $(\varphi_i : Y \to X_i)$ is a compatible family of continuous homomorphisms from a topological group Y, there is a unique continuous homomorphism $\varphi : Y \to X$ such that $\phi_i\varphi = \varphi_i$ for each i.

Proposition 3.3. [10, Proposition 1.1.4] *Let (X_i, ϕ_{ij}) be an inverse system of topological groups, indexed by I.*

(1). *There exists an inverse limit (X, ϕ_i) of (X_i, ϕ_{ij}), for which X is a topological group and the maps ϕ_i are continuous homomorphisms.*

(2). *If* $(X^{(1)}, \phi_i^{(1)})$ *and* $(X^{(2)}, \phi_i^{(2)})$ *are inverse limits of the inverse system, then there is an isomorphism* $\bar{\phi} : X^{(1)} \to X^{(2)}$ *such that* $\phi_i^{(2)}\bar{\phi} = \phi_i^{(1)}$ *for each* i.
(3). *Write* $G = \prod_{i\in I} X_i$ *with the product topology and for each* i *write* π_i *for the projection map from* G *to* X_i. *Define*

$$X = \{c \in G : \phi_{ij}\pi_j(c) = \pi_i(c) \ for\ all\ i, j\ with\ j \geq i\}$$

and $\phi_i = \pi_i|_X$ *for each* i. *Then* (X, ϕ_i) *is an inverse limit of* (X_i, ϕ_{ij}).

The above proposition shows that the inverse limit of an inverse system (X_i, ϕ_{ij}) exists and is unique up to isomorphism. A topological *profinite* group is by definition an inverse limit of finite topological groups. The inverse limit of a sequence of cyclic groups is called a *procyclic* group.

4. Proof of Theorem 2.8

For a d-dimensional limit-periodic potential V, there exist periodic potentials $\{P^{(k)}\}_{k\in\mathbb{Z}_+}$ converging to V uniformly. $P^{(k)}$ has the smallest periodicity vector $p^{(k)} = \left(p_1^{(k)}, \cdots, p_d^{(k)}\right)$. Let's consider hull($V$). hull($V$) has a strongly minimal $\mathbb{Z}^d$ action $T^n = S_n$ with $\alpha_j = T^{(0,\cdots,1,\cdots,0)}(V)$ (the j-th component being 1). $F_V = \left\{\left(p_1^{(k)}, \cdots, p_d^{(k)}\right) : k \in \mathbb{Z}_+\right\}$ is a frequency integer vector set of hull(V).

From $(0, \cdots, 0)$ to $\left(p_1^{(k)} - 1, \cdots, p_d^{(k)} - 1\right)$, there are $h^{(k)} = p_1^{(k)}p_2^{(k)}\cdots p_d^{(k)}$ vectors. Denote these vectors by $\gamma^{(k1)}, \gamma^{(k2)}, \cdots, \gamma^{(kh^{(k)})}$ for writing convenience. Make $\gamma^{(11)} = (0, \cdots, 0)$ and furthermore $\gamma^{(ki)} = \gamma^{(mi)}$ when $k > m$ with neat ordering. $\gamma_j^{(k1)}$ means the j-th entry of the vector $\gamma^{(k1)}$. Write

$$H^{(ki)} = \left\{\sum_{j=1}^{d}\left(\gamma_j^{(ki)} + h_j p_j^{(k)}\right)\alpha_j : h_j \in \mathbb{Z}\right\}$$

and

$$U^{(ki)} = \overline{\left\{\sum_{j=1}^{d}\left(\gamma_j^{(ki)} + h_j p_j^{(k)}\right)\alpha_j : h_j \in \mathbb{Z}\right\}}$$

in the space hull(V) for $1 \leq i \leq h^{(k)}$. Next we will first show that

$$\bigcap_{k=m}^{\infty} U^{(ki)} = \left\{\sum_{j=1}^{d}\gamma_j^{(mi)}\alpha_j\right\}$$

for $1 \leq i \leq h^{(m)}$.

Assume $j = 1$. Like in the proof of Proposition 2.3, given any E, a compact open neighborhood of V, we can choose $\delta > 0$ such that $\text{dist}(\omega_E, \omega_{\text{hull}(V)\setminus E}) > \delta$ for every $\omega_E \in E$ and every $\omega_{\text{hull}(V)\setminus E} \in \text{hull}(V) \setminus E$. Choose $P^{(m)}$ so that $\|V - P^{(m)}\| \leq \delta/2$. We have $\left\|V - T^{\left(0,\cdots,p_j^{(m)},\cdots,0\right)}(V)\right\| = \left\|V - P^{(m)} + T^{\left(0,\cdots,p_j^{(m)},\cdots,0\right)}(P^{(m)}) - T^{(0,\cdots,p_j^{(m)},\cdots,0)}(V)\right\| \leq \delta$ for $j = 1, 2, \cdots, d$. By the defining property of δ, it follows that $p_j^{(m)}\alpha_j$ is contained in E. Then $U^{(m1)}$ is contained in E. So we can conclude that $\bigcap_{k=1}^{\infty} U^{(k1)} = \{V\} = \{\sum_{j=1}^{d}\gamma_j^{(11)}\alpha_j\}$. Note that for $1 \leq i \leq h^{(m)}$, $\gamma^{(ki)} = \gamma^{(mi)}$ when $k \geq m$ and $U^{(k+1)i} \subset U^{(ki)}$ since $p_i^{(k)}|p_i^{(k+1)}$. Similarly, we have

$$\bigcap_{k=m}^{\infty} U^{(ki)} = \left\{\sum_{j=1}^{d}\gamma_j^{(mi)}\alpha_j\right\}. \tag{4.1}$$

Denote $\text{hull}(V)/U^{(k1)}$ by $X^{(k)}$. Let $\phi_{ij} : X^{(j)} \to X^{(i)}$ by $\phi_{ij}(\omega U^{(j1)}) = \omega U^{(i1)}$ for $\omega \in \text{hull}(V)$ and $j > i$. Then $\{X^{(j)}, \phi_{ij}\}$ is an inverse system, and we write the inverse limit as X. We will show that $\text{hull}(V) \cong X$. Let $\psi : \text{hull}(V) \to X$ be the continuous mapping induced by the canonical continuous surjection $\psi_k : \text{hull}(V) \to X^{(k)}$. It is easy to see that ψ is a continuous surjection. To prove that ψ is a homeomorphism, it suffices then to prove that it is an injection, since $\text{hull}(V)$ is compact. Let $\omega_1, \omega_2 \in \text{hull}(V)$. There exists a open neighborhood O of ω_1 that excludes ω_2. Choose k large enough so that for any $\omega \in U^{(k1)}$ we have $\omega\omega_1 \in O$ (this is because $U^{(k1)}$ is a sufficiently small neighborhood of V). Then $\omega_2 \notin \omega_1 U^{(k1)}$, and $\psi_k(\omega_1) \neq \omega_2 U^{(k1)}$. So, $\psi(\omega_1) \neq \psi(\omega_2)$. Thus, ψ is an injection.

The only thing left is to show $\text{hull}(V)/U^{(k1)} \cong Z_{p_1^{(k)}} \times Z_{p_2^{(k)}} \times \cdots \times Z_{p_d^{(k)}}$, where $Z_{p_j^{(k)}}$ is the $p_j^{(k)}$ cyclic group (note that we only need to consider a large k). Clearly, $\text{orb}(V) = \bigcup_{i=1}^{h^{(k)}} H^{(ki)}$. So we have $\text{hull}(V) = \bigcup_{i=1}^{h^{(k)}} U^{(ki)}$. By (4.1) we have $U^{(ki)} \bigcap U^{(kj)} = \emptyset$ when k is large enough since $\text{hull}(V)$ is totally disconnected. So we have

$$\text{hull}(V)/U^{(k1)} = \left\{ \left[\sum_{j=1}^{d} \gamma_j^{(k1)} \alpha_j\right], \left[\sum_{j=1}^{d} \gamma_j^{(k2)} \alpha_j\right], \cdots, \left[\sum_{j=1}^{d} \gamma_j^{(kh^{(k)})} \alpha_j\right] \right\},$$

which has $h^{(k)}$ elements. We can map $\text{hull}(V)/U^{(k1)}$ to $Z_{p_1^{(k)}} \times Z_{p_2^{(k)}} \times \cdots \times Z_{p_d^{(k)}}$ by $\left[\sum_{j=1}^{d} \gamma_j^{(ki)} \alpha_j\right] \to (\gamma_1^{(ki)}, \gamma_2^{(ki)}, \cdots, \gamma_d^{(ki)})$. It is easy to see that such a map is an isomorphic map. The proof is done.

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