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Calcium Waves in Thin Visco-Elastic Cells

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Abstract. The model we consider treats the cell as a viscoelastic medium filling one of two kinds of thin domains ("shapes" of cells): the thin slab being a caricature of a tissue and the thin circular cylinder mimicking a long cell. This enables us to simplify the system of mechano-chemical equations. We construct abundant classes of explicit, but approximate, formulae for heteroclinic solutions to these equations.

Keywords and phrases: calcium waves, mechano-chemical coupling, thin domains

Mathematics Subject Classification: 35K57, 92C05

1. The model

A stimulation of a cell or a group of cells generates some aggregation of calcium ions Ca^{2+}. The nature of such a stimulation can be very diverse. For instance, it can be a sperm penetrating an egg or a mechanical or electric agent. In some types of cells the density of such an increase becomes so large that it starts to propagate as a wave. The role of the calcium waves is not totally clear; the common opinion is however that they enable many physiological processes; in particular they enable the transmission of information within a single cell or a group of cells. That is why it is so important to understand the leading mechanisms governing the calcium motion. When a calcium wave propagates through a cell it meets on its way microfilaments of protein, forming a network. The elevated concentration of calcium changes the properties of these filaments so that they exert a contraction force on the cell, what lowers locally the calcium concentration - the stretch activation effect. Hence, the process is of a mechano-chemical character.

There are many agents participating in the phenomenon of calcium waves. In this paper, we consider only two field quantities:

– $\mathbf{u}(t, \mathbf{x})$ - the displacement at time t of a point of cytogel being initially at the position $\mathbf{x}$,
– $c(t, \mathbf{x})$ - the concentration of free cytosolic calcium ions, normalized to unity, i. e. $c \in [0, 1]$.

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The material of the interior of a cell is a gel-like substance. The model by Murray and Oster [1], [2] treats it as a visco-elastic medium. It is assumed that the Reynolds number is small, what enables to ignore the inertial effects. The body force, resulting from passive resistance of the cytoskeleton, balances the elastic force, the viscous force, and the acto-myosin contractive stress - the traction. The force balance equation reads

$$\frac{\partial}{\partial x_j}\sigma_{ij} = ku_i,\ i = 1,2,3, \tag{1.1}$$

where

$$\sigma_{ij} = \left[\left(\frac{1-2\nu}{1-\nu} + \mu_1\frac{\partial}{\partial t}\right)\varepsilon_{ij} + \left(\frac{\nu}{1-\nu} + \mu_2\frac{\partial}{\partial t}\right)\theta\delta_{ij} + \tau\delta_{ij}\right],\ i = 1,2,3, \tag{1.2}$$

is the stress tensor. In Eq.(1.1) the Einstein summation convention over repeated indices is assumed, and all quantities are non-dimensional. In (1.2), ν is the constant Poisson ratio, μ_1, μ_2 are the constant shear and bulk viscosities, k is a non-negative parameter, ε_{ij} are the components of the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.3}$$

and $\theta = div\mathbf{u}$ is the dilatation. At last, $\tau = \tau(c)$ is the traction. It is the contribution to the stresses resulting from molecules called actins and myosins which react to the changes of the calcium concentration: when the calcium concentration starts to grow they exert a contraction stress on the cytoplasm, however if the calcium concentration reaches a sufficiently large value the network of acto-myosins starts to break down, consequently the traction disappears [1], [2]. That is why we assume that τ is completely determined by calcium concentration c and is such that

$$\begin{gathered}\tau(c) \geq 0, \quad 0 \leq c \leq 1,\\ \tau(0) = \tau(1) = 0.\end{gathered} \tag{1.4}$$

To our knowledge, there is no formula describing the dependence of the traction on the calcium concentration, and the authors used their own expressions (c.f. [1]-[6]).
The calcium conservation equation is of the form

$$\frac{\partial c}{\partial t} = D\varDelta c + f(c) + \gamma\theta, \tag{1.5}$$

where $\varDelta$ is the Laplacian with respect to $x = (x_1, x_2, x_3)$, $D > 0$ is the coefficient of diffusion, and the source term $f(c)$ describes the kinetics of calcium transportation into and out of the cytosol. Calcium is stored in special compartments which open when the local calcium concentration reaches a certain threshold value. Then some amount of calcium is secreted what raises the local value of its concentration and stimulates further secretion. Thus, the calcium waves are maintained by an autocatalytic mechanism. So, the function $f(c)$ must be bistable i.e. it must have two stable equilibria and one unstable between them ([1],[2],[7],[8]). More precisely, we assume that $f(c)$ is continuously differentiable with respect to $c \in [0,1]$ and such that

$$\begin{gathered}f(0) = f(c_T) = f(1) = 0,\ \text{ for only one } c_T \in (0,1),\\ f_c'(0) < 0, \quad f_c'(c_T) > 0 \quad \text{and} \quad f_c'(1) < 0\end{gathered} \tag{1.6}$$

In the last term, $\gamma\theta$ in Eq. (1.5) the quantity γ is a positive coefficient. If $\theta > 0$, then locally the volume of the cell grows and the calcium concentration increases, if $-1 < \theta < 0$, then locally the cell shrinks and consequently the calcium concentration decreases. This phenomenon is known as stretch activation ([1],[2],[7],[8]).

2. Calcium waves in R^3

We start from presenting some remarks on calcium waves in the space free of any mechanical effects. These remarks will be used later as a sort of reference. When all mechanical effects are ignored, then the problem reduces to a study of a simplified version of Eq. (1.5)

$$\frac{\partial c}{\partial t} = D\Delta c + f(c) \tag{2.1}$$

with the bistable f. In this chapter we will study plane travelling wave solutions to Eq. (2.1) propagating in the direction of the x_1-axis. Therefore, it is natural to assume that the calcium concentration c is a function of t and x_1 only. More precisely, we assume that

$$c = c(x_1, t) = c(\xi), \quad \xi = x_1 - Ut, \tag{2.2}$$

where the parameter U is the speed of the wave. Substituting (2.2) into Eq. (2.1) we arrive at a ordinary differential equation of the form

$$D\frac{d^2c}{d\xi^2} + U\frac{dc}{d\xi} + f = 0, \tag{2.3}$$

We look for such solutions of this equation which satisfy the following limit conditions:

$$\lim_{\xi\to-\infty} c(\xi) = 0, \quad \lim_{\xi\to\infty} c(\xi) = 1 \tag{2.4}$$

and

$$\lim_{|\xi|\to\infty} \frac{dc}{d\xi} = 0. \tag{2.5}$$

A travelling wave is called a calcium wave if it is a wave of excitation, i.e. the propagating wave raises the calcium concentration. As we took the boundary conditions in the form of (2.4), the excitability of the wave demands the wave to go from the right to the left, what means that we are looking for travelling waves with negative speed U.

The analysis of the existence and uniqueness of travelling calcium waves in equations like (2.3) subject to the limit conditions (2.4), (2.5) can be found for instance in [9],[10],[11],[12]. We have

Theorem 2.1. *([9]). Let $f(c)$ be bistable on [0,1] and let*

$$\int_0^1 f(c)\, dc > 0. \tag{2.6}$$

Then, there exists only one, negative, value of the parameter U for which there is a unique, up to translations, travelling wave solution of the equation (2.3) *subject to the limit conditions* (2.4), (2.5). *Moreover, this solution is an increasing function of $\xi \epsilon R^1$.*

The assertion of this theorem makes justified the substitution

$$\frac{dc}{d\xi} = \frac{V(c)}{\sqrt{D}}, \tag{2.7}$$

where V is such a function that

$$V \in C^2([0,1]), \quad V(0) = V(1) = 0, \quad V(c) > 0 \quad \text{for} \quad c \in (0,1), \\ V_c'(0) > 0, \quad V_c'(1) < 0. \tag{2.8}$$

Inserting (2.7) into Eq. (2.3) we obtain an equation for V:

$$V\left(V_c' + \frac{U}{\sqrt{D}}\right) + f = 0. \tag{2.9}$$

The thesis of Theorem 2.1 guaranties the solvability of this equation subject to the conditions (2.4), (2.5). Unfortunately, this theorem does not say, given *f,* how to determine neither the speed U nor the wave profile $c(\xi)$. In this paper we invert this problem and assume that the function V is given. Then Eq. (2.9) becomes the definition of the function *f.* From experimental point of view, such an approach is even more natural since it is quite easy to measure the shape of the calcium wave, what enables an experimental determination of the function V and, as a consequence, determination of the function *f.* Setting in Eq. (2.9) $c = c_T$, we obtain

$$U = -\sqrt{D}V_c'(c_T). \tag{2.10}$$

The threshold value c_T has to be such that

$$V_c'(c_T) > 0$$

in order to be the wave speed U negative. Inserting (2.10) into (2.9) we obtain

$$f(c) = V(c)\left[V_c'(c_T) - V_c'(c)\right]. \tag{2.11}$$

The function f defined by the right hand side of Eq. (2.11) satisfies all conditions (1.6) but this one which demands the bistable function to have only one zero $c = c_T$ in the open interval (0,1). The zeroes of f are solutions of the equation

$$V_c'(c) = V_c'(c_T) \tag{2.12}$$

The conditions (2.8) do not guarantee in general that $c = c_T$ is the only solution of Eq. (2.12) in the interval (0,1). For sake of simplicity we assume additionally that

$$V_c''(c) < 0 \quad \text{on} \quad [0,1]. \tag{2.13}$$

Then, the only solution of Eq. (2.12) is $c = c_T$. Moreover, there is only one $c_0 \in (0,1)$ such that $V_c'(c_0) = 0$. Hence, $V_c'(c) > 0$ for $c \in [0, c_0)$ and $V_c'(c) < 0$ for $c \in (c_0, 1]$. Therefore, we obtain the following estimate for c_T :

$$0 < c_T < c_0. \tag{2.14}$$

As an example we take Richards' curve [13]:

$$V(c) = c(1 - c^m), \quad m > 0 \tag{2.15}$$

The graphs of the function $V(c)$ for $m = 0.5; 1; 2; 3; 4; 5; 100$ are given in Fig. 1. The values of this function grow with the value of the parameter m.

Its derivative vanishes at $c_0 = (m+1)^{-\frac{1}{m}}$. Hence,

$$U = -\sqrt{D}\left(1 - (1+m)c_T^m\right), \quad 0 < c_T < c_0. \tag{2.16}$$

The solution of Eq. (2.7) with V as in (2.15) is of the form

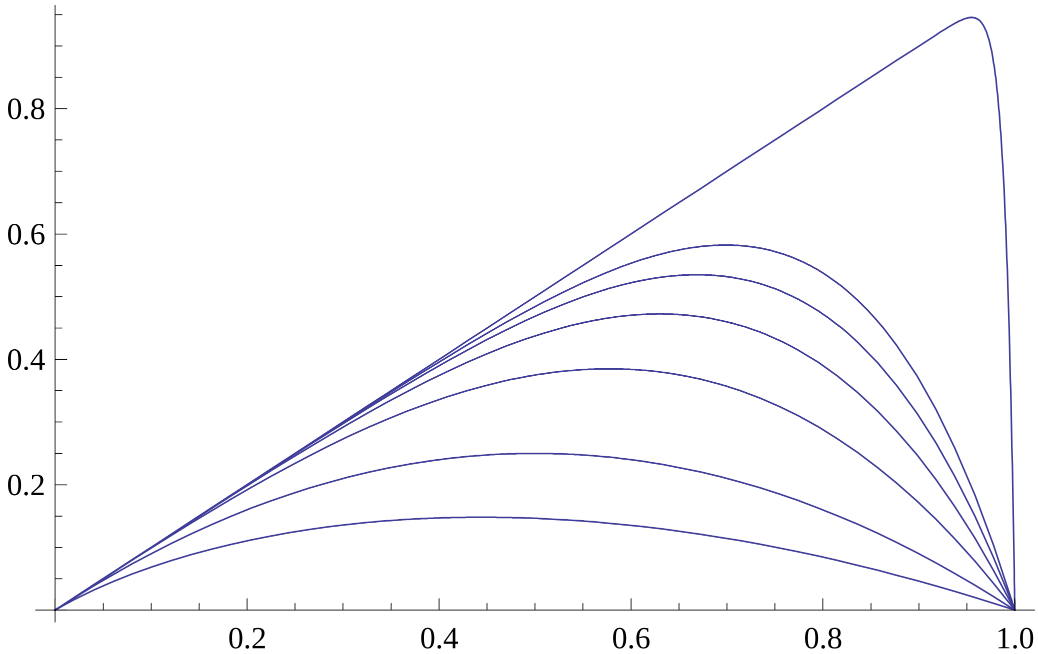

FIGURE 1. *Graphs of the function V defined by (2.15)*

$$c\,(\xi) = \exp\left(\frac{1}{\sqrt{D}}\xi\right)\left(m + \exp\left(\frac{m}{\sqrt{D}}\xi\right)\right)^{-\frac{1}{m}}. \tag{2.17}$$

The asymptotic behaviour of this solution is as follows

$$c(\xi) = \begin{cases} \left(\frac{1}{m}\right)^{\frac{1}{m}} \exp\left(m\xi\right) & \text{as} \quad \xi \to -\infty, \\ 1 - \exp\left(-\frac{m}{\sqrt{D}}\xi\right) & \text{as} \quad \xi \to +\infty. \end{cases} \tag{2.18}$$

We see from the above and from Fig.2 that for $m > 1$ the back part of the wave is steeper than the front part. Such solutions describe correctly, at least at qualitative level, the profiles of waves of excitation.

Finally, some graphs of the source function f defined by (2.11) with V given (2.15) are presented in Fig.3. For graphical reasons we took the threshold value $c_T = 0.5$. Let us notice that the positive part of the function f representing the induced calcium release is growing with the increase of m.

3. The mechano-chemical model in domains with boundaries

If the cell or tissue is contained in a volume Ω, at least partially limited by the boundary $\partial\Omega$, then some boundary conditions are necessary. We assume that the boundary $\partial\Omega$ is unloaded. More precisely, let $\mathbf{n} = \mathbf{n}\,(\mathbf{x}, t)$ be the outward vector normal to the boundary $\partial\Omega$ at the point $\mathbf{x} \in \partial\Omega$ and time t. The boundary conditions we take are as follows

$$\sigma_{ij} n_j \mid_{\partial\Omega} = 0, \quad i = 1, 2, 3 \tag{3.1}$$

We need also a boundary condition for the calcium concentration. The internal concentration of calcium in the cell is low, whereas in the extracellular matrix is much higher. Usually, the supply of calcium from the internal stores is insufficient and the cell pumps in the calcium from the outside

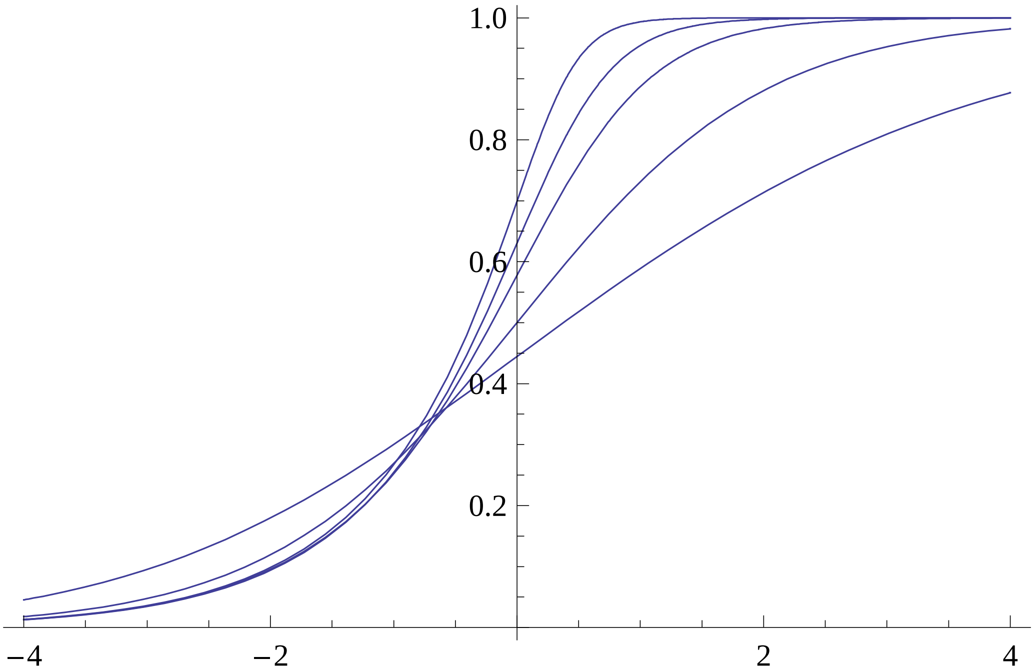

FIGURE 2. *The profiles of the calcium waves given by (2.17)) for $m = 0.5; 1; 2; 3; 5$. They steepen with the growth of m*

through special calcium channels. The large difference in concentrations does not play any role, since the influx of calcium is strictly controlled by the cell. It takes place only if the cell needs more calcium when stimulated, and it is stopped if the concentration of calcium reaches the point of saturation. More information of calcium dynamics can be found in the monograph by Keener and Sneyd [2]. In the present paper we mimic this process by imposing the Robin boundary condition. This condition relates, at the boundary, the value of the calcium flux in the direction of the outward normal to the boundary and the value of the calcium concentration on the interior side of the cell boundary. It can be formulated as follows

$$D\frac{\partial c}{\partial \mathbf{n}} = R\left(c, \mathbf{x}, t\right), \quad \mathbf{x} \in \partial \Omega, \tag{3.2}$$

where R is a given function representing this admitted amount of calcium which the cell takes from its surrounding or removes from its interior. If $R\left(c, \mathbf{x}, t\right) > 0$ at a point of the boundary, then calcium flows in the interior of the cell, if $R\left(c, \mathbf{x}, t\right) < 0$ then some amount of calcium is removed from the cell. We will precise the form of R later as now we do not need its details.

3.1. Thin slab

Now we consider the motion of calcium in visco-elastic material filling a thin slab with plane parallel walls, i.e. $(x_1, x_2, x_3) \in R^2 \times [-h, h]$, with the plane $(x_1, x_2, 0)$ being the plane of symmetry. We limit ourselves to a simpler case when the displacements within any plane parallel to the walls of the slab are potential. We assume also that the phenomenon under investigation is symmetric with respect to the x_3-axis. Under these assumptions we can write

$$u_\alpha\left(x_1, x_2, x_3\right) = \frac{\partial \varphi(x_1,x_2,\varsigma)}{\partial x_\alpha}, \quad \alpha = 1, 2, \quad u_3 = w\left(x_1, x_2, \varsigma\right) x_3,\ c = c\left(x_1, x_2, \varsigma\right), \\ \varsigma = \left(\frac{x_3}{h}\right)^2. \tag{3.3}$$

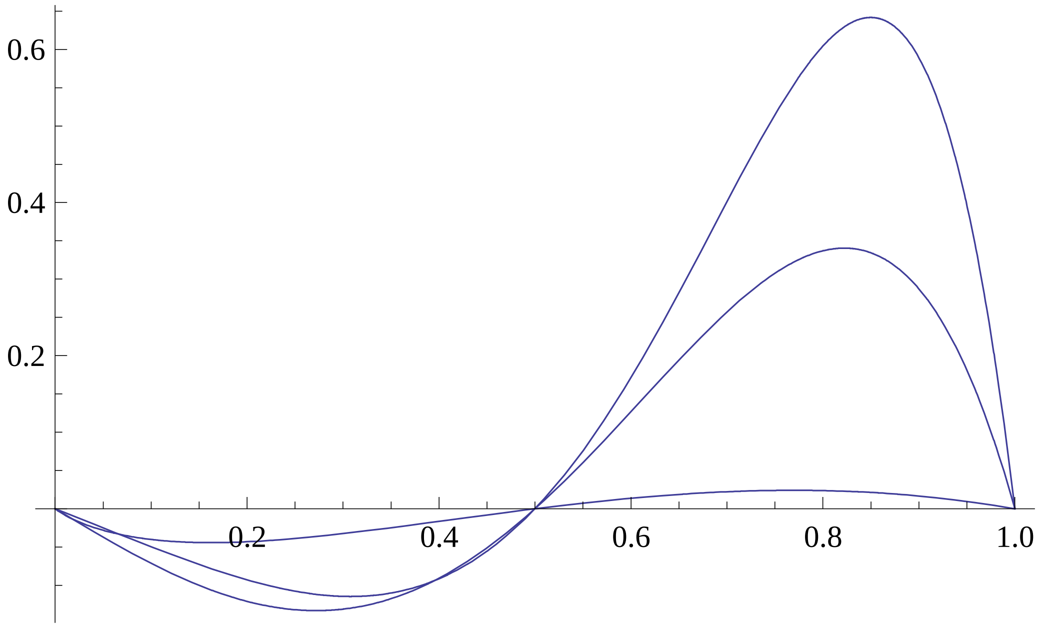

FIGURE 3. *The forms of the function f given by (2.11), (2.15) for* $m = 0.5; 2; 3$ *and* $c_T = 0.5$

Under this assumption the force balance equations (1.1) reduce to two equations

$$\left(2\varsigma\tfrac{\partial}{\partial\varsigma}+1\right)\tfrac{\partial}{\partial\varsigma}\left(\tfrac{1-2\nu}{1-\nu}+\mu_1\tfrac{\partial}{\partial t}\right)\varphi = h^2\Big[k\,\varphi-\tau-\left(1+(\mu_1+\mu_2)\tfrac{\partial}{\partial t}\right)\Delta_2\varphi \\ -\tfrac{1}{2}\left(\tfrac{1}{1-\nu}+(\mu_1+2\mu_2)\tfrac{\partial}{\partial t}\right)\left(2\varsigma\tfrac{\partial}{\partial\varsigma}+1\right)w\Big], \tag{3.4}$$

and

$$\tfrac{\partial}{\partial\varsigma}\Big[\left(\tfrac{1}{1-\nu}+(\mu_1+2\mu_2)\tfrac{\partial}{\partial t}\right)\Delta_2\varphi+2\left(2\varsigma\tfrac{\partial}{\partial\varsigma}+1\right)\left(1+(\mu_1+\mu_2)\tfrac{\partial}{\partial t}\right)w+2\tau\Big] \\ = h^2\Big[k\,w-\tfrac{1}{2}\left(\tfrac{1-2\nu}{1-\nu}+\mu_1\tfrac{\partial}{\partial t}\right)\Delta_2 w\Big], \tag{3.5}$$

for two unknown functions φ and w, where $\Delta_2 = \frac{\partial}{\partial x_1^2}+\frac{\partial}{\partial x_2^2}$ is the two-dimensional Laplacian. The reaction-diffusion equation can be written as

$$D\left(2\varsigma\frac{\partial}{\partial\varsigma}-1\right)\frac{\partial c}{\partial\varsigma} = \frac{h^2}{2}\left(\frac{\partial c}{\partial t}-D\Delta_2 c-f-\gamma\,\theta\right), \tag{3.6}$$

where

$$\theta = \Delta_2\varphi + w. \tag{3.7}$$

Now, we consider **the boundary conditions** (3.1). The vector normal to the surfaces limiting the layer is $\mathbf{n} = (0,0,1)$. So, the boundary conditions reduce to the following equations

$$\left[2\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)\frac{\partial\varphi}{\partial\varsigma}+h^2\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)w\right]_{\varsigma=1} = 0, \tag{3.8}$$

and

$$\left[\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)\Delta_2\varphi+\left(2\varsigma\frac{\partial}{\partial\varsigma}+1\right)\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)w+\tau\right]_{\varsigma=1}=0. \tag{3.9}$$

We give the boundary conditions for Eq. (3.2) the following more precise form

$$\left[D\frac{\partial c}{\partial\varsigma}-\frac{h^2}{2}\psi(c)\right]_{\varsigma=1}=0 \tag{3.10}$$

where $\psi=\psi(c)$ is a given function with as many derivatives with respect to c as necessary.

Since the slab is thin, we assume that it is possible to expand any function in a power series with respect to the small parameter h^2:

$$q\left(x_1,x_2,t,\varsigma:h^2\right)=\sum_{k=0}^{\infty}q_k\left(x_1,x_2,t,\varsigma\right)h^{2k}, \tag{3.11}$$

We assume also that any function $P(q)=P\left(q\left(x_1,x_2,t,\varsigma;h^2\right)\right)$ can be expanded as follows

$$P=\sum_{k=0}^{\infty}P_k\left(q_0,q_1,\ldots,q_k\right)h^{2k}, \tag{3.12}$$

where the first coefficients are

$$P_0=P(q_0),\quad P_1(q_0,q_1)=P_q'(q_0)\,q_1,\quad P_2(q_0,q_1,q_2)=P_q'(q_0)\,q_2+\frac{1}{2}P_q''(q_0)\,q_1^2,\quad \text{etc..} \tag{3.13}$$

It can be deduced [14], [15] that the quantities φ_0,w_0,c_0 representing the lowest order approximations to φ,w,c, respectively, do not depend on ς and satisfy the following equations

$$\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)\Delta_2\varphi_0+\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)w_0+\tau_0=k\,\varphi_0, \tag{3.14}$$

$$\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)\Delta_2\varphi_0+\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)w_0+\tau_0=0, \tag{3.15}$$

and

$$\frac{\partial c_0}{\partial t}=D\Delta_2c_0+f(c_0)+\psi(c_0)+\gamma\,\theta_0, \tag{3.16}$$

where

$$\theta_0=\Delta_2\varphi_0+w_0. \tag{3.17}$$

3.2. Thin circular fibre

In this subsection we consider the transportation of calcium in a long thin fibre of a circular cross section of the radius h. We assume that the motion possesses the cylindrical symmetry. The coordinate system we use is as follows: the x_1-axis coincides with the central axis of the cylinder and the x_2- and x_3-axes are perpendicular to it. We assume that the displacements are of the form

$$\begin{aligned}&u_1=\tfrac{\partial\varphi(x_1,\varsigma,t)}{\partial x_1},\quad u_2=w(x_1,\varsigma,t)\,x_2,\quad u_3=w(x_1,\varsigma,t)\,x_3,\\&\varsigma=\left(\tfrac{\sqrt{x_2^2+x_3^2}}{h}\right)^2,\quad 0\le\sqrt{x_2^2+x_3^2}\le h.\end{aligned} \tag{3.18}$$

Under assumptions (3.18) the force balance equations (1.1) reduce to two following ones

$$2\left(\varsigma\frac{\partial}{\partial\varsigma}+1\right)\frac{\partial}{\partial\varsigma}\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)\Delta_1\varphi$$
$$=h^2\left[k\,\varphi-\tau-\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)\Delta_1\varphi-\left(\frac{1}{1-\nu}+(\mu_1+2\mu_2)\frac{\partial}{\partial t}\right)\left(\varsigma\frac{\partial}{\partial\varsigma}+1\right)w\right], \tag{3.19}$$

$$\frac{\partial}{\partial\varsigma}\left[\left(\frac{1}{1-\nu}+(\mu_1+2\mu_2)\frac{\partial}{\partial t}\right)\Delta_1\varphi+4\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)\left(\varsigma\frac{\partial}{\partial\varsigma}+1\right)w+2\tau\right]$$
$$=h^2\left[\left[k\,w-\frac{1}{2}\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)\Delta_1 w\right]. \tag{3.20}$$

We have introduced the symbol $\Delta_1=\frac{\partial^2}{\partial x_1^2}$ of one-dimensional "Laplacian" for sake of consistency of notation with the previous subsections. The calcium equation becomes

$$D\left(\varsigma\frac{\partial}{\partial\varsigma}+1\right)\frac{\partial c}{\partial\varsigma}=\frac{h^2}{4}\left[\frac{\partial c}{\partial t}-D\Delta_1 c-f-\gamma\,\theta\right], \tag{3.21}$$

where

$$\theta=\Delta_1\varphi+2\left(\varsigma\frac{\partial}{\partial\varsigma}+1\right)w. \tag{3.22}$$

The vector normal to the boundary is $\mathbf{n}=(0,x_2,x_3)$. So, **the boundary conditions** take the form

$$\left[2\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)\frac{\partial\varphi}{\partial\varsigma}+h^2\left(\frac{1-2\nu}{1-\nu}+\mu_1\frac{\partial}{\partial t}\right)w\right]_{\varsigma=1}=0, \tag{3.23}$$

$$\left[\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)(\Delta_1\varphi+w)+\left(2\varsigma\frac{\partial}{\partial\varsigma}+1\right)\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)w+\tau\right]_{\varsigma=1}=0, \tag{3.24}$$

$$\left[D\frac{\partial c}{\partial\varsigma}-\frac{h^2}{4}\psi\right]_{\varsigma=1}=0. \tag{3.25}$$

Equations (3.19) - (3.24) are very similar to equations (3.4) - (3.10), since they differ by numerical values of some parameters. Hence, the same sort of analysis which we employed previously can applied to the present case as well. Similarly to the previous case, we look for the solutions in the form of asymptotic expansion

$$q\left(x_1,\varsigma,t\right)=\sum_{k=0}^{\infty}q_k\left(x_1,\varsigma,t\right)h^{2k}, \tag{3.26}$$

for q representing φ w or c, and, to the lowest order approximations, we can deduce that φ_0, w_0, c_0 do not depend on ς and satisfy the equations

$$\left(1+(\mu_1+\mu_2)\frac{\partial}{\partial t}\right)\Delta_1\varphi_0+2\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)w_0+\tau_0=k\,\varphi_0, \tag{3.27}$$

$$\left(\frac{\nu}{1-\nu}+\mu_2\frac{\partial}{\partial t}\right)\Delta_1\varphi_0+\left(\frac{1}{1-\theta}+(\mu_1+2\mu_2)\frac{\partial}{\partial t}\right)w_0+\tau_0=0, \tag{3.28}$$

$$\frac{\partial c}{\partial t}=D\Delta_1 c_0+f\left(c_0\right)+\psi\ \left(c_0\right)+\gamma\theta_0, \tag{3.29}$$

where

$$\theta_0=\Delta_1\varphi_0+2w_0. \tag{3.30}$$

3.3. Unified equations

Let d denote the dimensionality of the problem: $d = 2$ for thin slabs, and $d = 1$ for thin fibres. The systems (3.14) - (3.17) and (3.27) - (3.30) can be written jointly as follows [14]

$$\left(1 + (\mu_1 + \mu_2) \frac{\partial}{\partial t}\right) \Delta_d \varphi + (3 - d) \left(\frac{\nu}{1 - \nu} + \mu_2 \frac{\partial}{\partial t}\right) w + \tau = k\ \varphi, \tag{3.31}$$

$$\left(\frac{\nu}{1 - \nu} + \mu_2 \frac{\partial}{\partial t}\right) \Delta_d \varphi + \left(\frac{1 + \nu\ (1 - d)}{1 - \theta} + (\mu_1 + (3 - d)\, \mu_2) \frac{\partial}{\partial t}\right) w + \tau = 0, \tag{3.32}$$

and

$$\frac{\partial c}{\partial t} = D \Delta_d c + f\,(c) + \psi\ (c) + \gamma \theta \tag{3.33}$$

with

$$\theta = \Delta_d \varphi + (3 - d)\, w, \tag{3.34}$$

where, we removed the subscript "0" by $\varphi_0, w_0, c_0, \theta_0, \tau_0$. If we add Eq. (3.31) to Eq. (3.32) multiplied by $3 - d$, we obtain

$$\left(\frac{1 + \nu\ (2 - d)}{1 - \nu} + (\mu_1 + \mu_2\,(4 - d)) \frac{\partial}{\partial t}\right) \theta + (4 - d)\, \tau = k \varphi, \tag{3.35}$$

whereas, subtracting Eq. (3.32) from Eq. (3.31) and using Eq. (3.34) we obtain

$$\left(\frac{1 - 2\nu}{1 - \tau} + \mu_1 \frac{\partial}{\partial t}\right) ((4 - d)\, \Delta_d \varphi - \theta) = k\,(3 - d)\, \varphi. \tag{3.36}$$

We look for solutions of the system (3.33) - (3.36) in the form of travelling waves:

$$(c, w, \varphi, \theta) = (c, w, \varphi)\,(\xi)\,, \quad \xi = x_1 - Ut, \tag{3.37}$$

where $U < 0$ is a constant, and the functions c, w, φ, θ must be found. We impose the following limit value conditions

$$\lim_{\xi \to -\infty} c\,(\xi) = 0, \quad \lim_{\xi \to \infty} c\,(\xi) = 1, \tag{3.38}$$

$$\lim_{\xi \to \mp\infty} (\varphi, \theta, u, w) = (0, 0, 0, 0)\,, \tag{3.39}$$

$$\lim_{\xi \to \mp\infty} \frac{d}{d\xi} (\varphi, \theta, c, u, w) = (0, 0, 0, 0, 0)\,. \tag{3.40}$$

Inserting (3.37) into Eqs (3.33) - (3.36) we obtain the following system of ordinary differential equations

$$D \frac{d^2 c}{d\xi^2} + U \frac{dc}{d\xi} + f\,(c) + \psi\ (c) + \gamma \theta = 0, \tag{3.41}$$

$$\theta = \frac{d^2 \varphi}{d\xi^2} + (3 - d)\, w, \tag{3.42}$$

$$\left(\frac{1 + \nu\ (2 - d)}{1 - \nu} - U\,(\mu_1 + \mu_2\,(4 - d)) \frac{d}{d\xi}\right) \theta + (4 - d)\, \tau = k \varphi, \tag{3.43}$$

$$\left(\frac{1 - 2\nu}{1 - \nu} - U \mu_1 \frac{d}{d\xi}\right) \frac{d^2 \varphi}{d\xi^2} - k \frac{3 - d}{4 - d} \varphi = \frac{1}{4 - d} \left(\frac{1 - 2\nu}{1 - \nu} - U \mu_1 \frac{d}{d\xi}\right) \theta. \tag{3.44}$$

4. Calcium waves in medium without restoring forces

Four equations (3.41) - (3.44) relate five quantities: c, u, w, τ, and θ. Hence, one of them must be treated as given. We consider two problems:

- Problem 4.1: the traction function τ is treated as the given quantity satisfying conditions (1.4)- find c,u,w and θ.
- Problem 4.2: the dilatation function θ is the given quantity - find c,u,w and τ. In this section we consider the simplest case when $k = 0$ in Eqs (3.41) - (3.44); the case $k > 0$ will be analyzed in the next section

Problem 4.1. We assume temporarily that the function of calcium concentration wave, $c = c(\xi)$, travelling with the speed $U < 0$ is known. Next, we assume that for any such a function the traction $\tau = \tau(c)$ is given. Eq. (3.44) in the case of $k = 0$ reduces to

$$\left(\frac{1-2\nu}{1-\nu} - U\mu_1\frac{d}{d\xi}\right)\left(\frac{d^2\varphi}{d\xi^2} - \frac{\theta}{4-d}\right) = 0, \tag{4.1}$$

whose bounded solution on real axis is identically equal to zero thus yielding

$$\frac{d^2\varphi}{d\xi^2} = \frac{\theta}{4-d}. \tag{4.2}$$

Using this we obtain the following expression for w

$$w = \frac{\theta}{4-d}. \tag{4.3}$$

When $k = 0$, the solution of Eq. (3.43) satisfying the limit conditions

$$\theta(-\infty) = \theta(\infty) = 0, \tag{4.4}$$

due to (1.4), is

$$\theta(\xi) = (4-d)\frac{1-\nu}{1+\nu(2-d)}\omega_0\, e^{\omega_0\,\xi}\int_{-\infty}^{\xi}\tau(c(\varsigma))\,e^{-\omega_0\,\varsigma}d\varsigma, \tag{4.5}$$

where

$$\omega_0 = \frac{1+\nu(2-d)}{(1-\nu)\,U\,(\mu_1+\mu_2(4-d))}. \tag{4.6}$$

The expression for the u-component of the displacement can be obtained directly as the result of integration of (4.2) with respect to ξ over the interval $(-\infty,\xi)$, but this integration can be complicated. Because of that we take a different way. Eq. (3.43) becomes an identity if we set in it $k = 0$ and for θ we take the formula (4.7). Integrating this identity over the interval $(-\infty, \xi)$, with the use of (1.4), we get

$$\frac{1+\nu(2-d)}{1-\nu}\int_{-\infty}^{\xi}\theta(\varsigma)\,d\varsigma - U(\mu_1+\mu_2(4-d))\,\theta + (4-d)\int_{-\infty}^{\xi}\tau(c(\varsigma))\,d\varsigma = 0. \tag{4.7}$$

By the definition of the potential φ we obtain from Eq. (4.2)

$$u = \frac{1}{4-d}\int_{-\infty}^{\xi}\theta(\varsigma)\,d\varsigma. \tag{4.8}$$

Making use of (4.8) in Eq. (4.7) we get finally

$$u(\xi) = -\frac{1-\nu}{1+\nu(2-d)}\int_{-\infty}^{\xi}\tau(c(\varsigma))\,d\varsigma + \frac{\theta}{\omega_0\,(4-d)}. \tag{4.9}$$

Inserting (4.5) into (4.3) and (4.8), we obtain closed expressions for the displacements.

Remark The displacement u does not take the form of a pulse since owing to (4.4) we obtain from (4.8)

$$u(\infty) = -\frac{1-\nu}{1+\nu(2-d)} \int_{-\infty}^{\infty} \tau(c(\varsigma))\, d\varsigma < 0.$$

If we insert Eq. (4.5) into Eq. (3.41) we receive an integro-differential equation for the calcium concentration c. A similar problem was considered by Flores *et al.* [6]. Applying their method it is possible to prove the solvability of such an equation at least for small values of γ, but constructing a closed expression for it is highly unrealistic. Hence we are forced to take such assumptions which will enable obtaining closed formulae, though approximate ones. To this end we assume that both viscosities of the medium are small. In such a case Eq. (4.5) gives

$$\begin{aligned} \theta &= -(4-d)\tfrac{1-\nu}{1+\nu(2-d)} \int_0^{\infty} e^{-\varsigma} \tau\left(c\left(\xi + \tfrac{\varsigma}{\omega_0}\right)\right) d\varsigma \\ &= -(4-d)\tfrac{1-\nu}{1+\nu(2-d)} \left[\tau(c) + \tfrac{1}{\omega_0}\tfrac{d\tau(c)}{d\xi}\right] + O\left(\omega_0^2\right). \end{aligned} \tag{4.10}$$

.

Using this approximate formula in Eq.(3.41) we obtain an ordinary differential equation of the form

$$D\frac{d^2c}{d\xi^2} + \left(U - \gamma\ (4-d)\frac{1-\nu}{1+\nu(2-d)}\frac{1}{\omega_0}\frac{d\tau}{dc}\right)\frac{dc}{d\xi} + f - \gamma(4-d)\frac{1-\nu}{1+\nu(2-d)}\tau = 0, \tag{4.11}$$

which must be solved subject to the conditions (3.38), (3.38). We will not enter into the solvability question of this equation since it is discussed elsewhere, but we present a class of its solutions. To this we take

Assumption 1.

1. The calcium concentration function $c = c(\xi)$ is a solution of the equation

$$\frac{dc}{d\xi} = pV(c), \tag{4.12}$$

   where p is positive parameter, which will be suitably chosen, and V is a function satisfying conditions (2.8) and (2.13),
2. The function $f(c) + \psi\ (c)$ is given by (2.11),
3. The traction is taken in the form

$$\tau = qV = \frac{q}{p}\frac{dc}{d\xi}, \tag{4.13}$$

where q is a positive constant. Inserting (2.11), (4.12), and (4.13) into Eq. (4.11) we obtain

$$\begin{aligned} &Dp^2VV_c' + p\left(U - q\gamma\ (4-d)\tfrac{1-\nu}{1+\nu(2-d)}\tfrac{1}{\omega_0}V_c'\right)V \\ &+V\left(V_c'(c_T) - V_c'\right) - q\gamma(4-d)\tfrac{1-\nu}{1+\nu(2-d)}\ V = 0. \end{aligned}$$

Hence

$$p = \sqrt{\frac{1 - q\gamma\ (\mu_1 + \mu_2(4-d))\left(V_c'(c_T) - q\gamma\frac{1-\nu}{1+\nu(2-d)}\right)\left(\frac{1-\nu}{1+\nu(2-d)}\right)^2}{D}}, \tag{4.14}$$

and

$$U=-\frac{V_c'\left(c_T\right)-q\gamma\frac{1-\nu}{1+\nu(2-d)}}{\sqrt{1-q\gamma\ \left(\mu_1+\mu_2\left(4-d\right)\right)\left(V_c'\left(c_T\right)-q\gamma\frac{1-\nu}{1+\nu(2-d)}\right)\left(\frac{1-\nu}{1+\nu(2-d)}\right)^2}}\sqrt{D}. \tag{4.15}$$

The parameter p given by (4.14) is real and the wave speed is negative provided that

$$0<q\gamma\ \left(\mu_1+\mu_2\left(4-d\right)\right)\left(V_c'\left(c_T\right)-q\gamma\frac{1-\nu}{1+\nu\left(2-d\right)}\right)<\left(\frac{1+\nu\left(2-d\right)}{1-\nu}\right)^2. \tag{4.16}$$

We notice that the mechanical effects can eliminate the existence of any calcium wave. This can happen if any of the conditions (4.16) is not satisfied. Next, comparing (2.7) and (4.12), (4.14) we note that the mechanical effects widen the profile of the wave.
Eqs (4.10), (4.13) imply

$$\theta=-\left(4-d\right)\frac{q}{p}\frac{1-\nu}{1+\nu\left(2-d\right)}\left(\frac{dc}{d\xi}+\frac{1}{\omega_0}\frac{d^2c}{d\xi^2}\right)+O\left(\omega_0^{-2}\right),\ \text{ as }\ \omega_0\to\infty, \tag{4.17}$$

whereas if we use jointly Eqs (4.5), (4.12) and (4.17) then

$$u=-\frac{q}{p}\frac{1-\nu}{1+\nu\left(2-d\right)}\left(c+\frac{1}{\omega_0}\frac{dc}{d\xi}+\frac{1}{\omega_0^2}\frac{d^2c}{d\xi^2}\right)+O\left(\omega_0^{-3}\right),\quad \text{as }\ \omega_0\to\infty. \tag{4.18}$$

Problem 4.2. In the previous subsection we assumed that the given quantity was the traction. An approach like this seems to be very natural: determine the motion provided that you know the acting forces. However, in our case things are more complicated, because no traction formula of experimental origin is known, and various models can be found in the literature [1]-[6]. Therefore, an alternative approach assuming the knowledge of a quantity different from the traction can be of some value. We suppose that the dilatation is known from the experimental measurements (as a function of ξ) and satisfies conditions (4.4). Knowing it, one has to determine the dependence of traction forces on the calcium concentration. Under this assumption and supposing that $k=0$, Eq. (3.44) reduces to Eq. (4.1), so the formula (4.2) holds in this case as well. Consequently, the equalities (4.3) and (4.9) hold true either, and Eq. (3.43) becomes now the definition of the expression for the traction τ. So, the problem is trivially solved. Therefore, we consider an example in which we assume that θ is given by

$$\theta=-\left(4-d\right)\frac{q}{p}\frac{1-\nu}{1+\nu\left(2-d\right)}\left(\frac{dc}{d\xi}+\frac{1}{\omega_0}\frac{d^2c}{d\xi^2}\right), \tag{4.19}$$

i.e. by the main part of (4.17), c satisfies Eq. (4.12), and f is given by (2.11). Then the formulae (4.3), (4.9) imply

$$w=-\frac{q}{p}\frac{1-\nu}{1+\nu\left(2-d\right)}\left(\frac{dc}{d\xi}+\frac{1}{\omega_0}\frac{d^2c}{d\xi^2}\right), \tag{4.20}$$

$$u=-\frac{q}{p}\frac{1-\nu}{1+\nu\left(2-d\right)}\left(c+\frac{1}{\omega_0}\frac{dc}{d\xi}\right), \tag{4.21}$$

respectively, and Eq. (3.43) gives

$$\tau=\frac{q}{p}\left(\frac{dc}{d\xi}-\frac{1}{\omega_0^2}\frac{d^3c}{d\xi^3}\right). \tag{4.22}$$

The traction $\tau(c)$ given by (4.22) satisfies condition (1.4). As we can see, formulae (4.19) - (4.22) for w, u, and τ coincide with their counterparts obtained in the previous subsection up to terms of second order of inverse powers of ω_0, which in the previous subsection (but not now) were assumed to be small. At last, if we use (4.19) in Eq. (3.41) we obtain the same expressions for the parameter p and the wave speed U as previously.

5. Calcium waves in medium with strong restoring forces

For sake of simplicity we consider only the case when the dilatation function $\theta = \theta(\xi)$ such that $\theta(-\infty) = \theta(\infty) = 0$, is given. Another words we will study a counterpart of Problem 4.2, defined in the previous section. We begin the analysis from solving Eq. (3.44) for $\varphi(\xi)$. This is a linear ordinary differential equation with constant coefficients, which we write as

$$A\frac{d^3\varphi}{d\xi^3} + B\frac{d^2\varphi}{d\xi^2} - C\varphi = \psi, \tag{5.1}$$

where the coefficients A,B,C are defined by

$$A = -U\mu_1 > 0, \quad B = \frac{1-2\nu}{1-\nu} > 0, \quad C = k\frac{3-d}{4-d} > 0, \tag{5.2}$$

and the function ψ is as follows

$$\psi\left(\xi\right) = \frac{1}{4-d}\left(\frac{1-2\nu}{1-\nu} - U\mu_1\frac{d}{d\xi}\right)\theta. \tag{5.3}$$

The characteristic equation of Eq. (5.1) reads:

$$A\,\lambda^3 + B\lambda^2 - C = 0. \tag{5.4}$$

By means of the substitution

$$\lambda = \frac{B}{A}\kappa \tag{5.5}$$

we give Eq. (5.4) a simpler form

$$\kappa^3 + \kappa^2 - \eta = 0, \tag{5.6}$$

where

$$\eta = \frac{A^2C}{B^3} = k\left(U\mu_1\right)^2\frac{3-d}{4-d}\left(\frac{1-2\nu}{1-\nu}\right)^{-3} \geq 0. \tag{5.7}$$

From the general theory of cubic algebraic equations we obtain:

Proposition 5.1. *The following properties hold:*

1. *If* $0 < \eta < \frac{4}{27}$, *then Eq.* (5.6) *has three distinct real roots*
2. *if* $\eta = 0$, *then Eq.* (5.6) *has three real roots: one negative* $\kappa_1 = -1$ *and a double root* $\kappa_2 = 0, \quad \kappa_3 = 0$ *; if* $\eta = \frac{4}{27}$, *then Eq.*(5.6) *has three real roots: one double negative root* $\kappa_1 = -\frac{2}{3}, \quad \kappa_2 = -\frac{2}{3}$, *and one positive root* $\kappa_1 = \frac{1}{3}$
3. *if* $\eta > \frac{4}{27}$, *then Eq.*(5.6) *has one real root and two complex conjugate roots.*

In what follows we will need more precise results, namely the signs of the real parts of these roots. We have

Proposition 5.2. *If* $0 < \eta < \frac{4}{27}$, *then Eq.*(5.6) *has three distinct real roots: two of them are negative and one is positive.*

Proof. Let us consider the asymptotic approximations to the roots as $\eta \downarrow 0$. They are

$$\begin{aligned} \kappa_1 &= -1 + \eta + O\left(\eta^2\right), \\ \kappa_2 &= -\sqrt{\eta} - \tfrac{1}{2}\eta + O\left(\eta^{\frac{3}{2}}\right), \\ \kappa_2 &= \sqrt{\eta} - \tfrac{1}{2}\mu + O\left(\eta^{\frac{3}{2}}\right). \end{aligned} \tag{5.8}$$

Hence, two roots are negative and one is positive. None of them can change sign, because then it should become equal to zero for some $0 < \eta_0 < \frac{4}{27}$. But this is impossible since $\kappa = 0$ does not satisfy Eq. (5.6) for positive η. The proof is complete. □

Proposition 5.3. *If $\eta > \frac{4}{27}$, then Eq.* (5.6) *has one real positive root and two complex conjugate roots with negative real parts.*

Proof. Let us consider the asymptotic approximations to the roots as $\eta \uparrow \infty$. They are

$$\begin{aligned} \kappa_1 &= \left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)\sqrt[3]{\eta} - \tfrac{1}{3} + O\left(\eta^{-\frac{1}{3}}\right), \\ \kappa_2 &= \left(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)\sqrt[3]{\eta} - \tfrac{1}{3} + O\left(\eta^{-\frac{1}{3}}\right), \\ \kappa_3 &= \sqrt[3]{\eta} - \tfrac{1}{3} + O\left(\eta^{-\frac{1}{3}}\right). \end{aligned} \tag{5.9}$$

Hence, for large values of η, the complex roots have negative real parts. They cannot change sign, because then they should become purely imaginary for some $\eta_0 > \frac{4}{27}$. However, this is impossible because Eq. (5.6) does not admit imaginary roots for real η. The real root cannot change its sign due to the same reason, which was used in the proof of Proposition 5.2. The proof is complete. □

For completeness we present the last group of the asymptotic expressions

$$\begin{aligned} \kappa_1 &= -\tfrac{2}{3} + \tfrac{1}{9}\sqrt{3\left(\tfrac{4}{27} - \eta\right)} + O\left(\left|\tfrac{4}{27} - \eta\right|\right), \\ \kappa_2 &= -\tfrac{2}{3} - \tfrac{1}{9}\sqrt{3\left(\tfrac{4}{27} - \eta\right)} + O\left(\left|\tfrac{4}{27} - \eta\right|\right), \\ \kappa_3 &= \tfrac{1}{3} + O\left(\left|\tfrac{4}{27} - \eta\right|\right), \end{aligned} \tag{5.10}$$

for η close to 4/27. In our future calculations the Viète's formulae for Eq.(5.4) will be used

$$\begin{aligned} &\lambda_1 + \lambda_2 + \lambda_3 = -\tfrac{B}{A}, \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0, \\ &\lambda_1\lambda_2\lambda_3 = \tfrac{C}{A}. \end{aligned} \tag{5.11}$$

We have to distinguish two cases: $0 < \eta \neq \frac{4}{27}$ and $\eta = \frac{4}{27}$. For the first case the solution of Eq. (5.1) subject to the conditions $\varphi(-\infty) = \varphi(\infty) = 0$ reads:

$$\begin{aligned} \varphi^{(m)}(\xi) = \tfrac{1}{A}\Big\{ &-\tfrac{\lambda_1^m}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)} e^{\lambda_1\xi} \textstyle\int_{-\infty}^{\xi} \psi(\zeta)\, e^{-\lambda_1\zeta} d\zeta \\ &+\tfrac{\lambda_2^m}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} e^{\lambda_2\xi} \textstyle\int_{-\infty}^{\xi} \psi(\zeta)\, e^{-\lambda_2\zeta} d\zeta \\ &-\tfrac{\lambda_3^m}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} e^{\lambda_3\xi} \textstyle\int_{\xi}^{\infty} \psi(\zeta)\, e^{-\lambda_3\zeta} d\zeta \Big\}, \quad m = 0, 1, 2. \end{aligned} \tag{5.12}$$

where $\varphi^{(m)}$ denotes the m-th derivative of φ with respect to ξ, $\varphi^{(0)} \equiv \varphi$. Analogously, the solution of Eq.(5.1) for $\eta = \frac{4}{27}$ is

$$\begin{aligned} \varphi^{(m)}(\xi) = \tfrac{1}{A}\Big\{ &\tfrac{\lambda_1^m - m\lambda_1^{m-1}(\lambda_1-\lambda_3)}{(\lambda_1-\lambda_3)^2} e^{\lambda_1\xi} \textstyle\int_{-\infty}^{\xi} \psi(\varsigma)\, e^{-\lambda_1\varsigma} d\varsigma \\ &+\tfrac{\lambda_1^m}{\lambda_1-\lambda_3} e^{\lambda_1\xi} \textstyle\int_{-\infty}^{\xi} \psi(\varsigma)(\varsigma-\xi)\, e^{-\lambda_1\varsigma} d\varsigma + \tfrac{\lambda_3^m}{(\lambda_1-\lambda_3)^2} e^{\lambda_3\xi} \textstyle\int_{\xi}^{\infty} \psi(\varsigma)\, e^{-\lambda_3\varsigma} d\varsigma \Big\}, \quad m = 0, 1, 2. \end{aligned} \tag{5.13}$$

Using (5.13) in (3.42) and (3.43) we obtain closed expressions for the functions w and τ, respectively. Also, we can obtain the displacement u since it is equal to the derivative of φ with respect to ξ. All these functions are determined uniquely and their asymptotic values are:
$u(-\infty) = u(\infty) = 0$, $w(-\infty) = w(\infty) = 0$ and $\tau(-\infty) = \tau(\infty) = 0$.

We collect our considerations in the form of

Theorem 5.4. *Given the dilatation function $\theta(\xi) = \Theta(c(\xi))$, $\Theta(0) = \Theta(1) = 0$ such that the function $h(c) \equiv f(c) + \psi(c) + \gamma\,\Theta(c)$ is bistable on the closed interval [0,1], and*

$$\int_0^1 [f(c) + \psi(c) + \gamma\,\Theta(c)]\,dc > 0,$$

then

1. *There exists only one value of the parameter $U < 0$ for which there exists a unique, up to a translation, solution of Eq.* (3.41) *satisfying the limit conditions* (3.38) - (3.40)*;*
2. *The functions $u(\xi)$, $w(\xi)$ and $\tau(\xi)$, $\xi \in (-\infty, \infty)$ are determined uniquely, up to a translation in ξ, and vanish at $\mp\infty$.*

The expressions (5.12) or (5.13) for φ and consequently, those for u ,w , τ are complicated nonlinear functionals acting on the calcium concentration $c(\xi)$, which, in turn, exists but its formula is not known. Hence, in order to shed some light on these functions we have to deduce approximate expressions for $\varphi(\xi)$ and its derivatives. This is our present aim.
First, assuming that $\psi \in C^2(\mathrm{R})$ we can rewrite the expression for (5.12) for the solution of Eq. (5.1) in the form

$$\begin{aligned}\varphi^{(m)}(\xi) = \tfrac{1}{A}\Big\{&-\tfrac{1}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}e^{\lambda_1\xi}\textstyle\int_{-\infty}^{\xi}\psi^{(m)}(\zeta)\,e^{-\lambda_1\zeta}d\zeta\\ &+\tfrac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}e^{\lambda_2\xi}\textstyle\int_{-\infty}^{\xi}\psi^{(m)}(\zeta)\,e^{-\lambda_2\zeta}d\zeta\\ &-\tfrac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{\lambda_3\xi}\textstyle\int_{\xi}^{\infty}\psi^{(m)}(\zeta)\,e^{-\lambda_3\zeta}d\zeta\Big\},\quad m=0,1,2.\end{aligned}\tag{5.14}$$

Using this representation of the solution one can prove the following.

Theorem 5.5. *Let:*

1. *$\psi \in C^4(R)$ and let $\eta_1 > \frac{4}{27}$ be an arbitrary but fixed constant,*
2. *There exists a constant $K > 0$ such that for every pair $(\xi, \varsigma) \in R \times R$ the following inequality*

$$\left|\psi^{(m)}(\varsigma) - \psi^{(m)}(\xi) - (\varsigma-\xi)\,\psi^{(m)}(\xi) - \frac{(\varsigma-\xi)^2}{2}\psi^{(m+2)}(\xi)\right| \le K\,|\varsigma-\xi|^{2+\alpha} \tag{5.15}$$

holds for some constant $\alpha > 0$. Then there is a constant $M > 0$ independent of A,B and C such that for every $\xi \in R$ one of the following estimates

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le KM\frac{B^{1+\frac{\alpha}{2}}}{C^{2+\frac{\alpha}{2}}} \quad \text{for} \quad 0 < \eta < \eta_1, \tag{5.16}$$

or

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le KM\frac{A^{\frac{2+\alpha}{3}}}{C^{\frac{5+\alpha}{3}}} \quad \text{for} \quad \frac{4}{27} < \eta_1 < \eta \tag{5.17}$$

is satisfied.

Proof. The proof of the estimates (5.16), (5.17) is a little bit lengthy therefore we moved it to the Appendix. □

We will use the estimates (5.16), (5.17) in the case when the parameter k characterizing the passive cell restoring force (i. e. the coefficient C in the notation (5.2)) is large, the Poisson ratio ν is fixed (i.e. the coefficient B), and the shear viscosity μ_1 is finite (the coefficient A). The estimates (5.16), (5.17) justify neglecting terms of order $o\left(k^{-1}\right)$ in (5.14), what leads to the following approximate formulae

1. for the longitudinal displacement

$$u = \frac{d\varphi}{d\xi} = -\frac{1}{k}\frac{1}{3-d}\left(\frac{1-2\nu}{1-\nu} - U\mu_1\frac{d}{d\xi}\right)\frac{d\theta}{d\xi}, \tag{5.18}$$

2. for the quantity w characterizing the transversal displacements (c.f. (3.41))

$$w = \frac{\theta}{3-d} - \frac{1}{k}\frac{1}{(3-d)^2}\left(\frac{1-2\nu}{1-\nu} - U\mu_1\frac{d}{d\xi}\right)\frac{d^2\theta}{d\xi^2}, \tag{5.19}$$

and for the traction

$$\tau = -\frac{1}{3-d}\frac{1+\nu(1-d)}{1-\nu}\left(\theta - \frac{1}{\omega_1}\frac{d\theta}{d\xi}\right), \tag{5.20}$$

where

$$\omega_1 = \frac{1+\nu(1-d)}{(1-\nu)\,U\left(\mu_1+\mu_2(3-d)\right)}. \tag{5.21}$$

Let us notice that, in the considered approximation, the traction does not depend on k . Hence, as it should be, it is not related to the passive restoring ones. Finally, we consider an example. We assume that points 1 and 2 of Assumption 1 hold, and point 3 is replaced by the following

$$\theta = -(3-d)\frac{q}{p}\frac{1-\nu}{1+\nu(1-d)}\left(\frac{dc}{d\xi} + \frac{1}{\omega_1}\frac{d^2c}{d\xi^2}\right), \tag{5.22}$$

where $q > 0$ is a parameter. Under the assumption (5.22) we obtain from Eq.(5.20) the following formula for the traction, which satisfies condition (1.4)

$$\tau = \frac{q}{p}\left(\frac{dc}{d\xi} - \frac{1}{\omega_1^2}\frac{d^3c}{d\xi^3}\right). \tag{5.23}$$

Let us note that $\omega_1 \to \infty$ under the additional assumption $\mu_1 + \mu_2 \to 0$. Hence, in such a case Eq. (5.23) coincides with (4.13) up higher order terms.

Inserting (4.12) and (5.22) into Eq. (3.41) we obtain as the result of the same procedure we used in Section 5. Problem 4.1 the following expression for the characteristic of the wave thickness (c.f. (4.13))

$$p = \sqrt{\frac{1 - q\gamma\ \left(\mu_1+\mu_2(3-d)\right)\left(V_c'(c_T) - q\gamma\frac{1-\nu}{1+\nu(1-d)}\right)\left(\frac{1-\nu}{1+\nu(1-d)}\right)^2}{D}}, \tag{5.24}$$

and for the wave speed (c.f. (4.14))

$$U = -\frac{V_c'(c_T) - q\gamma\frac{1-\nu}{1+\nu(1-d)}}{\sqrt{1 - q\gamma\ \left(\mu_1+\mu_2(3-d)\right)\left(V_c'(c_T) - q\gamma\frac{1-\nu}{1+\nu(1-d)}\right)\left(\frac{1-\nu}{1+\nu(1-d)}\right)^2}}\sqrt{D}. \tag{5.25}$$

The parameter p given by (5.23) is real and the wave speed is negative provided that

$$0 < q\gamma\ \left(\mu_1+\mu_2(3-d)\right)\left(V_c'(c_T) - q\gamma\frac{1-\nu}{1+\nu(1-d)}\right) < \left(\frac{1+\nu(1-d)}{1-\nu}\right)^2. \tag{5.26}$$

Appendix. Proof of Theorem 5.5

By the Viète's formulae (5.11) we can give the expression (5.14) the following form

$$\varphi^{(m)}(\xi) = -\frac{\psi^{(m)}(\xi)}{C} +$$
$$\frac{1}{A}\left\{-\frac{1}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}\int_{-\infty}^{\xi} e^{-\lambda_1(\varsigma-\xi)}\left[\psi^{(m)}(\varsigma)-\psi^{(m)}(\xi)-(\varsigma-\xi)\psi^{(m+1)}(\xi)-\frac{1}{2}(\varsigma-\xi)^2\psi^{(m+2)}(\xi)\right]d\varsigma\right.$$
$$+\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\int_{-\infty}^{\xi} e^{-\lambda_2(\varsigma-\xi)}\left[\psi^{(m)}(\varsigma)-\psi^{(m)}(\xi)-(\varsigma-\xi)\psi^{(m+1)}(\xi)-\frac{1}{2}(\varsigma-\xi)^2\psi^{(m+2)}(\xi)\right]d\varsigma$$
$$\left.-\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\int_{\xi}^{\infty} e^{-\lambda_3(\varsigma-\xi)}\left[\psi^{(m)}(\varsigma)-\psi^{(m)}(\xi)-(\varsigma-\xi)\psi^{(m+1)}(\xi)-\frac{1}{2}(\varsigma-\xi)^2\psi^{(m+2)}(\xi)\right]d\varsigma\right\},$$
$$m=0,1,2. \tag{.1}$$

We must consider some cases.

Case 1. $\eta \in (0, \eta_0]$, where η_0 is an arbitrary fixed real number such that $0 < \eta_0 < \frac{4}{27}$. Owing to Assumption 2 of the Theorem we can write

$$\left|\varphi^{(m)}(\xi)+\frac{f^{(m)}(\xi)}{C}\right| \le \frac{K}{A}\left\{\frac{1}{|\lambda_2-\lambda_1||\lambda_1-\lambda_3|}\int_{-\infty}^{\xi}\left|e^{-\lambda_1(\varsigma-\xi)}\right|(\xi-\varsigma)^{2+\alpha}d\varsigma\right.$$
$$\left.+\frac{1}{|\lambda_2-\lambda_1||\lambda_2-\lambda_3|}\int_{-\infty}^{\xi}\left|e^{-\lambda_2(\varsigma-\xi)}\right|(\xi-\varsigma)^{2+\alpha}d\varsigma-\frac{1}{|\lambda_1-\lambda_3||\lambda_2-\lambda_3|}\int_{\xi}^{\infty}e^{-\lambda_3(\varsigma-\xi)}(\varsigma-\xi)^{2+\alpha}d\varsigma\right\}, \tag{.2}$$
$$m=0,1,2.$$

For $\eta \in (0, \eta_0]$, $\lambda_1, \lambda_2$are negative real numbers, and λ_3 is positive (Proposition 4.1). Evaluating the integrals in (.2) we obtain

$$\left|\varphi^{(m)}(\xi)+\frac{\psi^{(m)}(\xi)}{C}\right| \le K\frac{B^{1+\frac{\alpha}{2}}}{C^{2+\frac{\alpha}{2}}}F_1(\eta), \tag{.3}$$

where $\Gamma(x)$ is the Euler gamma function:

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt,$$

and

$$F_1(\eta)=\Gamma(3+\alpha)\,\eta^{2+\frac{\alpha}{2}}\left[\frac{1}{(\kappa_2-\kappa_1)(\kappa_3-\kappa_1)\kappa_1^{3+\alpha}}+\frac{1}{(\kappa_2-\kappa_1)(\kappa_3-\kappa_2)\kappa_2^{3+\alpha}}\right.$$
$$\left.+\frac{1}{(\kappa_3-\kappa_1)(\kappa_3-\kappa_2)\kappa_3^{3+\alpha}}\right].$$

By the expansions (5.8), the function $F_1(\eta)$ is bounded on the considered interval. Hence, there is a constant $M = M(\eta_0)$ such that $0 < F_1(\eta) \le M, \quad 0 < \eta \le \eta_0 < \frac{4}{27}$. Therefore we conclude that the estimate (5.16) is proved in this case.

Case 2. $\eta \in \left[\eta_0, \frac{4}{27}\right)$.

The denominator of the first two terms in (5.14) contain the difference $\lambda_2 - \lambda_1$, which in the case under consideration can be a small quantity. Because of that we must rewrite it in a little bit different form showing that these singularities cancel and that the limit as $\lambda_2 \to \lambda_1$ or, equivalently, as $\eta \to \frac{4}{27}$ of expression (5.14) is finite. The new form is as follows

$$\varphi^{(m)}(\xi) = -\frac{\psi^{(m)}(\xi)}{C} +$$

$$\frac{1}{a}\left\{-\frac{1}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}\int_{-\infty}^{\xi} e^{-\lambda_1(\varsigma-\xi)}\left[f^{(m)}(\varsigma) - f^{(m)}(\xi) - (\varsigma-\xi) f^{(m+1)}(\xi) - \frac{1}{2}(\varsigma-\xi)^2 f^{(m+2)}(\xi)\right]d\varsigma\right.$$

$$+\frac{1}{(\lambda_2-\lambda_3)}\int_{-\infty}^{\xi}\frac{e^{-\lambda_1(\varsigma-\xi)}-e^{-\lambda_2(\varsigma-\xi)}}{\lambda_2-\lambda_1}\left[f^{(m)}(\varsigma) - f^{(m)}(\xi) - (\varsigma-\xi) f^{(m+1)}(\xi) - \frac{1}{2}(\varsigma-\xi)^2 f^{(m+2)}(\xi)\right]d\varsigma$$

$$\left.-\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\int_{\xi}^{\infty} e^{-\lambda_3(\varsigma-\xi)}\left[f^{(m)}(\varsigma) - f^{(m)}(\xi) - (\varsigma-\xi) f^{(m+1)}(\xi) - \frac{1}{2}(\varsigma-\xi)^2 f^{(m+2)}(\xi)\right]d\varsigma\right\},$$

$$m = 0,1,2. \tag{.4}$$

For η from this interval, λ_1, λ_2 are negative real numbers and λ_3 is positive. Applying Assumption 2 we obtain the following estimate:

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le \frac{K}{a}\left\{\frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}\int_{-\infty}^{\xi} e^{-\lambda_1(\varsigma-\xi)}(\xi-\varsigma)^{2+\alpha}d\varsigma\right.$$

$$\left.+\frac{1}{(\lambda_3-\lambda_2)}\int_{-\infty}^{\xi}\frac{e^{-\lambda_1(\varsigma-\xi)}\left(1-e^{-(\lambda_2-\lambda_1)(\varsigma-\xi)}\right)}{(\lambda_2-\lambda_1)(t-\varsigma)}(\xi-\varsigma)^{3+\alpha}d\varsigma + \frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}\int_{\xi}^{\infty} e^{-\lambda_3(\varsigma-\xi)}(\xi-\varsigma)^{2+\varepsilon}d\varsigma\right\},$$

$$m = 0,1,2. \tag{.5}$$

Using the inequality $1-e^{-x} \le x, \quad \text{for} \quad x \ge 0$ and performing the necessary integrations we obtain

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le \Gamma(3+\alpha)\frac{K}{A}\left[\frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}\left(\frac{1}{\lambda_1^{3+\alpha}} + \frac{1}{\lambda_3^{3+\alpha}}\right) + \frac{3+\alpha}{(\lambda_3-\lambda_2)}\frac{1}{\lambda_1^{4+\alpha}}\right]$$

$$= KM\frac{B^{1+\frac{\alpha}{2}}}{C^{2+\frac{\alpha}{2}}}F_2(\eta), \quad m = 0,1,2,$$

where

$$F_2(\eta) = \Gamma(3+\alpha)\frac{\eta^{2+\frac{\alpha}{2}}}{\kappa_3-\kappa_2}\left[\frac{1}{\kappa_3-\kappa_1}\left(\frac{1}{\kappa_1^{3+\alpha}} + \frac{1}{\kappa_3^{3+\varepsilon}}\right) + \frac{3+\alpha}{\kappa_1^{4+\alpha}}\right] \le M_3 = M_3(\eta_0)$$

is a function bounded for $\eta \in \left[\eta_0, \frac{4}{27}\right)$ by a constant denoted as M_2. Hence, also in this case the inequality (5.16) is proved. This estimate being true for $\eta < \frac{4}{27}$ holds true for the limit as $\eta \to \frac{4}{27}$.

Case 3. $\eta \in \left(\frac{4}{27}, \eta_1\right]$. The inequality (.8) implies

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le \frac{K}{A}\left\{-\frac{1}{|\lambda_2-\lambda_3|^2}\int_{-\infty}^{t} e^{-Re\lambda_1(\varsigma-\xi)}(\xi-\varsigma)^{2+\alpha}d\varsigma\right.$$

$$\left.+\frac{1}{|\lambda_2-\lambda_3|}\int_{-\infty}^{\xi}\frac{e^{-Re\lambda_1(\varsigma-\xi)}\left|1-e^{-(\lambda_2-\lambda)_1(\varsigma-\xi)}\right|}{|\lambda_2-\lambda_1|(t-\varsigma)}(\zeta-\varsigma)^{3+\alpha}d\varsigma - \frac{1}{|\lambda_2-\lambda_3|^2}\int_{\xi}^{\infty} e^{-\lambda_3(\varsigma-\xi)}(\varsigma-\xi)^{2+\alpha}d\varsigma\right\}.$$

We make use of the inequality $\left|1-e^{ix}\right| \le x \quad \text{for} \quad x \ge 0$ and performing simple integrations we get

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \le KM\frac{B^{1+\frac{\alpha}{2}}}{C^{2+\frac{\alpha}{2}}}F_3(\eta),$$

where

$$F_3(\eta) = \Gamma(3+\alpha)\eta^{2+\frac{\alpha}{2}}\left[\frac{1}{(\omega+\kappa_3)^2+\beta^2}\left(\frac{1}{\omega^{3+\alpha}} + \frac{1}{\kappa_3^{3+\alpha}}\right) + \frac{3+\alpha}{\omega^{4+\alpha}\sqrt{(\omega+\kappa_3)^2+\beta^2}}\right],$$

$$\eta \in \left[\frac{4}{27}, \eta_1\right]$$

is a bounded function of the variable $\eta \in \left(\frac{4}{27}, \eta_1\right)$. Hence, the estimate (5.16) is proved.

Case 4. $\eta \in [\eta_1, \infty)$.

Under this assumption λ_1, λ_2 are complex conjugate numbers with negative real part. That is why

$$\left|e^{-\lambda_1(\varsigma - t)}\right| = \left|e^{-\lambda_2(\varsigma - t)}\right| = e^{Re\lambda_1(\varsigma - t)}. \tag{.6}$$

We write

$$\kappa_1 = -\omega + i\beta, \quad \kappa_2 = -\omega - i\beta, \quad \omega, \beta > 0 \tag{.7}$$

Evaluating the integrals present in (.2) and using the notation (.7) we can write

$$\left|\varphi^{(m)}(\xi) + \frac{\psi^{(m)}(\xi)}{C}\right| \leq KM \frac{A^{\frac{2+\alpha}{3}}}{C^{\frac{5+\alpha}{3}}} F_4(\eta) \tag{.8}$$

where

$$F_2(\eta) = \Gamma(3+\alpha)\,\eta^{\frac{5+\alpha}{3}} \left[\frac{2}{\omega^{3+\alpha}\beta\sqrt{(\omega+\kappa_3)^2+\beta^2}} + \frac{1}{(\omega+\kappa_3)^2+\beta^2}\frac{1}{\kappa_3^{3+\alpha}}\right], \quad \eta \in [\eta_1, \infty).$$

By the equalities (5.9) we conclude that the function $F_4(\eta)$ is bounded on the considered interval. Thus, there exists a constant $M_4 = M_2(\eta_1)$ that $0 < F_4(\eta) \leq M_4$. Hence, this and (.7) imply that the estimate (5.17) holds true in this case as well. The proof of the theorem is complete.

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References

[1] J. D. Murray, G. F. Oster. *Generation of biological pattern and form.* IMA Journal of Mathematics Applied in Medicine and Biology, 1 (1984), 51-75.

[2] D, C. Lane, J. D. Murray, V. S. Manoranjan. *Analysis of wave phenomena in a morphogenic mechanochemical model and an application to post-fertilization waves in eggs.* IMA Journal of Mathematics Applied in Medicine and Biology, 4 (1987), 309-331.

[3] C. Brière, B.C. Goodwin. *Effects of calcium input/output on the stability of a system for calcium-regulated viscoelastic strain fields,* Journal of Mathematical Biology, vol. 28, 585-593, 1990.Hart

[4] B. Kaźmierczak, M. Dyzma. *Mechanical effects coupled with calcium waves.* Archives of Mechanics, 62(2) (2010), 121-133.

[5] B. Kaźmierczak, Z. Peradzyński. *Calcium waves with fast buffers and mechanical effects,* Journal of Mathematical Biology, 62 (2011), 1-38.

[6] G. Flores, A. Minzoni, K.Mischaikov, V. Moll. *Post-fertilization travelling waves on eggs, Nonlinear Analysis: Theory, Methods and Applications.* 36(1) (1999), 45-62.

[7] J. D. Murray. *Mathematical Biology.* 2^{nd} ed. Springer, Berlin, 1993.

[8] J. Keener, J. Sneyd. *Mathematical Physiology.* Springer, New York, 1998

[9] P. Fife. *Mathematical Aspects of Reacting and Diffusing Systems.* Lecture Notes in Biomathematics, Vo. 28, Springer, New York, 1979.

[10] B. Kaźmierczak, V. Volpert. *Existence of heteroclinic orbits for systems satisfying monotonicity conditions.* Nonlinear Analysis, 55 (2003), 467-491

[11] B. Kaźmierczak, V. Volpert. *Travelling calcium waves in systems with non diffusing buffers.* Math. Mod. Meth. Appl. Sci., 18 (2008), 883-912.

[12] B. Kaźmierczak, V. Volpert. *Calcium waves in systems with immobile buffers a a limit of waves for systems with non zero diffusion.* Nonlinearity 21 (2008), 71-96

[13] F. J. Richards. *A flexible growth function for empirical use.* Journal of Experimental Botany, 10(1959), 290-300.

[14] Z. Peradzyński. *Diffusion of calcium in biological tissues and accompanying mechano-chemical effects,* Archives of Mechanics, 62(2) (2010), 423-440.

[15] K. Piechór. *Reaction-diffusion equation modelling calcium waves with fast buffering in visco-elastic environment.* Archives of Mechanics, 64(5) (2012), 477-509.