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Existence of Reaction-Diffusion Waves with Nonlinear Boundary Conditions

N. Apreutesei[1], A. Tosenberger[2], V. Volpert[2] *
[1] Department of Mathematics, "Gheorghe Asachi" Technical University
Bd. Carol. I, 700506 Iasi, Romania
[2] Institut Camille Jordan, UMR 5208 CNRS, University Lyon 1, 69622 Villeurbanne, France

Abstract. The paper is devoted to a reaction-diffusion equation in an infinite two-dimensional strip with nonlinear boundary conditions. The existence of travelling waves is proved in the bistable case by the Leray-Schauder method. It is based on a topological degree for elliptic problems in unbounded domains and on a priori estimates of solutions.

Keywords and phrases: reaction-diffusion equation, nonlinear boundary condition, travelling wave, Leray-Schauder method

Mathematics Subject Classification: 35K57

1. Formulation of the problem

In this work we consider the reaction-diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v + f(v), \tag{1.1}$$

with nonlinear boundary conditions:

$$y = 0 : \frac{\partial v}{\partial y} = 0, \quad y = 1 : \frac{\partial v}{\partial y} = g(v) \tag{1.2}$$

in the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < 1\}$. Such models arise in various applications including mathematical models of atherosclerosis [2] and other inflammatory diseases. In this case, the variable v corresponds to the concentration of white blood cells in the tissue. The nonlinear boundary condition describes the cell influx through the boundary. This influx depends on cell concentration in the tissue. This self-amplifying mechanism can result in the development of chronic inflammation and spreading of the inflammation in space. In the context of atherosclerosis, domain Ω corresponds to the blood vessel wall (intima) where the disease develops.

We will study the existence of a travelling wave solution of this problem. This is a solution of the form $v(x, y, t) = u(x - ct, y)$. It satisfies the equation

*Corresponding author. E-mail: volpert@math.univ-lyon1.fr

$$\Delta u + c\frac{\partial u}{\partial x} + f(u) = 0 \tag{1.3}$$

with the boundary conditions

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g(u). \tag{1.4}$$

Here c is an unknown constant, the wave speed. Everywhere below we will assume that the functions f and g are continuous together with their third derivatives. In some cases, these conditions can be weakened.

The case where $g(u) \equiv 0$ is well studied in the literature. In particular, it can have a one-dimensional solution, which depends only on the variable x along the axis of the strip. In this case, we obtain the reaction-diffusion equation

$$u'' + cu' + f(u) = 0, \tag{1.5}$$

where prime denotes the derivative with respect to x. Suppose that $f(u_\pm) = 0$ for some u_+ and u_-. Let us recall that the case where $f'(u_\pm) < 0$ is called *bistable*. If one of these two derivatives is negative and another one is positive, then it is a *monostable* case. If there exists a solution of equation (1.5) with the limits $u(\pm\infty) = u_\pm$, then it is unique in the bistable case; in the monstable case, there is a continuous family of solutions. The existence of such solutions is determined by the function $f(u)$ (see [5] and the references therein).

In this work we study problem (1.3), (1.4) with a function g different from zero. We will look for solutions with the limits

$$\lim_{x\to\pm\infty} u(x, y) = u_\pm(y), \quad 0 < y < 1, \tag{1.6}$$

where $u_\pm(y)$ are some functions which satisfy the problem in the cross section:

$$u'' + f(u) = 0, \quad 0 < y < 1, \quad u'(0) = 0, \quad u'(1) = g(u(1)). \tag{1.7}$$

As above, we introduce the bistable and the monostable cases. Consider problem (1.7) linearized about solutions $u_\pm(y)$ and the corresponding eigenvalue problems:

$$v'' + f'(u_\pm(y))v = \lambda v, \quad 0 < y < 1, \quad v'(0) = 0, \quad v'(1) = g'(u_\pm(1))v(1). \tag{1.8}$$

If both of them have all eigenvalues in the left-half plane, then we call it the bistable case. If one of these problems has all eigenvalues in the left-half plane and another one has some eigenvalues in the right-half plane, then it is the monostable case.

Investigation of problem (1.3), (1.4) relies on the properties of the corresponding operators. It will be shown that in the bistable case where the essential spectrum of the corresponding linearized operator lies in the left-half plane, the operator satisfies the Fredholm property. Moreover we can introduce a topological degree. These tools allow us to use various methods to prove the existence of solutions. We will use the Leray-Schauder method based on the topological degree and a priori estimates of solutions. It is a continuation of the previous work [1] where more restrictive conditions on the functions f and g were imposed. It was assumed that they had the same zeros.

Let us note that a reaction-diffusion system of equations with nonlinear boundary conditions suggested as a model of atherosclerosis was studied in [2] in the monostable case. The method of proof is different in this case and it cannot be applied in the bistable case. However we can expect that it is applicable for the scalar equation in the monostable case. The scalar equation with nonlinear boundary condition and with $f(u) \equiv 0$ was considered in [4]. However, behavior of solutions at infinity in [4] was not specified. In this work we study problem (1.3), (1.4) in the bistable case.

2. Solutions in the cross-section

2.1. General case

In this section we will study the problem

$$\frac{d^2w}{dy^2} + f(w) = 0, \quad w'(0) = 0, \;\; w'(L) = g(w(L)) \tag{2.1}$$

in the interval $0 < y < L$. We will suppose here that the functions f and g are continuous together with their first derivatives. We can reduce the second-order equation to the system of two first-order equations

$$w' = p, \quad p' = -f(w),$$

and then to the equation

$$\frac{dp}{dw} = -\frac{f(w)}{p} \; .$$

We can solve this equation analytically. We will consider for simplicity only monotone solutions and denote $w_+ = \max w(y)$, $w_- = \min w(y)$. In the case of decreasing solutions $w_+ = w(0),\; w_- = w(L)$, and the boundary conditions become

$$p(w_+) = 0, \quad p(w_-) = g(w_-)$$

(Figure 1). Under the assumption that

$$\int_w^{w_+} f(u)du \geq 0, \quad w_- \leq w \leq w_+,$$

we obtain

$$p(w) = -\sqrt{2\int_w^{w_+} f(u)du}. \tag{2.2}$$

From the second boundary condition

$$g(w_-) = -\sqrt{2\int_{w_-}^{w_+} f(u)du}. \tag{2.3}$$

Thus, for any given w_+ such that $f(w_+) > 0$, we find w_- as a solution of equation (2.3). Further, we solve the differential equation (2.2), where $p(w) = w'$, and obtain

$$L = \int_{w_-}^{w_+} \frac{dv}{\sqrt{2\int_v^{w_+} f(u)du}} \; .$$

Hence we found the length of the interval as a function of the maximal value of solution. Depending on the functions f and g, solution can exist, it can be unique or non-unique, or it may not exist. The case of increasing solutions can be studied in a similar way. The spectrum of the problem linearized about the solutions can be completely in the left-half plane or it can be partially in the right-half plane.

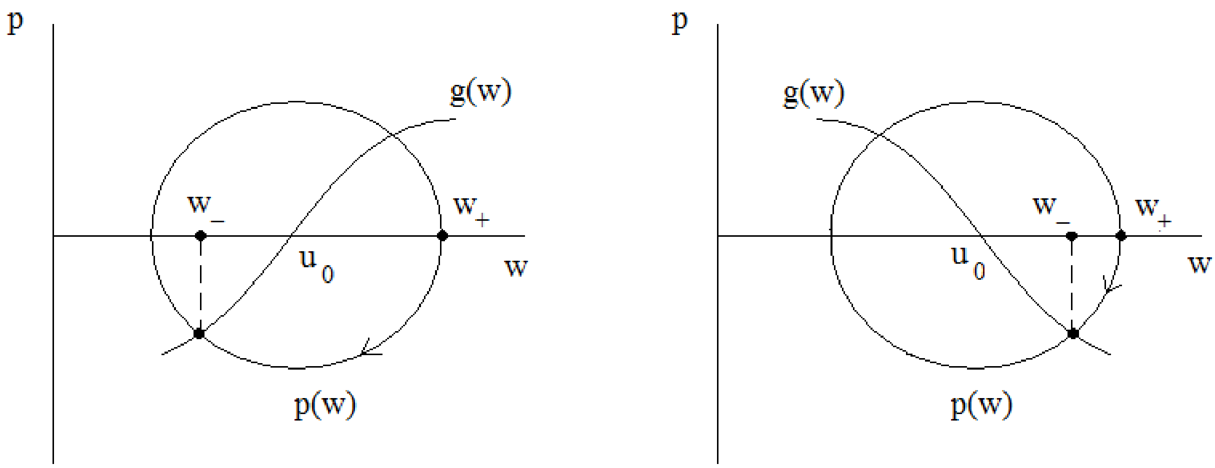

FIGURE 1. Graphical solution of problem (1.7). The function $p(w) = w'(y)$ satisfies the boundary conditions, $p(w_+) = 0, p(w_-) = g(w_-)$. Two examples presented here, with an increasing and a decreasing function g are discussed in the text.

2.2. Constant solutions

2.2.1. Existence

In the next section, when we study the wave existence, we will consider problems which depend on parameters. So we will discuss here problem (2.1) where $g = \delta g_0$ and δ is a positive parameter. Suppose that functions $f(y)$ and $g(y)$ are continuous together with their first derivatives and such that

$$f(u_\pm) = g(u_\pm) = 0, \quad f'(u_\pm) < 0, \; g'(u_\pm) < 0 \tag{2.4}$$

for some u_+ and u_-, and that these functions have a single zero u_0 in the interval $u_+ < u < u_-$,

$$f(u_0) = g(u_0) = 0, \quad f'(u_0) > 0, \; g'(u_0) > 0. \tag{2.5}$$

Lemma 2.1. *Let functions f and g satisfy conditions (2.4), (2.5). Then there exists L_0 such that problem (2.1) with $u_+ < w(0) < u_-$ has only constant solutions for any $L \le L_0$ and any positive δ.*

Proof. The trajectory $p(w)$ corresponding to the solution of this problem is shown schematically in Figure 1 (left). If we take $w(0) = w_+$, then $w_- < u_0$, and the value of L is limited from below. It is similar for the symmetric case where $p(w) > 0$.

Let us note that it is different if $g'(u_0) < 0$ (Figure 1 (right)). The points w_- converges to w_+ as $\delta \to 0$, and L also converges to 0. □

2.2.2. Stability

Let us discuss stability of constant solutions. We begin with the case where $f(u) \equiv 0$. Then from the first boundary condition in (1.7) we obtain $u =$ const, from the second one, $g(u) = 0$. Denote a zero of the function g by u^*. Let us analyze the eigenvalue problem

$$v'' = \lambda v, \;\; v'(0) = 0, \;\; v'(1) = g'(u^*) v(1). \tag{2.6}$$

Since the principal eigenvalue of this problem is real [6] (in fact, they are all real because the problem is self-adjoint), it is sufficient for what follows to consider real λ. It can be easily verified that $\lambda = 0$ is not an eigenvalue of this problem if $g'(u^*) \neq 0$. Let us find conditions when the eigenvalue λ is positive. Set $\mu = \sqrt{\lambda}$ for a positive λ. Then from the equation and the first boundary condition we obtain

$$v(y) = k(e^{\mu y} + e^{-\mu y}).$$

From the second boundary condition it follows that

$$\mu = g'(u^*)\ \frac{e^{\mu} + e^{-\mu}}{e^{\mu} - e^{-\mu}}\ .$$

This equation has a positive solution for $g'(u^*) > 0$, that is for $u^* = u_0$. In this case there is a positive eigenvalue of problem (2.6). All eigenvalues are negative for $u^* = u_\pm$ since $g'(u_\pm) < 0$.

If $f(u)$ is different from zero, then the corresponding eigenvalue problem, instead of (2.6), writes

$$v'' + f'(u^*)v = \lambda v, \ \ v'(0) = 0, \ \ v'(1) = g'(u^*)v(1). \tag{2.7}$$

If $f'(u^*) > 0$, then the principal eigenvalue of this problem is greater than the principal eigenvalue of problem (2.6), and it remains positive. This is the case for $u^* = u_0$. If $u^* = u_\pm$, then the eigenvalues are negative.

3. Property of the operators

3.1. Fredholm property

Consider the operator corresponding to problem (1.3), (1.4) and linearized about a solution $u(x, y)$:

$$Av = \Delta v + c\frac{\partial v}{\partial x} + a(x, y)v, \quad (x, y) \in \Omega, \tag{3.1}$$

$$Bv = \begin{cases} \frac{\partial v}{\partial y} & , \ y = 0 \\ \frac{\partial v}{\partial y} - b(x)v & , \ y = 1 \end{cases}, \tag{3.2}$$

where $\Omega = \{-\infty < x < \infty, \ \ 0 < y < 1\}$, and

$$a(x, y) = f'(u(x, y)), \quad b(x) = g'(u(x, 1)).$$

The operator $L = (A, B)$ acts from the space $E = C^{2+\alpha}(\bar{\Omega})$ into the space $F = C^{\alpha}(\bar{\Omega}) \times C^{1+\alpha}(\partial\Omega)$. Consider the limiting operators

$$A^{\pm}v = \Delta v + c\frac{\partial v}{\partial x} + a_\pm(y)v, \quad (x, y) \in \Omega, \tag{3.3}$$

$$B^{\pm}v = \begin{cases} \frac{\partial v}{\partial y} & , \ y = 0 \\ \frac{\partial v}{\partial y} - b_\pm v & , \ y = 1 \end{cases} \tag{3.4}$$

and the corresponding equations

$$A^{\pm}v = 0, \quad B^{\pm}v = 0. \tag{3.5}$$

Here

$$a_\pm(y) = \lim_{x\to\pm\infty} a(x, y), \quad b_\pm = \lim_{x\to\pm\infty} b(x).$$

Denote by $\tilde{v}(\xi, y)$ the partial Fourier transform of $v(x, y)$ with respect to x. Then from (3.5) we obtain

$$\tilde{v}'' + (-\xi^2 + ci\xi + a_\pm(y))\tilde{v} = 0, \quad 0 < y < 1, \tag{3.6}$$

$$\tilde{v}'(\xi, 0) = 0, \;\; \tilde{v}'(\xi, 1) = b_\pm \tilde{v}(\xi, 1). \tag{3.7}$$

Since we consider the bistable case, then the eigenvalue problem

$$v'' + a_\pm(y)v = \lambda v, \;\; 0 < y < 1, \quad v'(0) = 0, \;\; v'(1) = b_\pm v(1) \tag{3.8}$$

has all eigenvalues in the left-half plane. Therefore for each $\xi \in \mathbb{R}$, problem (3.6), (3.7) has only zero solution. Hence $v(x, y) \equiv 0$, and thus we have proved that limiting problems do not have nonzero bounded solutions. This is also true for the formally adjoint operator. Therefore the operator L satisfies the Fredholm property. It remains also true if the operator acts from $W^{2,2}_\infty(\Omega)$ into $L^2_\infty(\Omega) \times W^{1/2,2}_\infty(\partial\Omega)$ ([7], page 163) where the ∞-spaces are defined as follows. Let E be a Banach space with the norm $\| \cdot \|$ and ϕ_i be a partition of unity. Then E_∞ is the space of functions for which the expression

$$\|u\|_\infty = \sup_i \|u\phi_i\|$$

is bounded. This is the norm in this space.

Theorem 3.1. *If both problems (3.8) have all eigenvalues in the left-half plane, then the operator* $L = (A, B)$ *acting from* $C^{2+\alpha}(\bar{\Omega})$ *into* $F = C^\alpha(\bar{\Omega}) \times C^{1+\alpha}(\partial\Omega)$ *or from* $W^{2,2}_\infty(\Omega)$ *into* $L^2_\infty(\Omega) \times W^{1/2,2}_\infty(\partial\Omega)$ *satisfies the Fredholm property.*

3.2. Properness and topological degree

Consider the nonlinear operator in the domain Ω

$$T_0(w) = \Delta w + c\frac{\partial w}{\partial x} + f(w), \quad (x, y) \in \Omega, \tag{3.9}$$

and the boundary operator

$$Q_0(w) = \begin{cases} \frac{\partial w}{\partial y} & , \; y = 0 \\ \frac{\partial w}{\partial y} - g(w) & , \; y = 1 \end{cases}. \tag{3.10}$$

Let $w = u + \psi$, where $\psi(x, y)$ is an infinitely differentiable function such that $\psi(x, y) = u_+(y)$ for $x \geq 1$ and $\psi(x, y) = u_-(y)$ for $x \leq -1$. Set

$$T(u) = T_0(u + \psi) = \Delta u + c\frac{\partial u}{\partial x} + f(u + \psi) + \Delta\psi + c\,\frac{\partial \psi}{\partial x}\;, \quad (x, y) \in \Omega, \tag{3.11}$$

$$Q(u) = Q_0(u + \psi) = \begin{cases} \frac{\partial u}{\partial y} & , \; y = 0 \\ \frac{\partial u}{\partial y} - g(u + \psi) + \frac{\partial \psi}{\partial y} & , \; y = 1 \end{cases}. \tag{3.12}$$

We consider the operator $P = (T, Q)$ acting in weighted spaces,

$$P = (T, Q) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

with the weight function $\mu(x) = \sqrt{1 + x^2}$. The norm in the weighted space is defined as follows:

$$\|u\|_{\infty,\mu} = \|u\mu\|_\infty.$$

In the bistable case where all eigenvalues of problems (1.8) lie in the left-half plane, the operator P is proper in the weighted spaces and the topological degree can be defined [7].

4. A priori estimates

4.1. Auxiliary results

We begin with some auxiliary results. Consider the problem

$$\Delta u + c\frac{\partial u}{\partial x} + f(u) = 0, \tag{4.1}$$

$$y = 0 : \frac{\partial u}{\partial y} = 0, \quad y = 1 : \frac{\partial u}{\partial y} = g(u). \tag{4.2}$$

We look for the solutions with the limits

$$\lim_{x\to\pm\infty} u(x,y) = u_\pm(y), \quad 0 < y < 1 \tag{4.3}$$

at infinity, $u_-(y) > u_+(y)$. The proofs of the following lemmas are similar to those in [1].

Lemma 4.1. *Let $U_0(x,y)$ be a solution of problem (4.1), (4.2) such that $\frac{\partial U_0}{\partial x} \le 0$ for all $(x,y) \in \bar{\Omega}$. Then the last inequality is strict.*

Lemma 4.2. *Let $u_n(x,y)$ be a sequence of solutions of problem (4.1), (4.2) such that $u_n \to U_0$ in $C^1(\bar{\Omega})$, where $U_0(x,y)$ is a solution monotonically decreasing with respect to x. Then for all n sufficiently large $\frac{\partial u_n}{\partial x} < 0$, $(x,y) \in \bar{\Omega}$.*

We will now determine the sign of the speed of the wave connecting a stable and an unstable solutions. This result will be used below for estimates of solutions.

Lemma 4.3. *Suppose $u_0(y)$ is a solution of problem (1.7) in the cross section of the domain, and $u_+(y) < u_0(y) < u_-(y)$. Assume, next, that the corresponding eigenvalue problem*

$$v'' + f'(u_0)v = \lambda v, \quad v'(0) = 0, \quad v'(1) = g'(u_0(1))v(1) \tag{4.4}$$

has some eigenvalues in the right-half plane. If a monotone with respect to x function $w(x,y)$ satisfies the problem

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{4.5}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = g(w), \tag{4.6}$$

$$\lim_{x\to-\infty} w(x,y) = u_-(y), \quad \lim_{x\to\infty} w(x,y) = u_0(y), \tag{4.7}$$

then $c > 0$. If

$$\lim_{x\to-\infty} w(x,y) = u_0(y), \quad \lim_{x\to\infty} w(x,y) = u_+(y),$$

instead of (4.7), then $c < 0$.

Lemma 4.4. *If problem (4.1)-(4.3) has a solution w, then the value of the speed admits the estimate $|c| \le M$, where the constant M depends only on $\max_{u\in[u_+,u_-]} |f'(u)|, |g'(u)|$.*

4.2. Functionalization of the parameter

Let $w_0(x, y)$ be a solution of problem (4.1)-(4.3). Then the functions

$$w_h(x, y) = w_0(x + h, y), \quad h \in \mathbb{R}$$

are also solutions of this problem. The existence of the family of solutions does not allow one to use directly the topological degree because there is a zero eigenvalue of the linearized problem and a uniform a priori estimate of solutions in the weighted spaces does not occur.

In order to overcome this difficulty, we replace the unknown parameter c, the wave speed, by a functional $c(w_h)$. This approach was suggested in [3] for periodic solutions of ordinary differential systems of equations, and then used for travelling waves in [5]. This functional determines a function of h, $s(h) = c(w_h)$. We will construct this functional in such a way that $s'(h) < 0$ and $s(h) \to \pm\infty$ as $h \to \mp\infty$. Then instead of the family of solutions we obtain a single solution for the value of h for which $c = s(h)$.

Let

$$\rho(w_h) = \int_\Omega (w_0(x + h, y) - u_+(y)) r(x) dx dy,$$

where $r(x)$ is an increasing function satisfying the conditions:

$$r(-\infty) = 0, \quad r(+\infty) = 1, \quad \int_{-\infty}^0 r(x) dx < \infty.$$

Since $w_0(x, y)$ is a decreasing function of x, then $\rho(w_h)$ is a decreasing function of h, and

$$\rho(w_h) \to \begin{cases} 0 & , h \to +\infty \\ +\infty & , h \to -\infty \end{cases}.$$

Hence the function $s(h) = c(w_h) = \ln \rho(w_h)$ possesses the required properties.

4.3. Estimates of solutions

We consider next the problem

$$\Delta w + c\frac{\partial w}{\partial x} + f_\tau(w) = 0, \tag{4.8}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = g_\tau(w), \tag{4.9}$$

$$w(\pm\infty, y) = u_\pm(y), \tag{4.10}$$

where the functions f and g depend on the parameter $\tau \in [0, 1]$. Everywhere below we will assume that the functions $f_\tau(w), g_\tau(w)$ are bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. These conditions allow the construction of the topological degree [7].

The proof of the following lemma is given in the appendix.

Lemma 4.5. *Suppose that solution $w(x, y)$ of problem (4.8)-(4.10) satisfies the estimate $|w| \le M$ with some positive constant M, and*

$$|f_\tau^{(i)}(w)|, \quad |g_\tau^{(i)}(w)| \le K \text{ for } |w| \le M, \quad i = 0, 1, 2, 3,$$

where K is a positive constant. Then the Hölder norm $C^{2+\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ of the solution is bounded by a constant which depends only on K, M and c.

Denote by w_τ a solution of problem (4.8)-(4.10). We need to obtain a uniform estimate of the solution $u_\tau = w_\tau - \psi$ in the norm of the space $W^{2,2}_{\infty,\mu}(\Omega)$. Here $\psi(x,y)$ is an infinitely differentiable function such that $\psi(x,y) = u_+(y)$ for $x \geq 1$ and $\psi(x,y) = u_-(y)$ for $x \leq -1$. Since $u \in C^{2+\alpha}(\bar{\Omega})$, then the norm $W^{2,2}_{\infty}(\Omega)$ of the solution is also uniformly bounded. However, the boundedness of the norm in the weighted space does not follow from this and should be proved. In order to obtain the estimate, it is sufficient to prove that the solution is bounded in the weighted space, that is

$$\sup_{(x,y)\in\Omega} |(w_\tau(x,y) - \psi(x,y))\mu(x)| \leq M \tag{4.11}$$

with some constant M independent of τ. If this estimate is satisfied, then the derivatives of the solution up to the order two are also bounded. Indeed, the function $u_\tau = w_\tau - \psi$ satisfies the problem

$$\Delta u + c\frac{\partial u}{\partial x} + f(u+\psi) + \gamma(x,y) = 0,$$

$$y = 0: \frac{\partial u}{\partial y} = 0, \quad y = 1: \frac{\partial u}{\partial y} = g(u+\psi),$$

where $\gamma(x,y) = \Delta\psi + c\frac{\partial\psi}{\partial x}$. Then the function $v_\tau = u_\tau\mu$ satisfies the problem

$$\Delta v + (c - 2\mu_1)\frac{\partial v}{\partial x} + (-c\mu_1 + 2\mu_1^2 - \mu_2)v + (f(u+\psi) - f(\psi))\mu + (\gamma + f(\psi))\mu = 0, \tag{4.12}$$

$$y = 0: \frac{\partial v}{\partial y} = 0, \quad y = 1: \frac{\partial v}{\partial y} = (g(u+\psi) - g(\psi))\mu + g(\psi)\mu, \tag{4.13}$$

where

$$\mu_1 = \frac{\mu'}{\mu}, \quad \mu_2 = \frac{\mu''}{\mu}$$

are bounded infinitely differentiable functions converging to zero at infinity. Since

$$|(f(u+\psi) - f(\psi))\mu| \leq \sup_s |f'(s)||u\mu|, \quad |(g(u+\psi) - g(\psi))\mu| \leq \sup_s |g'(s)||u\mu|,$$

then, by virtue of (4.11), the functions

$$\Phi(u,x) = (f(u+\psi) - f(\psi))\mu + (\gamma + f(\psi))\mu, \quad \Psi(u,x) = (g(u+\psi) - g(\psi))\mu + g(\psi)\mu$$

are bounded together with their second derivatives. Therefore solutions of problem (4.12), (4.13) are uniformly bounded in the space $C^{2+\alpha}(\Omega)$. Then the norm $W^{2,2}_{\infty}(\Omega)$ is also bounded.

It remains to prove estimate (4.11). Consider first of all the behavior of solutions at the vicinity of infinity. By virtue of the Fredholm property, $|w_\tau(x,y) - u_\pm(y)|$ decay exponentially as $x \to \pm\infty$. The decay rate is determined by the principal eigenvalue of the corresponding operators in the cross-section of the cylinder. They can be estimated independently of τ.

Let $\epsilon > 0$ be small enough, $N_-(\tau)$ and $N_+(\tau)$ be such that $|w_\tau(x,y) - u_+(y)| \leq \epsilon$ for $x \geq N_+(\tau)$ and $|w_\tau(x,y) - u_-(y)| \leq \epsilon$ for $x \leq N_-(\tau)$. For a polynomial weight function $\mu(x)$ there exists a constant K independent of $\tau \in [0,1]$ such that

$$|w_\tau(x,y) - u_\pm(y)|\mu(x) \leq K, \quad x \gtrless N_\pm(\tau), \quad \tau \in [0,1].$$

Since the functions $w_\tau(x,y)$ are uniformly bounded, then (4.11) will follow from the uniform boundedness of the values $N_\pm(\tau)$.

First, let us note that the difference between them is uniformly bounded. Indeed, if this is not the case and $N_+(\tau) - N_-(\tau) \to \infty$ as $\tau \to \tau_0$ for some τ_0, then there are two solutions of problem (4.8), (4.9) for $\tau = \tau_0$, w_1 and w_2 with the limits

$$w_1(x,y) \to \begin{cases} u_-(y)\ , \ x \to -\infty \\ u_0(y)\ , \ x \to +\infty \end{cases}, \quad w_2(x,y) \to \begin{cases} u_0(y)\ , \ x \to -\infty \\ u_+(y)\ , \ x \to +\infty \end{cases}.$$

These solutions are obtained as limits of the solution w_τ as $\tau \to \tau_0$. In order to obtain them, consider a sequence of functions $w_{\tau_k}(x,y)$, $\tau_k \to \tau_0$ and two sequences of shifted functions: $w_{\tau_k}(x + N_-(\tau_k), y)$ and $w_{\tau_k}(x + N_+(\tau_k), y)$. The first sequence gives in the limit the first solution, the second limit gives the second solution.

The existence of such solutions contradicts Lemma 4.3 since the first one affirms that the speed is positive while the second one that it is negative.

Next, if one of the values $|N_\pm(\tau)|$ tends to infinity as $\tau \to \tau_0$, then the modulus $|c(w_h)|$ of the functional introduced in Section 4.1 also tends to infinity as $\tau \to \tau_0$. This contradicts a priori estimates of the wave speed. Thus, we have proved the following theorem.

Theorem 4.6. *Let the functions $f_\tau(w), g_\tau(w)$ be bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. If there exists a solution w_τ of problem (4.8)-(4.10) such that $u_\tau = w_\tau - \psi \in W^{2,2}_{\infty,\mu}(\Omega)$, then the norm $\|u_\tau\|_{W^{2,2}_{\infty,\mu}(\Omega)}$ is bounded independently of τ and of the solution w_τ.*

5. Leray-Schauder method

5.1. Model problem

Consider the problem

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{5.1}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = 0, \tag{5.2}$$

$$w(\pm\infty, y) = u_\pm, \tag{5.3}$$

where we put 0 instead of $g(w)$ in the boundary condition, u_+ and u_- are some numbers such that $f(u_\pm) = 0$, $f'(u_\pm) < 0$. Suppose that there exists a single zero u_0 of the function f in the interval (u_+, u_-), $f'(u_0) > 0$. Less restrictive conditions on the function f can also be considered. In this case the problem

$$w'' + cw' + f(w) = 0, \quad w(\pm\infty) = u_\pm$$

has a solution $w_0(x)$ for a unique value of c (see, e.g., [5]). This function is also a solution of problem (5.1)-(5.3). The uniqueness of this solution as a solution of the two-dimensional problem is proved in the following lemma.

Lemma 5.1. *There exists a unique monotone in x solution of problem (5.1)-(5.3) up to translation in space.*

Proof. Suppose that there exist two different monotone solutions of problem (5.1)-(5.3), (w_1, c_1) and (w_2, c_2). We recall that the corresponding values of the speed c can be different. Consider the equation

$$\frac{\partial v}{\partial t} = \Delta v + c_1\frac{\partial v}{\partial x} + f(v) \tag{5.4}$$

with the boundary condition (5.2). The function $w_1(x,y)$ is a stationary solution of this problem. It is proved in [6] that it is globally stable with respect to all initial conditions $v(x,y,0)$, which are monotone with respect to x and such that the norm $\|v(x,y,0) - w_1(x,y)\|_{L^2(\Omega)}$ is bounded.

Consider the initial condition $v(x, y, 0) = w_2(x, y)$. It is monotone and the L^2 norm of the difference $w_2 - w_1$ is bounded since these functions approach exponentially their limits at infinity. According to the stability result, the solution converges to $w_1(x + h, y)$ with some h. On the other hand, the solution writes $u(x, y, t) = w_2(x - (c_2 - c_1)t, y)$, and it cannot converge to w_1. This contradiction proves the lemma. □

We consider next the problem (4.8)-(4.10) and the corresponding operators

$$T_\tau(u) = \Delta(u + \psi) + c(u + \psi)\frac{\partial(u + \psi)}{\partial x} + f_\tau(u + \psi), \quad (x, y) \in \Omega, \tag{5.5}$$

$$Q_\tau(u) = \begin{cases} \frac{\partial u}{\partial y} & , \ y = 0 \\ \frac{\partial u}{\partial y} - g_\tau(u + \psi) & , \ y = 1 \end{cases}, \tag{5.6}$$

$$P_\tau = (T_\tau, Q_\tau) : W^{2,2}_{\infty,\mu}(\Omega) \to L^2_{\infty,\mu}(\Omega) \times W^{1/2,2}_{\infty,\mu}(\partial\Omega).$$

Suppose that $g_\tau(u) \equiv 0$ for $\tau = 0$. Then the equation

$$P_\tau(u) = 0 \tag{5.7}$$

has a unique solution $u_0 = w_0 - \psi$ for $\tau = 0$. The index of this solution, that is the topological degree of this operator with respect to a small neighborhood of the solution, equal 1. Indeed, the index equals $(-1)^\nu$, where the ν is the number of positive eigenvalues of the linearized operator [5], [7]. In the case under consideration, the linearized operator has all eigenvalues in the left-half plane [6].

5.2. Wave existence

As above, we assume that the functions $f_\tau(w), g_\tau(w)$ are bounded and continuous together with their derivatives of the third order with respect to w and of the second order with respect to τ. We begin with a general result on wave existence.

Theorem 5.2. *Let the problem*

$$\frac{d^2 w}{dy^2} + f_\tau(w) = 0, \quad w'(0) = 0, \quad w'(L) = g_\tau(w(L)) \tag{5.8}$$

have solutions $u^\tau_\pm(y)$ such that

$$u^\tau_+(y) < u^\tau_-(y), \quad 0 \le y \le L$$

and the eigenvalue problems

$$\frac{d^2 v}{dy^2} + f'_\tau(u^\tau_\pm)v = \lambda v, \quad v'(0) = 0, \quad v'(L) = g'_\tau(u^\tau_\pm)v(L) \tag{5.9}$$

have all eigenvalues in the left-half plane for any $\tau \in [0, 1]$. Suppose that for any other solution $u^\tau_0(y)$ of problem (5.8), the eigenvalue problem

$$\frac{d^2 v}{dy^2} + f'_\tau(u^\tau_0)v = \lambda v, \quad v'(0) = 0, \quad v'(L) = g'_\tau(u^\tau_0)v(L) \tag{5.10}$$

has some eigenvalues in the right-half plane. If the problem

$$\Delta w + c\frac{\partial w}{\partial x} + f_\tau(w) = 0, \tag{5.11}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = L : \frac{\partial w}{\partial y} = g_\tau(w), \tag{5.12}$$

$$\lim_{x \to \pm\infty} w(x, y) = u_\pm^\tau(y), \quad 0 < y < L, \tag{5.13}$$

considered in the domain $\Omega = \{-\infty < x < \infty, \ 0 < y < L\}$, *has a unique solution monotone with respect to* x *for* $\tau = 0$, *then it also has a unique monotone solution for any* $\tau \in [0, 1]$.

Proof. The proof of the theorem is based on the Leray-Schauder method. We consider equation (5.7). The topological degree for the operator $P_\tau(u)$ is defined (Section 3).

Denote by Γ_m the ensemble of solutions of equation (5.7) for all $\tau \in [0, 1]$ such that for any $u \in \Gamma_m$ the function $w = u + \psi$ is monotone with respect to x. Let Γ_n be the set of all solutions for which the function $w = u + \psi$ is not monotone with respect to x. Then the distance d between these two sets in the space $E = W^{2,2}_{\infty,\mu}(\Omega)$ is positive. Indeed, suppose that this is not true. Then there exist two sequences $u_k \in \Gamma_m$ and $v_k \in \Gamma_n$ such that $\|u_k - v_k\|_E \to 0$ as $k \to \infty$. From Lemma 5.2 it follows that the functions $w_k = v_k + \psi$ are monotone with respect to x for k sufficiently large. This contradiction shows that the convergence cannot occur.

From Theorem 4.6, applicable for solutions from Γ_m, it follows that the set Γ_m is bounded in E. Moreover, by virtue of properness of the operator P_τ it is compact. Hence there exists a bounded domain $G \subset E$ such that $\Gamma_m \subset G$ and $\Gamma_n \cap \bar{G} = \oslash$.

Consider the topological degree $\gamma(P_\tau, G)$. Since

$$P_\tau(u) \neq 0, \quad u \in \partial G,$$

then it is well defined. Since $\gamma(P_0, G) = 1$ (Section 6.1), then $\gamma(P_\tau, G) = 1$ for any $\tau \in [0, 1]$. Hence problem (5.11)-(5.13) has a monotone solution for any $\tau \in [0, 1]$.

It remains to verify its uniqueness. We recall that

$$\gamma(P_\tau, G) = \sum_i \text{ind } u_i,$$

where ind u_i is the index of a solution u_i and the sum is taken with respect to all solutions $u_i \in G$. Since $\gamma(P_\tau, G) = 1$ and ind $u_i = 1$ (cf. Section 5.1), then the solution is necessarily unique. □

The previous theorem uses some assumptions about the solutions $u_\pm^\tau$ and u_0^τ of problem (5.8) in the cross-section. We will now consider some particular cases where these conditions can be verified.

Theorem 5.3. *Let* u_+ *and* u_- *be some constants and the following conditions be satisfied:*

1. $f(u_\pm) = 0, \ f'(u_\pm) < 0$, $g(u_\pm) = 0, \ g'(u_\pm) < 0$,

2. $f(u_0) = 0, \ f'(u_0) > 0$, $g(u_0) = 0, \ g'(u_0) > 0$ *for some* $u_0 \in (u_+, u_-)$, *and there are no other zeros of these functions in this interval.*

Then for all positive L *sufficiently small, the problem*

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{5.14}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = L : \frac{\partial w}{\partial y} = g(w), \tag{5.15}$$

$$\lim_{x \to \pm\infty} w(x, y) = u_\pm \tag{5.16}$$

considered in the domain $\Omega = \{-\infty < x < \infty, \ 0 < y < L\}$ *has a unique solution monotone with respect to* x.

This theorem follows from the previous one, where we set $g_\tau = \tau g$, and from Lemmas 2.2 and 5.1.

Theorem 5.4. *Let the function $g(w)$ satisfy conditions of the previous theorem. Then for all positive L, the problem*

$$\Delta w + c\frac{\partial w}{\partial x} = 0, \tag{5.17}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = L : \frac{\partial w}{\partial y} = g(w), \tag{5.18}$$

$$\lim_{x\to\pm\infty} w(x,y) = u_\pm \tag{5.19}$$

considered in the domain $\Omega = \{-\infty < x < \infty, \ \ 0 < y < L\}$ has a unique solution monotone with respect to x.

Proof. The proof consists of two steps. First, we consider sufficiently small L and use the result of the previous theorem as a starting point for the deformation $f_\tau = (1-\tau)f$. For $\tau = 1$ we obtain $f_\tau(w) \equiv 0$. At the next step, we increase the width L of the domain. It is equivalent to the change of variables $y = \sigma\eta$ in the equation and in the boundary condition. The problem in the cross-section has only constant solutions. We can use the results of Section 2.2 about their stability and Theorem 5.2. □

In the last theorem we consider the case of small boundary conditions where the solution is close to a one-dimensional solution.

Theorem 5.5. *Suppose that $f(u_\pm) = 0,\ f'(u_\pm) < 0$ and for some c_0 there exists a monotone solution $w(x)$ of the problem*

$$w'' + c_0 w' + f(w) = 0, \quad w(\pm\infty) = u_\pm.$$

Then for all ϵ sufficiently small, the problem

$$\Delta w + c\frac{\partial w}{\partial x} + f(w) = 0, \tag{5.20}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = L : \frac{\partial w}{\partial y} = \epsilon g(w), \tag{5.21}$$

$$\lim_{x\to\pm\infty} w(x,y) = u_\pm^\epsilon(y) \tag{5.22}$$

considered in the domain $\Omega = \{-\infty < x < \infty, \ \ 0 < y < L\}$ has a unique solution monotone with respect to x. Here $u_\pm^\epsilon(y)$ are solutions of the problem

$$\frac{\partial w}{\partial y} + f(w) = 0, \quad w'(0) = 0, \ \ w'(L) = \epsilon g(w(L)),$$

$u_\pm^\epsilon(y) \to u_\pm$ as $\epsilon \to 0$ uniformly in y.

The proof of this theorem follows from the property of topological degree: a solution with nonzero index persists under small deformation of the operator.

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Appendix. Proof of Lemma 4.5

In order to simplify the presentation, we will suppose throughout this proof that the solution w of problem (4.8)-(4.10) exponentially converges to 0 at infinity together with its first derivatives, and

$$\int_{\Omega} |w(x,y)| dxdy \le M.$$

In general, if it is not the case, we subtract some given sufficiently smooth function with the limits $u_\pm(y)$ as $x \to \pm\infty$. Exponential convergence of solution to its limits at infinity follows from the Fredholm property of the corresponding operator.

We multiply equation (4.8) by w and integrate over Ω. Taking into account the boundary conditions, we obtain the estimate

$$\int_{\Omega} |\nabla w|^2 dxdy \le C.$$

Here and below we denote by C any constant which depends only on K, M and c. Hence $\partial w/\partial y \in L^2(\Omega)$.

Set $v = \partial w/\partial y$. Then this function satisfies the problem

$$\Delta v + c\frac{\partial v}{\partial x} + f'_\tau(w)v = 0, \tag{5.1}$$

$$y = 0 : v = 0, \quad y = 1 : v = g_\tau(w). \tag{5.2}$$

Here and below f'_τ and g'_τ denotes the derivatives of these functions with respect to w. Put $\phi = f'_\tau(w)v$ and consider the auxiliary problems

$$\Delta v_\pm + c\frac{\partial v_\pm}{\partial x} + \phi = 0, \tag{5.3}$$

$$y = 0 : v_\pm = 0, \quad y = 1 : v_\pm = \pm K. \tag{5.4}$$

Then from the maximum principle

$$v_-(x,y) \le v(x,y) \le v_+(x,y), \quad (x,y) \in \Omega.$$

Since the function $f'_\tau(w)$ is bounded, then $\phi \in L^2(\Omega)$. Therefore problems (5.3), (5.4) are solvable in $H^2(\Omega)$, and their norms depend only on K, M and c. By virtue of embedding theorems (on bounded subsets), the functions $v_\pm(x,y)$ are bounded and, consequently, solution v of problem (5.1), (5.2) admits the estimate:

$$\sup_{(x,y)\in\Omega} |v(x,y)| \le C. \tag{5.5}$$

Next, we multiply equation (5.1) by v and integrate over Ω. Taking into account that $v=0$ at $y=0$ and

$$\frac{\partial v}{\partial y} = g_\tau'(w) g_\tau(w), \quad y=1,$$

we obtain the estimate

$$\int_\Omega |\nabla v(x,y)|^2 dxdy \le C. \tag{5.6}$$

Hence $\partial v/\partial y \in L^2(\Omega)$.

Set $z = \partial v/\partial y$. Then this function satisfies the equation

$$\Delta z + c\frac{\partial z}{\partial x} + f_\tau'(w) z + f_\tau''(w) v^2 = 0. \tag{5.7}$$

Since the boundary condition for z at $y=0$ is not defined, we extend this problem by symmetry and consider it in the domain

$$\widehat{\Omega} = \{-\infty < x < \infty, \ \ -1 < y < 1\}$$

with the boundary conditions

$$|y| = 1 : z = g_\tau'(w) g_\tau(w). \tag{5.8}$$

Put

$$\zeta = f_\tau'(u) z + f_\tau''(u) v^2$$

and consider the auxiliary problems

$$\Delta z + c\frac{\partial z}{\partial x} + \zeta = 0. \tag{5.9}$$

$$|y| = 1 : z = \pm K^2. \tag{5.10}$$

As before, $z_- \le z \le z_+$, where z is a solution of problem (5.7), (5.8) and $z_\pm$ are solutions of problems (5.9), (5.10).

Since $v, z \in L^2(\Omega)$ and v is bounded, then $\zeta \in L^2(\Omega)$. As above, we prove that the functions $z_\pm$ are bounded. Hence

$$\sup_{(x,y)\in\Omega} \Big|\frac{\partial^2 w}{\partial y^2}\Big| \le C. \tag{5.11}$$

Having proved this estimate, we return to equation (4.8) which we consider as a second-order ordinary differential equation (y is a fixed parameter):

$$U'' + cU' + H = 0,$$

where $U(x) = w(x,y)$, prime denotes the derivative with respect to x,

$$H(x) = \frac{\partial^2 u}{\partial y^2} + f_\tau(w(x,y)).$$

By virtue of (5.11) and boundedness of the function f_τ, $H(x)$ is also bounded. Multiplying the last equation by U and integrating from $-\infty$ to ∞, we estimate the first derivative U' in the L^2-norm. Next, we multiply the same equation by U' and integrate from $-\infty$ to x. This gives an estimate of U' in the supremum norm. From the estimate of the first derivative and the equation it follows the estimate of the second derivative U''. Hence

$$\sup_{(x,y)\in\Omega} \left|\frac{\partial w}{\partial x}\right|, \left|\frac{\partial^2 w}{\partial x^2}\right| \le C. \tag{5.12}$$

Thus we have proved that $w \in C^2(\bar{\Omega})$. Finally, we write problem (4.8)-(4.10) in the form

$$\Delta w + c\frac{\partial w}{\partial x} + \beta(x,y) = 0, \tag{5.13}$$

$$y = 0 : \frac{\partial w}{\partial y} = 0, \quad y = 1 : \frac{\partial w}{\partial y} = \gamma(x,y), \tag{5.14}$$

where $\beta(= f_\tau) \in C^\alpha(\bar{\Omega})$ and $\gamma(= g_\tau) \in C^{1+\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. From a priori estimates of solutions it follows that $u \in C^{2+\alpha}(\bar{\Omega})$, and the norm of the solution depends on K, M and c.

□