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Dynamical Features in a Slow-fast Piecewise Linear Hamiltonian System

A. Kazakov[1], N. Kulagin[2], L. Lerman[1] *
[1] Lobachevsky State University of Nizhni Novgorod, Russia
[2] The State University of Management, Moscow, Russia

Abstract. We demonstrate that a piecewise linear slow-fast Hamiltonian system with an equilibrium of the saddle-center type can have a sequence of small parameter values for which a one-round homoclinic orbit to this equilibrium exists. This contrasts with the well-known findings by Amick and McLeod and others that solutions of such type do not exist in analytic Hamiltonian systems, and that the separatrices are split by the exponentially small quantity. We also discuss existence of homoclinic trajectories to small periodic orbits of the Lyapunov family as well as symmetric periodic orbits near the homoclinic connection. Our further result, illustrated by simulations, concerns the complicated structure of orbits related to passage through a non-smooth bifurcation of a periodic orbit.

Keywords and phrases: slow-fast system, Hamiltonian, piecewise linear, homoclinic, periodic, nonsmooth bifurcation

Mathematics Subject Classification: 34A36, 34C23, 34C37, 34C28, 34E15, 70K70, 74H65

1. Introduction

We consider a singularly perturbed Hamiltonian differential system (sometimes such systems are called *slow-fast*):

$$\varepsilon\dot{x}_1 = \frac{\partial H}{\partial y_1},\ \varepsilon\dot{y}_1 = -\frac{\partial H}{\partial x_1},\ \dot{x}_2 = \frac{\partial H}{\partial y_2},\ \dot{y}_2 = -\frac{\partial H}{\partial x_2},$$

where H is the Hamiltonian: usually a smooth or even an analytic function of all its variables, and ε is a small positive parameter. Such systems serve as mathematical models for many phenomena in physics. In particular, one can reduce to this form the equation which arises in the theory of surface water waves when a weak surface tension is taken into account [2].

$$\varepsilon^2 u'''' + u'' + F^{'}(u) = 0,\quad F = -u^2/2 + u^3/3. \tag{1.1}$$

For this equation the following question was posed in [2], and is of substantial interest for many similar equations (see, for instance [1, 8]): on formally setting $\varepsilon = 0$, the second order differential equation is obtained. The latter equation has a solution $u(x)$, which decays to zero at $|x| \to \infty$ (solitary wave).

*Corresponding author. E-mail: lermanl@mm.unn.ru

Then a natural question arises: does the initial equation possess such solutions for small positive ε? The question turned out to be rather difficult, namely for small $\varepsilon > 0$ the equation always has the solution that decays to zero as $x \to \infty$ as well as the solution that decays to zero as $x \to -\infty$. In the four-dimensional phase space of the (Hamiltonian) system related to this equation (see below) these two solutions correspond to orbits on the, respectively, stable and unstable separatrices of the equilibrium of the saddle-center type. If a two-dimensional cross-section is chosen within the level of the Hamiltonian to which all these separatrices belong, the separatrices cut this cross-section at two points which are separated by the distance, exponentially small with respect to the value of $\varepsilon > 0$. The assertion on its exponential smallness follows from [20] but its positivity was proven for the first time in [3]. Thus for a given analytic system a solution of the type we seek is absent.

In this paper we show that for a piecewise smooth system, even for a piecewise linear continuous one, the effect of separatrix splitting indeed occurs, but not for all small values of $\varepsilon > 0$: there is a sequence of values $\varepsilon_n \to +0$ for which the system has a solitary wave, or the related Hamiltonian system in the four-dimensional phase space has a homoclinic orbit to the saddle-center equilibrium. We also show that main ingredients of the orbit behavior for a smooth Hamiltonian with a loop to the saddle-center are preserved in the piecewise linear continuous system. In the last section we discuss also some elements of the behavior that are implied by the absence of smoothness.

We study the equation (1.1) with a continuous piecewise linear function F':

$$F'(u) = \begin{cases} -u/a, \text{if } u \le a; \\ (u-1)/(1-a), \text{if } u \ge a. \end{cases}$$

Henceforth we assume that the parameter a varies in the interval $0 < a < 1$.

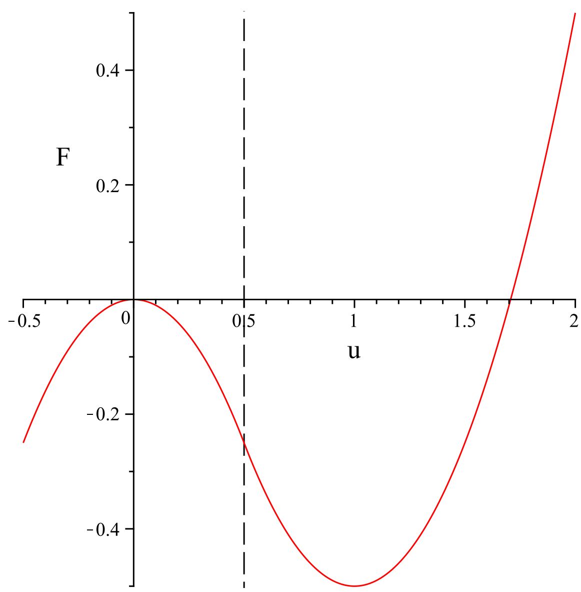


FIGURE 1. Graph of $F(u)$, $a = 0.5$

Below, for convenience, we denote the independent variable by t instead of x. The equation under study belongs to the class of higher Euler-Lagrange-Poisson equations [4]. It is reduced to the Hamiltonian form by the change of variables

$$u = x_2,\ u' = x_1,\ \varepsilon u'' = y_1,\ -u' - \varepsilon^2 u''' = y_2,$$

so that the system

$$\varepsilon x_1' = y_1, \ \ \varepsilon y_1' = -x_1 - y_2, \ x_2' = x_1, \ y_2' = F'(x_2) \tag{1.2}$$

with the Hamiltonian $H = (x_1^2 + y_1^2)/2 + x_1 y_2 - F(x_2)$ is obtained.

Right hand sides of equations are continuous piecewise linear functions, therefore the Lipschitz condition is satisfied for the system and the standard existence and uniqueness theorem holds. Solutions are differentiable functions of the independent variable t. The orbits of linear systems coming from both half-spaces $x_2 < a$, $x_2 > a$ upon the switching hyperplane $x_2 = a$ without tangency are continued through this plane, giving smooth orbits. We emphasize that for such orbits the Poincaré maps or successor maps on an appropriate cross-sections are smooth. For orbits which are tangent to the switching plane (grazing orbits) the situation is more complicated and requires a special study. Here the Poincaré map is not smooth and methods of non-smooth systems should be utilized [6]. Especially, this concerns the study of bifurcations where for smooth systems one needs to calculate certain quantities depending on higher derivatives, like the coefficients of normal forms. In the system under study the tangency of orbits on the switching plane takes place along the 2-dimensional plane $x_1 = 0, x_2 = a$. We will examine such bifurcations later on.

It is worth remarking that for the properly chosen constants for both half-spaces, the Hamiltonian becomes C^1-smooth function of its variables, hence its value is preserved along the orbit as in usual smooth Hamiltonian systems.

For small ε the system under consideration is a singularly perturbed or a slow-fast system [5, 13]. For such systems it is sometimes useful to consider two related degenerate systems that are derived in the singular limit $\varepsilon \to 0$. This limit can be reached by two options and it yields slow and fast systems. The slow system is a two-dimensional system (the Hamiltonian one in this case) that is obtained, if one sets formally $\varepsilon = 0$ in the two first equations, and from these two algebraic equations expresses the fast variables (x_1, y_1) through the slow ones (x_2, y_2). The resulting functions are inserted into r.h.s. of the third and the fourth equations of (1.2). To derive the fast system, one should start with a linear rescaling of time units $t/\varepsilon = \tau$, then the small parameter ε enters the last two equations as a factor. Then, on setting $\varepsilon = 0$, the slow variables x_2, y_2 become two parameters, and the first two equations yield the two-dimensional Hamiltonian system (here, the linear one) which depends on these parameters, here y_2^0. In the fast system the relations $y_1 = 0, x_1 = -y_2^0$ give the coordinates of the equilibrium, being a center, and all orbits of the fast system are periodic. In the whole phase space the plane of slow motions for the system with the fast time τ and $\varepsilon = 0$ is the plane of equilibria and 2-planes $x_2 = x_2^0, y_2 = y_2^0$ define invariant foliation of the phase space; on each plane with the fixed y_2^0 there is a system with a unique center. As we will see later, the structure of the system with non-zero ε is essentially different.

One of our goals will be the study of the system near the object called a "ghost separatrix loop" [13]. It is the closure of a homoclinic orbit on the slow manifold, which is easily found at $\varepsilon = 0$. This separatrix loop is called a "ghost" since it may not really exist in the whole system for small $\varepsilon > 0$.

It is worth mentioning that the system (1.2), in addition to being Hamiltonian, is reversible w.r.t. involution L:

$$L : (x_1, x_2, y_1, y_2) \to (-x_1, x_2, y_1, -y_2),$$

that is the system is transformed to itself under the change $X \to L(X)$ and the simultaneous change $t \to -t$. The set $Fix(L)$ of fixed points of the involution forms the plane $x_1 = y_2 = 0$. Recall that the orbits cutting this plane are called symmetric. In particular, the equilibria which belong to $Fix(L)$ are symmetric. If the orbit intersects $Fix(L)$ at a unique point, this orbit is symmetric nonclosed (for instance, a separatrice loop of a symmetric equilibrium). The orbit intersects $Fix(L)$ at two points if and only if it is periodic [15]. For this case intersection points divide the orbit into two pieces interchangeable by the involution.

In our study we essentially use the ideas and methods developed for non-slow-fast smooth Hamiltonian systems near a separatrix loop of the saddle-center [17] (see also [14, 16, 19]).

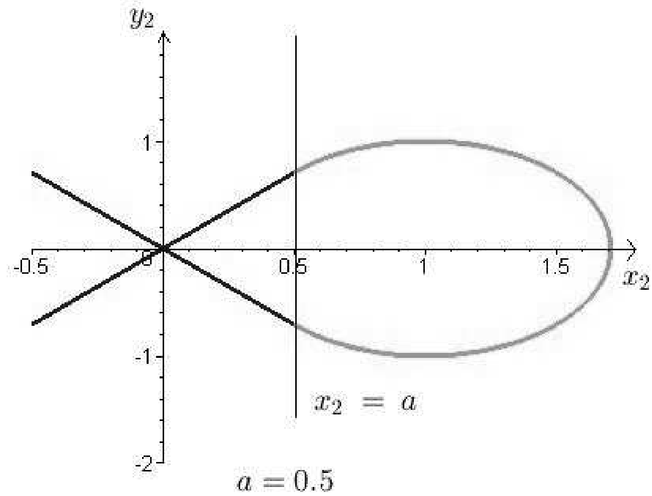


FIGURE 2. Separatrix loop on the slow manifold

2. The structure of slow system

Let us consider first the structure of the slow system. To this purpose we set $\varepsilon = 0$ in (1.2) and identify the plane of slow motions $y_1 = 0$, $x_1 = -y_2$. On inserting these relations into the third and fourth equations of the system, we get a reversible Hamiltonian system with one degree of freedom, i.e. the slow system:

$$x_2' = -y_2,\ y_2' = F^{'}(x_2).$$

The function $F'(x_2)$ has different representations on the half-planes $x_2 \le a$, $x_2 \ge a$. For $x_2 \le a$ one has

$$x_2' = -y_2,\ y_2' = -x_2/a. \tag{2.1}$$

The system (2.1) has a saddle equilibrium at the point $(0, 0)$ and the Hamiltonian

$$h = \frac{x_2^2}{2a} - \frac{y_2^2}{2}.$$

Separatrices correspond to $h = 0$, i.e. coincide with two half-lines $x_2 = \pm\sqrt{a}y_2$ restricted to the half-plane $x_2 \le a$.

For $x_2 \ge a$ the system is as follows:

$$x_2^{'} = -y_2,\ y_2^{'} = (x_2 - 1)/(1 - a). \tag{2.2}$$

It has a center at the point $(1, 0)$, its orbits are arcs of ellipses $(x_2 - 1)^2 + (1 - a)y_2^2 = c_2 \ge 0$ restricted to the half-plane $x_2 \ge a$. Whenever an ellipse intersects the line $x_2 = a$, the orbit is continued by the orbit of the left system passing through the related point (provided that the ellipse has no tangency with the line $x_2 = a$).

Existence of a homoclinic orbit to the saddle for the slow system is easily seen. The point of intersection of the outgoing (to the right side) separatrix with the switching line $x_2 = a$ is $(a, -\sqrt{a})$, and the ellipse of the right system through this point with $c_2 = 1 - a$ intersects the line $x_2 = a$ for the second time at the symmetric point $(a, \sqrt{a})$ through which the incoming separatrix is passing (see Fig. 2).

Each orbit inside the loop is symmetric periodic, it encloses the center. The orbits lying outside of the loop are symmetric non-closed, they leave any compact set on the plane. For our further goals it is useful to compute the passage time for a homoclinic orbit from the point $(a, -\sqrt{a})$ till it cuts the symmetry line $y_2 = 0$. This time is

$$T_0 = \sqrt{1-a}\left[\pi - \arctan(\sqrt{\frac{a}{1-a}})\right].$$

3. On orbits of the system at $\varepsilon > 0$

Now we consider the whole system ($\varepsilon > 0$) in the half-space $x_2 \le a$

$$\varepsilon x_1^{'} = y_1,\ \varepsilon y_1^{'} = -x_1 - y_2,\ x_2^{'} = x_1,\ y_2^{'} = -x_2/a, \tag{3.1}$$

with the Hamiltonian $H = (x_1^2 + y_1^2)/2 + x_1 y_2 + x_2^2/2a$. The point $O = (0,0,0,0)$ is an equilibrium of the saddle-center type, that is, it has a pair of nonzero real eigenvalues and a pair of pure imaginary ones. Its characteristic equation is

$$\varepsilon^2\lambda^4 + \lambda^2 - a^{-1} = 0$$

with the roots

$$\pm\Lambda = \pm\sqrt{\frac{\sqrt{1+4\varepsilon^2 a^{-1}} - 1}{2\varepsilon^2}}, \quad \pm i\lambda = \pm i\sqrt{\frac{\sqrt{1+4\varepsilon^2 a^{-1}} + 1}{2\varepsilon^2}}.$$

So, a pair of nonzero real eigenvalues tend to eigenvalues of the slow system as $\varepsilon \to 0$, but the pair of fast pure imaginary ones are of the order $1/\varepsilon$. Function $\Lambda(\varepsilon)$ is an analytic function of ε in a neighborhood of $\varepsilon = 0$, $\Lambda(0) = 1/\sqrt{a}$.

The system under study has the following solutions

$$\begin{aligned}
x_1 &= \Lambda[c_1 e^{\Lambda t} - c_2 e^{-\Lambda t}] - \lambda[m\sin(\lambda t) + l\cos(\lambda t)],\\
x_2 &= c_1 e^{\Lambda t} + c_2 e^{-\Lambda t} + m\cos(\lambda t) - l\sin(\lambda t),\\
y_1 &= \varepsilon\Lambda^2[(c_1 e^{\Lambda t} + c_2 e^{-\Lambda t}] - \varepsilon\lambda^2[m\cos(\lambda t) - l\sin(\lambda t)],\\
y_2 &= -\Lambda(1+\varepsilon^2\Lambda^2)[c_1 e^{\Lambda t} - c_2 e^{-\Lambda t}] + \lambda(1-\varepsilon^2\lambda^2)[m\sin(\lambda t) + l\cos(\lambda t)],
\end{aligned}$$

and an additional integral $K = \Lambda^2(\varepsilon^2\lambda^2 x_2 + \varepsilon y_1)^2 - (\varepsilon^2\Lambda^2 x_1 - y_2)^2$. Since O is the center-saddle, there is a family of Lyapunov periodic orbits corresponding to pure imaginary eigenvalues. This family is given by setting $c_1 = c_2 = 0$ and is described by $(2\pi/\lambda)$-periodic functions

$$\begin{aligned}
x_1 &= -\lambda[m\sin(\lambda t) + l\cos(\lambda t)],\\
x_2 &= m\cos(\lambda t) - l\sin(\lambda t),\\
y_1 &= -\varepsilon\lambda^2[m\cos(\lambda t) - l\sin(\lambda t)],\\
y_2 &= \lambda(1-\varepsilon^2\lambda^2)[m\sin(\lambda t) + l\cos(\lambda t)],
\end{aligned}$$

or in a more compact form with $\cos\varphi = m/\sqrt{m^2+l^2}$, $\sin\varphi = l/\sqrt{m^2+l^2}$:

$$\begin{aligned}
x_1 &= -\lambda\sqrt{m^2+l^2}\sin(\lambda t + \varphi),\\
x_2 &= \sqrt{m^2+l^2}\cos(\lambda t + \varphi),\\
y_1 &= -\varepsilon\lambda^2\sqrt{m^2+l^2}\cos(\lambda t + \varphi),\\
y_2 &= \lambda(1-\varepsilon^2\lambda^2)\sqrt{m^2+l^2}\sin(\lambda t + \varphi).
\end{aligned} \tag{3.2}$$

Each Lyapunov periodic orbit is the saddle one on the related level of Hamiltonian, and its stable and unstable manifold are both cylinders. The union of stable cylinders along with the stable manifold (a line) of the saddle-center itself (when $c_1 = m = l = 0$) makes up the center-stable manifold W^{cs} of the saddle-center ($c_1 = 0$). Similarly, one obtains the center-unstable manifold W^{cu} ($c_2 = 0$). Excluding time-dependent functions, we get the 3-plane corresponding to W^{cu} (more precisely, to its part lying in the half-space $x_2 < a$):

$$(1-\varepsilon^2\lambda^2)x_1 + \Lambda\varepsilon^2\lambda^2 x_2 + \varepsilon\Lambda y_1 + y_2 = 0.$$

Lyapunov periodic orbit γ_c as well its stable and unstable manifolds belong to the level $H = c > 0$. The value of c is bounded from above by the value at which the related Lyapunov periodic orbit touches the 3-plane $x_2 = a$. To get the explicit expression for the unstable manifold of γ_c, let us find y_2 from the equation $H = c$ as a function of x_1, x_2, y_1:

$$y_2 = \frac{2c - x_1^2 - y_1^2 - x_2^2/a}{2x_1}. \tag{3.3}$$

Now we insert this function into the equation for the center-unstable manifold and set there $x_2 = a$. Then we get a curve in the 3-plane $x_2 = a$ lying on the trace of level $H = c$ in 3-plane $x_2 = a$; this curve is the trace of $W^u(\gamma_c)$. Its projection onto the plane (x_1, y_1) is of the form

$$2(1-\varepsilon^2\lambda^2)x_1^2 + 2\Lambda\varepsilon^2\lambda^2 a x_1 + 2\varepsilon\Lambda x_1 y_1 + 2c - a - x_1^2 - y_1^2 = 0,$$

where, after inserting the expressions for eigenvalues, we get

$$\sqrt{1+4\varepsilon^2 a^{-1}}x_1^2 - 2\varepsilon\Lambda x_1 y_1 + y_1^2 - 2\varepsilon^2\lambda^2\Lambda a x_1 = 2c - a, \tag{3.4}$$

that is an ellipse, if $c > 0$. The trace of unstable manifold $W^u(O)$ on $x_2 = a$ is the point $(\Lambda a, a, \varepsilon\Lambda^2 a, -\Lambda(1+\varepsilon^2\Lambda^2)a)$. For $c = 0$ the point $(\Lambda a, \varepsilon\Lambda^2 a)$ is a unique solution of (3.4). Parametric representation of this curve is given by

$$x_1 = \Lambda a + \xi_1 \cos\tau - \xi_2 \sin\tau,\ y_1 = \varepsilon\Lambda^2 a + \eta_1\cos\tau - \eta_2\sin\tau, \tag{3.5}$$

where $X_1 = (\xi_1, \eta_1)^\top$, $X_2 = (\xi_2, \eta_2)^\top$ are two eigenvectors of the matrix A of the quadratic form

$$(\sqrt{1+4\varepsilon^2 a^{-1}}\xi^2 - 2\varepsilon\Lambda\xi_1\eta_1 + \eta_1^2)/2 = (AX, X)/2$$

with eigenvalues

$$\kappa_\pm = \frac{1+\sqrt{1+4\varepsilon^2/a}}{2} \pm \varepsilon\sqrt{\frac{1}{a}}.$$

Thus, the larger semi-axis of the ellipse is $\sqrt{2c/\kappa_-}$ and the lesser one is $\sqrt{2c/\kappa_+}$. The expression for $y_2(\tau)$ is rendered by inserting $x_1(\tau), y_1(\tau)$ into (3.3).

To find symmetric separatrix loops to O, homoclinic orbits to γ_c and symmetric periodic orbits near the loop, let us write down the symmetric solutions of the left system in the half-space $x_2 < a$. These solutions cut the plane $Fix(L)$. Since the system is autonomous, we can assume that these solutions intersect $Fix(L)$ at $t = 0$. Setting $t = 0$ in the general solution, we get the relation between parameters c_1, c_2, l:

$$\Lambda[c_1 - c_2] - l\lambda = 0,\ -\Lambda(1+\varepsilon^2\Lambda^2)[c_1 - c_2] + l\lambda(1-\varepsilon^2\lambda^2) = 0.$$

The determinant of this homogeneous system w.r.t. variables $c_1 - c_2, l$ does not vanish, hence $c_1 = c_2$ and $l = 0$. Therefore symmetric solutions in $x_2 \le a$ can be written as

$$\begin{aligned}
x_1 &= c_1\Lambda\sinh(\Lambda t) - m\lambda\sin(\lambda t),\\
x_2 &= c_1\cosh(\Lambda t) + m\cos(\lambda t),\\
y_1 &= c_1\varepsilon\Lambda^2\cosh(\Lambda t) - m\varepsilon\lambda^2\cos(\lambda t),\\
y_2 &= -c_1\Lambda(1+\varepsilon^2\Lambda^2)\sinh(\Lambda t) + m\lambda(1-\varepsilon^2\lambda^2)\sin(\lambda t).
\end{aligned} \tag{3.6}$$

In the half-space $x_2 \le a$ the Hamiltonian has the form

$$H = \frac{x_1^2 + y_1^2}{2} + x_1 y_2 + \frac{x_2^2}{2a}.$$

Among symmetric solutions let us choose those which belong to the level $H = c$. By fixing c, one gets the relation between c_1 and m:

$$c = \sqrt{1 + 4\varepsilon^2/a}\left[\frac{c_1^2 \Lambda^2}{2} + \frac{m^2\lambda^2}{2}\right]. \tag{3.7}$$

On the auxiliary plane (c_1, m) this curve is an ellipse, if $c > 0$. The trace of the family of symmetric solutions at $t = 0$ gives a curve if positive c is fixed. This curve E is also an ellipse since the transformation from initial values (x_2^0, y_1^0) at $t = 0$ to (c_1, m) is linear and non-degenerate for positive ε. The ellipse E is indeed the ellipse for $c \in (0, a/2)$ (it belongs to the half-space $x_2 < a$) but it touches $x_2 = a$ for $c = a/2$ at the point $(0, a, 0, 0)$ and in the half-plane (x_2, y_1), $x_2 < a$, there is only a part of this ellipse. Therefore, for these values of c the trace on $x_2 = a$ of symmetric solutions from the left side is a non-closed curve.

In particular, at $c = 0$ the unique solution of this equation is $(c_1, m) = (0, 0)$, that is, there are no other symmetric solutions in this level except for the separatrices. As we will see below, the reason of that is purely topological: the level $H = 0$ for $x_2 \le a$ intersects $Fix(L)$ only at the point $(0, 0, 0, 0)$. For $c > 0$ we have a closed 1-parameter family of symmetric solutions, they cut the 3-plane $x_2 = a$ along two disjoint ellipses, of which one is near the trace of $W^u(O)$, while another one is near the trace of $W^s(O)$.

Now consider the system in the half-space $x_2 \ge a$:

$$\varepsilon x_1{}' = y_1,\ \varepsilon y_1{}' = -x_1 - y_2,\ x_2{}' = x_1,\ y_2{}' = (x_2 - 1)/(1 - a). \tag{3.8}$$

Its Hamiltonian, being C^1-extension of the Hamiltonian from $x_2 \le a$, has the form

$$H = \frac{x_1^2 + y_1^2}{2} + x_1 y_2 - \frac{(x_2 - 1)^2}{2(1 - a)} + \frac{1}{2}.$$

The point $P = (0, 1, 0, 0)$ is the elliptic equilibrium of the system with eigenvalues

$$\pm i\Omega = \pm i\sqrt{\frac{1 + \sqrt{1 - 4\varepsilon^2/(1 - a)}}{2\varepsilon^2}},\ \pm i\omega = \pm i\sqrt{\frac{1 - \sqrt{1 - 4\varepsilon^2/(1 - a)}}{2\varepsilon^2}}.$$

Solutions of the system are of the form:

$$\begin{aligned} x_1 &= -\Omega[b\sin(\Omega t) + c\cos(\Omega t)] - \omega[d\sin(\omega t) + e\cos(\omega t)],\\ x_2 &= 1 + b\cos\Omega t - c\sin\Omega t + d\cos\omega t - e\sin\omega t,\\ y_1 &= -\varepsilon\Omega^2[b\cos(\Omega t) - c\sin(\Omega t)] - \varepsilon\omega^2[d\cos(\omega t) - e\sin(\omega t)],\\ y_2 &= \Omega(1 - \varepsilon^2\Omega^2)[b\sin(\Omega t) + c\cos(\Omega t)] +\\ &\qquad \omega(1 - \varepsilon^2\omega^2)[d\sin(\omega t) + e\cos(\omega t)]. \end{aligned} \tag{3.9}$$

Setting $b = c = 0$, we obtain the Lyapunov family of periodic orbits with frequency ω, whereas at the values $d = e = 0$ one gets another Lyapunov family of periodic orbits with frequency Ω. Each of these two families fills its own 2-plane; two planes intersect transversely at P. Of course, genuine periodic orbits in these planes are those that belong completely to the half-space $x_2 \ge a$.

If one takes an initial point in the half-space $x_2 \ge a$ out of the union of these 2-planes, then this point and the orbit through it belong to the invariant 2-torus, provided that such torus lies entirely in the half-space. These tori are obtained by fixing the values of $H = c$ and positive definite quadratic integral

$$K_1 = \frac{[(1 - \varepsilon^2\Omega^2)x_1 + y_2]^2}{2} + \varepsilon^2\omega^2\frac{[\varepsilon\Omega^2(x_2 - 1) + y_1]^2}{2}.$$

Among the obtained solutions, let us choose symmetric ones. They intersect at $t = 0$ the plane $Fix(L)$. By the same reasoning as above, we get:

$$\begin{aligned} x_1 &= -\Omega b\sin\Omega t - \omega d\sin\omega t\\ x_2 &= 1 + b\cos\Omega t + d\cos\omega t\\ y_1 &= -\varepsilon[\Omega^2 b\cos\Omega t + \omega^2 d\cos\omega t]\\ y_2 &= \Omega(1 - \varepsilon^2\Omega^2)b\sin\Omega t + \omega(1 - \varepsilon^2\omega^2)d\sin\omega t. \end{aligned} \tag{3.10}$$

In the level $H = c$ one gets the relation between b, d:

$$c = \sqrt{1 - 4\varepsilon^2/(1-a)}\left[\frac{b^2\Omega^2}{2} - \frac{d^2\omega^2}{2}\right] + \frac{1}{2}.$$

In particular, for $c = 0$ we obtain:

$$b^2\Omega^2 - d^2\omega^2 + \frac{1}{\sqrt{1 - 4\varepsilon^2/(1-a)}} = 0. \tag{3.11}$$

It is worth remarking that the switch between the left and the right systems occurs along the 3-plane $x_2 = a$. Since $x_2' = x_1$, the flow orbits starting on this plane for $x_1 > 0$ pass from the left half-space into the right one, but if $x_1 < 0$, they pass in the opposite direction. So, along the 2-plane $x_1 = 0, x_2 = a$ the flow orbits touch the switching plane. It is essential for the further considerations to know how the flow orbits starting on this 2-plane behave: they can return to the same half-space from which they come, or they can cross the switching plane. If the right hand sides of the system would be C^1-smooth, the choice would be decided by the sign of the second derivatives in t, but here we need to examine this question directly from the solutions.

4. Topology of levels $H = c$ near the saddle-center

It is known [17] that the study of orbit behavior of a Hamiltonian system near a homoclinic loop to a saddle-center relies on the topology of levels for the Hamilton function near the loop, in particular, of the level containing the loop. Therefore let us describe this topology for our simple case for values $c = 0, c > 0, c < 0$. Locally near the loop this level, if the loop exists, consists of two pieces. One piece is a neighborhood of the "separatrix cross" in half-space $x_2 \le a$, another piece is a neighborhood of the outer segment of the loop lying in half-space $x_2 \ge a$. Both separatrices in half-space $x_2 \le a$ belong to the level $H = 0$, therefore we consider the topology of $H = 0$ using the equation:

$$\frac{{x_1}^2 + {y_1}^2}{2} + x_1 y_2 + \frac{{x_2}^2}{2a} = 0.$$

Let us rewrite the equation in the form $y_1^2 + x_2^2/a = -x_1^2 - 2x_1y_2$. To get the equality, it is necessary to fulfill the inequality $x_1(x_1 + 2y_2) \le 0$. Starting from here, it is easy to define the topology of the set $H = 0$. Its projection onto the plane (x_1, y_2) consists of two sectors $x_1 \ge 0$, $x_1 + 2y_2 \le 0$ and $x_1 \le 0$, $x_1 + 2y_2 \ge 0$. The first sector belongs to the fourth quadrant of the plane but the second sector lies in the second quadrant. Cut the sector by straight-lines $y_2 = x_1 + b$, $b \in (-d, d) = I$. Each of the lines intersects the sector along a segment. Over each point of this segment, except for its extreme points, an ellipse in the plane (x_2, y_1) lies. When approaching the extreme points of the segment, the ellipse shrinks to a point. Thus, we have a two-dimensional sphere over every segment. As $b \to 0$, this sphere contracts to the point. So, in $\mathbb{R}^4$ over each closed sector $0 \le b \le d$ and $-d \le b \le 0$ a 3-dimensional ball lies. Two sectors give two 3-balls glued together at its inner points, giving $O = (0, 0, 0, 0)$. If we cut the set obtained at this point and delete the boundary spheres we get two disjoint sets, each of them is homeomorphic to the spherical layer $S^2 \times I$. Thus we have described the topology of the level $H = 0$, more precisely its part that belongs to the half-space $x_2 \le a$. Intersection of each component of the level with the switching plane $x_2 = a$ consists of two smooth 2-dimensional submanifolds given by solutions of the equation $y_1^2 = -a - x_1(x_1 + 2y_2)$. One submanifold belongs to the half-space $y_2 \ge \sqrt{a}$, and another one lies in $y_2 \le -\sqrt{a}$. It is worth remarking that due to such structure of the level $H = 0$, all orbits of the systems that start on the 3-plane $x_2 = a$ on one submanifold and go to the neighborhood of O, cannot reach again the same 3-plane passing through a small neighborhood of O.

The local stable manifold of O is a segment through O. O divides this segment into two separatrices: one of them goes to the half-space $x_2 > 0$ and cuts the 3-plane $x_2 = a$, another one goes to $x_2 < 0$ and

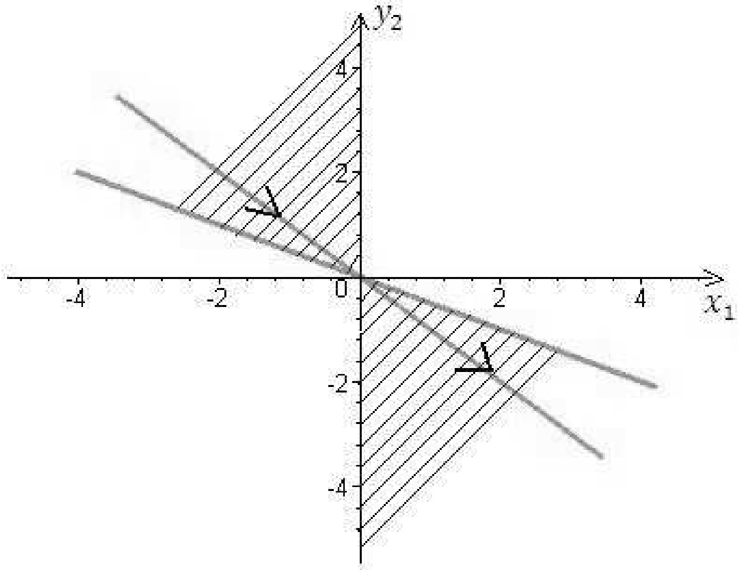


FIGURE 3. Projection of $H = 0$ onto the plane (x_1, y_2). Lines with arrows are projections of stable/unstable separatrices.

escapes to infinity. The same holds for unstable separatrices. All separatrices are given parametrically by

$$x_1 = -\Lambda c e^{-\Lambda t},\ x_2 = c e^{-\Lambda t},\ y_1 = \varepsilon \Lambda^2 c e^{-\Lambda t},\ y_2 = \Lambda(1 + \varepsilon^2 \Lambda^2) c e^{-\Lambda t},$$

(the stable separatrix), and the unstable separatrix has the form

$$x_1 = \Lambda c e^{\Lambda t},\ x_2 = c e^{\Lambda t},\ y_1 = \varepsilon \Lambda^2 c e^{\Lambda t},\ y_2 = -\Lambda(1 + \varepsilon^2 \Lambda^2) c e^{\Lambda t}.$$

Projections of separatrices on the plane (x_1, y_2) belong to the related sectors (see fig. 3).

In coordinates (x_1, y_2) the stable separatrix with positive x_2 lies on the semi-line, $y_2 = -(1 + \varepsilon^2 \Lambda^2) x_1$, $x_1 \le 0$. Unstable separatrix with positive x_2 corresponds to the same straight line but to its semi-axis $x_1 \ge 0$. It is worth remarking that the second separatrices on the related stable (respectively unstable) line are projected onto the same line $y_2 = -(1 + \varepsilon^2 \Lambda^2) x_1$, but on its other semi-axis: the stable ones do so to $x_1 \ge 0$ and unstable ones – to $x_1 \le 0$. This implies, as a corollary, that in the level $H = 0$ the orbits passing through the 3-plane $x_2 = a$ near the trace of the stable separatrix stay in the same connection component of this level.

Thus one can assert that zero level of the Hamiltonian does not contain periodic orbits and multi-round homoclinic orbits in a neighborhood of the homoclinic loop. But such orbits can exist on the levels $H = c > 0$, since the level for positive c becomes connected (it acquires a crosspiece near O).

The level $H = c$ in the half-space $x_2 \le a$ consists of the points obeying the equation $(x_1^2 + y_1^2)/2 + x_1 y_2 + x_2^2/2a = c$. As above, let us rewrite this equation in the form $y_1^2 + x_2^2/a = 2c - x_1^2 - 2x_1 y_2$. Then the inequality $2c - x_1(x_1 + 2y_2) \ge 0$ should hold. For $c > 0$ the projection of the related set onto the plane (x_1, y_2) is the region bounded by branches of hyperbola $y_2 = c/x_1 - x_1/2$ and containing the origin. Let us cut this region by segments $y_2 = x_1 + b$, $b \in (-d, d)$. Over each segment in $\mathbb{R}^4$ we get again a 2-sphere and their union in b gives the set homeomorphic to $S^2 \times [-d, d]$ (spherical layer).

So, we conclude that in the levels $H = c > 0$ orbits can exist which start on 3-plane $x_2 = a$, pass near O and return to 3-plane $x_2 = a$. Therefore in these levels periodic and homoclinic orbits of roundness 1 and higher may exist. Recall that in the level $H = c > 0$ there is the unique saddle Lyapunov periodic orbit γ_c. Its stable and unstable manifolds (their halves) cut 3-plane $x_2 = a$. Consider the trace of $W^s(\gamma_c)$. Its intersection with $x_2 = a$ is a closed curve lying on the disk $\{H = c\} \cap \{x_2 = a\}$. Its projection on the plane (x_1, y_1) is the ellipse (see (3.4)). For points inside this closed curve, the orbits go to a neighborhood of O and after that they follow the unstable separatrix and return to $x_2 = a$ inside the closed curve being the trace of $W^u(\gamma_c)$.

For $H = c < 0$ the projection of the level onto the plane (x_1, y_2) consists of two disjoint components bounded by the hyperbola $y_2 = c/x_1 - x_1/2$. Cutting this set by segments $y_2 = x_1 + b$, $|b| \geq b_0$ we conclude, like above, that each component is homeomorphic to a 3-ball. This representation shows that all orbits of the system in the levels $H = c$, $c < 0$ starting at 3-plane $x_2 = a$ near the trace of stable separatrix leave for infinity with no returns to $x_2 = a$.

5. Homoclinic orbits to saddle-center

In this section we show existence of one-round homoclinic orbits to the saddle-center for a certain sequence of small parameter $\varepsilon_n \to 0$. Such an orbit contains as pieces those separatrices of the saddle-center that intersect 3-plane $x_2 = a$. Involution L interchanges these stable and unstable separatrices. Besides this, a homoclinic orbit includes a mandatory travel in the half-space $x_2 > a$. Due to reversibility of the system w.r.t. L, a homoclinic orbit has to be symmetric. Without loss of generality one may suppose the related solution to cut the plane $Fix(L)$ at $t = 0$ in the half-space $x_2 > a$. Thus, it has to belong to the previously found one-parameter family of symmetric solutions (3.10) and, when cutting the 3-plane $x_2 = a$ at some $t = T < 0$, should pass through the trace of $W^u(O) : (\Lambda a, a, \varepsilon\Lambda^2 a, -\Lambda(1 + \varepsilon^2\Lambda^2)a)$.

For some values of the parameter ε multi-round symmetric homoclinic orbits to O may also exist, they intersect 3-plane $x_2 = a$ more than twice. For instance, symmetric 2-round homoclinic orbits to O behave as follows: unstable separatrix of O intersects switching plane, after that it is continued by some asymmetric orbit from the right half-space and intersects $x_2 = a$ for the second time, it is continued by a symmetric orbit of the left system that hits $Fix(L)$ near equilibrium O. The symmetric image of the curve constructed gives the second part of a symmetric 2-round homoclinic orbits to O: this orbit intersects $Fix(L)$ one time but intersects four times 3-plane $x_2 = a$.

Below we search for 1-round homoclinic orbits to O. Let us write down the system to find the value T:

$$\begin{aligned}
&\Lambda a = -\Omega b \sin(\Omega T) - \omega d \sin(\omega T),\\
&a = 1 + b\cos(\Omega T) + d\cos(\omega T),\\
&\varepsilon\Lambda^2 a = -\varepsilon\Omega^2 b\cos(\Omega T) - \varepsilon\omega^2 d\cos(\omega T),\\
&-(\Lambda + \varepsilon^2\Lambda^3)a = \Omega(1 - \varepsilon^2\Omega^2)b\sin(\Omega T) + \omega(1 - \varepsilon^2\omega^2)d\sin(\omega T).
\end{aligned} \tag{5.1}$$

The system breaks into two independent subsystems w.r.t. $b\cos(\Omega T)$, $d\cos(\omega T)$ and $b\sin(\Omega T)$, $d\sin(\omega T)$. Solutions of the related systems are:

$$\begin{aligned}
&b\cos(\Omega T) = \frac{(1-a)\omega^2 - a\Lambda^2}{\Omega^2 - \omega^2},\; d\cos(\omega T) = \frac{a\Lambda^2 - (1-a)\Omega^2}{\Omega^2 - \omega^2},\\
&b\sin(\Omega T) = \left(\frac{a\Lambda}{\Omega}\right)\frac{\omega^2 + \Lambda^2}{\Omega^2 - \omega^2},\; d\sin(\omega T) = -\left(\frac{a\Lambda}{\omega}\right)\frac{\Lambda^2 + \Omega^2}{\Omega^2 - \omega^2}.
\end{aligned} \tag{5.2}$$

From this we derive the system for the quantities T and ε:

$$\begin{aligned}
&\tan(\Omega T) = \left(\frac{a\Lambda}{\Omega}\right)\frac{\omega^2 + \Lambda^2}{(1-a)\omega^2 - a\Lambda^2} = \frac{\sqrt{a}}{\varepsilon}(1 + O(\varepsilon^2)),\\
&\tan(\omega T) = \left(\frac{a\Lambda}{\omega}\right)\frac{1 + (\Lambda/\Omega)^2}{(1-a) - a(\Lambda/\Omega)^2} = \sqrt{\frac{a}{1-a}}(1 + O(\varepsilon^2)).
\end{aligned} \tag{5.3}$$

Recall that for Λ, Ω, ω the following expansions in ε are valid:

$$\Lambda^2 = \frac{1}{a} - \frac{\varepsilon^2}{a^2} + O(\varepsilon^4),\ \Omega^2 = \frac{1}{\varepsilon^2} - \frac{1}{1-a} + O(\varepsilon^2),$$

$$\omega^2 = \frac{1}{1-a} + \frac{\varepsilon^2}{(1-a)^2} + O(\varepsilon^4).$$

In particular we see that the trace of $W^u(O)$ on the switching plane is an analytic curve w.r.t. small ε, it belongs to the level $H = 0$ and passes through the point $(x_1, x_2, y_1, y_2) = (\sqrt{a}, a, 0, -\sqrt{a})$ (this latter point coincides with the trace of unstable separatrix of the saddle on the slow manifold $y_1 = 0, x_1 + y_2 = 0$).

We find possible values $T_m(\varepsilon) < 0$ from the second equation in (5.3):

$$T_m(\varepsilon) = (1/\omega)\arctan[a/(1-a)]^{1/2}(1 + O(\varepsilon^2))) + m\pi/\omega,\ m \in -\mathbb{N}.$$

Let us recall that to find 1-round homoclinic orbits to O (they intersect 3-plane $x_2 = a$ precisely two times) one needs to take only those $T_m(\varepsilon)$ for which the related symmetric solution of the right system for values $T_m(\varepsilon) < t < 0$ does not intersect $x_2 = a$. Such a solution can do many "revolutions around elliptic point" while moving through the right half-space.

It is worth to notice that at $m = -1$ and $\varepsilon \to +0$ we get in the limit $\lim T_{-1}(\varepsilon)$ the quantity $-T_0$ being the passage time for the separatrix of slow system from the point $(x_2, y_2) = (1 + \sqrt{1-a}, 0)$ to point $(a, -\sqrt{a})$. Below we restrict ourselves by searching for homoclinic orbits with $m = -1$ as $\varepsilon \to +0$. Therefore we denote $T(\varepsilon) = T_{-1}(\varepsilon)$.

Let us insert the expression for $T(\varepsilon)$ into the first equation in (5.3). This yields the equation for finding parameters values ε_n:

$$\tan(\Omega_0(\varepsilon)T(\varepsilon)/\varepsilon) = a^{1/2}(1 + O(\varepsilon^2))/\varepsilon,\ \Omega_0(\varepsilon) = \varepsilon\Omega(\varepsilon) = 1 - \tfrac{\varepsilon^2}{2(1-a)} + \cdots,$$

or

$$\tan\left(1/\varepsilon(1 + O(\varepsilon^2))[1/\omega \arctan((a/(1-a))^{1/2}(1 + O(\varepsilon^2))) - \pi/\omega]\right)$$
$$= a^{1/2}(1 + O(\varepsilon^2))/\varepsilon.$$

Make the change $E = 1/\varepsilon$, $E \to \infty$ as $\varepsilon \to +0$. Then the equation turns into

$$\tan(ES(\frac{1}{E})) = Ed(E),$$

where

$$S(\frac{1}{E}) = (1 + O(\varepsilon^2))[\frac{1}{\omega}\arctan(\frac{a}{(1-a)^{1/2}}(1 + O(\varepsilon^2))) - \pi/\omega]) = -T_0 + O(\varepsilon^2).$$

From here, we derive the asymptotics for the values ε_n

$$\varepsilon_n \sim -\frac{2T_0}{\pi(1+2n)}, \quad n \in -\mathbb{N},$$

at which the system (1.2) has 1-round homoclinic orbit to O. It is worth remarking that as $\varepsilon \to 0$ the related homoclinic orbit moves closer and closer to the pieces of stable and unstable separatrices of the slow system but it rotates around a global piece of the slow separatrix more and more times in the half-space $x_2 > a$. This is the manifestation of the slow-fast nature of the system.

To demonstrate the shape of homoclinic solutions we present graphs of functions $x_2(t)$ at $m = -1$ and different ε_n for $a = 0.5$. Set $a = 0.5$ and take $n = -2, -12, -25$. We plot on the same Figure four curves: respectively, $x_2(t)$ and its derivatives up to the order 3 (see Fig. 4). Obviously, the higher the derivative is, the more oscillating function we get. This reflects the singular nature of the model.

Recall that for a homoclinic orbit the function $x_2(t, \varepsilon)$ has the representation on $t \in (-\infty, 0)$:

$$x_2(t) = \begin{cases} a\exp[\Lambda(t - T(\varepsilon))],\ \text{if}\ t < T(\varepsilon), \\ 1 + b\cos(\Omega t) + d\cos(\omega t),\ \text{if}\ 0 > t > T(\varepsilon), \end{cases}$$

where b, d are found from the relations (5.2). Its part for $t \in [0, \infty)$ is given by the evenness.

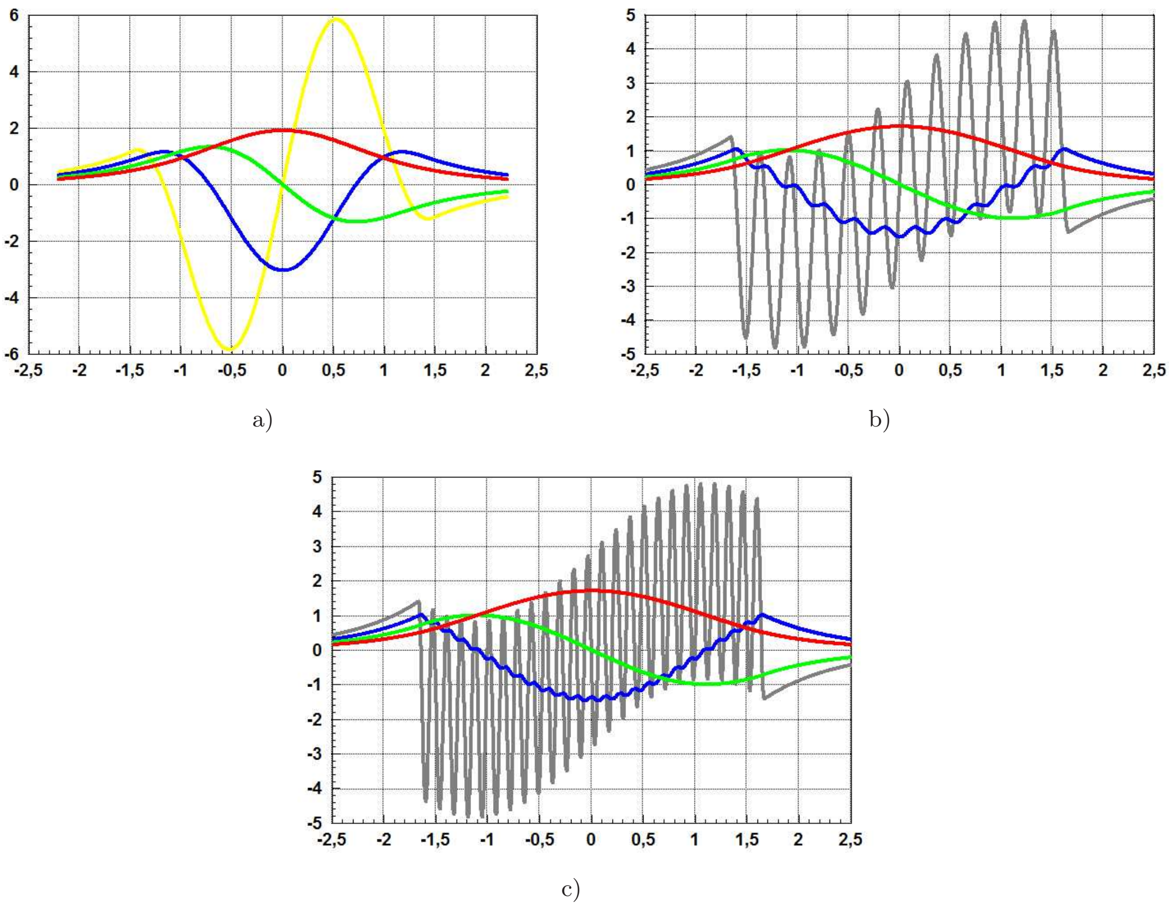


FIGURE 4. Graph of $x_2(t)$ (red) and its three derivatives for: a) $n = 2$, b) $n = 12$, c) $n = 25$

6. On homoclinic orbits to Lyapunov periodic orbits

Our next task is to show existence of transverse homoclinic orbits to periodic orbits γ_c for small positive c. For an analytic Hamiltonian it was done in [17] and then was repeated in [14,19] (with a series of other interesting results). If such a Poincaré homoclinic orbit (in terminology of [22]) exists we are in position described by Smale [21], Shilnikov [22] and others that implies the complicated dynamics of the related differential system, in particular, countably many saddle periodic orbits, multi-round homoclinic orbits to γ_c, almost periodic, recurrent and many others orbits extracted from the symbolic description.

To prove their existence recall that the trace of $W^{cu}(O)$ on 3-plane $x_2 = a$ within the level $H = c > 0$ for c small is a closed curve which projection onto the plane (x_1, y_1) is an ellipse (3.5). Let us show that this curve intersects the trace of the family of symmetric orbits from half-space $x_2 \geq a$. Through intersection points symmetric homoclinic orbits to Lyapunov periodic orbit pass.

To this end, we go along the line of thinking of the previous section and write down the equations for searching for time of intersection of symmetric orbits from the right system with the closed curve on $x_2 = a$. These equations read as follows:

$$\begin{aligned}
&x_1(\tau) = -\Omega b \sin \Omega t - \omega d \sin \omega t \\
&a = 1 + b \cos \Omega t + d \cos \omega t \\
&y_1(\tau) = -\varepsilon[\Omega^2 b \cos \Omega t + \omega^2 d \cos \omega t] \\
&y_2(\tau) = \Omega(1 - \varepsilon^2 \Omega^2) b \sin \Omega t + \omega(1 - \varepsilon^2 \omega^2) d \sin \omega t.
\end{aligned} \tag{6.1}$$

The left hand sides of these relations tend, as $c \to +0$, to the values $\Lambda a, a, \varepsilon\Lambda^2 a, -\Lambda(1+\varepsilon^2\Lambda^2)a$, respectively. As before, these equations break into two groups from where we get two equations

$$\begin{aligned} \tan(\omega T) &= \frac{(1-\varepsilon^2\Omega^2)x_1(\tau)+y_2(\tau)}{\omega[\varepsilon^2\Omega^2(a-1)+\varepsilon y_1(\tau)]}, \\ \tan(\Omega T) &= \frac{(1-\varepsilon^2\omega^2)x_1(\tau)+y_2(\tau)}{\varepsilon\Omega[\varepsilon\omega^2(a-1)+y_1(\tau)]}. \end{aligned} \tag{6.2}$$

The right hand side of the second equation is a 2π-periodic function of τ that is $\sqrt{c}$-close to its mean value being the right hand side of (5.3).

Let us remark that this system has a solution not always. The point is that for those ε when separatrices of the saddle-center are split, the trace of its stable separatrix on $x_2 = a$ and the trace of its unstable separatrix continued through half-space $x_2 > a$ are two different points and there is $c_*(\varepsilon) > 0$ such that for $0 < c \le c_*(\varepsilon)$ two closed curves being traces of $W^s(\gamma_c)$, $W^u(\gamma_c)$, respectively, do not intersect. At $c = c_*(\varepsilon)$ there is a tangency for the related curves and after that one gets the intersection, that is, homoclinic orbits appear.

Thus we conclude that the trace on $x_2 = a$ of the family of symmetric orbits from the right half-space $x_2 \ge a$ within the level $H = c$ intersects the trace of the unstable manifold of Lyapunov periodic orbit γ_c. Therefore, the orbit through this intersection point is symmetric (it cuts $Fix(L)$) homoclinic orbit of γ_c.

We illustrate the above reasoning by Fig. 5. Here the graphs of $x_2(t)$ are presented for certain values of parameters c and ε. a is fixed at 0.5 but the values c of the Hamiltonian vary. Near the homoclinic loop we found two symmetric homoclinic orbits to the related Lyapunov periodic orbit (two more are non-symmetric and interchanged by L). While varying c we observed their merging and disappearance.

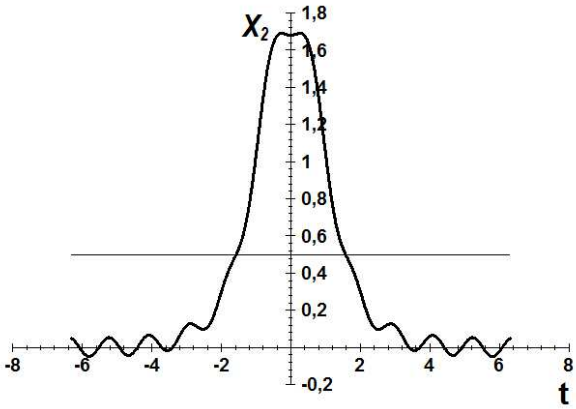
7. Geometry of the trace of outgoing separatrix as ε varies

As stated before, the intersection point of $W^u(O)$ with $x_2 = a$ has coordinates $(\Lambda a, a, \varepsilon\Lambda^2 a, -\Lambda(1+\varepsilon^2\Lambda^2)a)$. When the loop formation occurs, the outgoing separatrix should intersect due to its symmetry the plane $Fix(L) = \{x_1 = y_2 = 0\}$ in the half-space $x_2 > a$ on the level $H = 0$. The intersection of the set $H = 0$ with $Fix(L)$ is the curve $(x_2-1)^2/(1-a) - y_1^2 = 1$, moreover, in the half-plane $x_2 > a$ only one connected component of this hyperbola lies. At small ε the separatrix when moving in the right half-space will make fast oscillations around the piece of the former separatrix of the slow system. To illustrate geometrically the intersection/nonintersection of this separatrix with $Fix(L)$ we plot the trace of this separatrix within the cross-section $y_2 = 0$. The intersection of the trace with the plane $x_1 = 0$ gives homoclinic orbits to saddle-center.

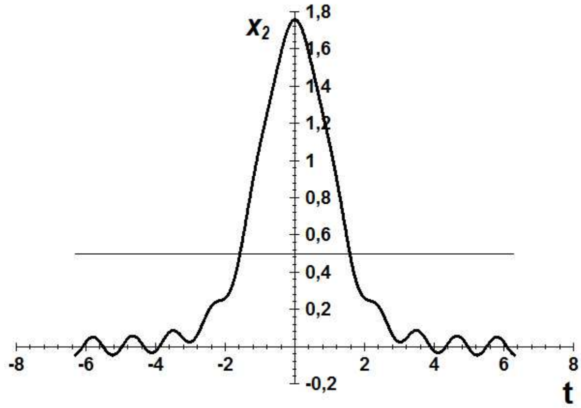
For simulation we take $a = 0.5$ and vary $\varepsilon \to +0$. Then we have to take the point $(\Lambda a, a, \varepsilon\Lambda^2 a, -\Lambda(1+\varepsilon^2\Lambda^2)a)$ as the initial one to find solutions of the right system, calculate the time $T(\varepsilon)$ when this solution intersects cross-section $y_2 = 0$ and insert this time to other relations. We get functions $(x_1(T), x_2(T), y_1(T))$. Varying ε and plotting the points we get what we need.

To find the solutions we seek, one needs to find values of constants b, c, d, e expressed via ε supposing that at $t = 0$ the related orbit starts at the trace of $W^u(O)$ on $x_2 = a$. Then we get

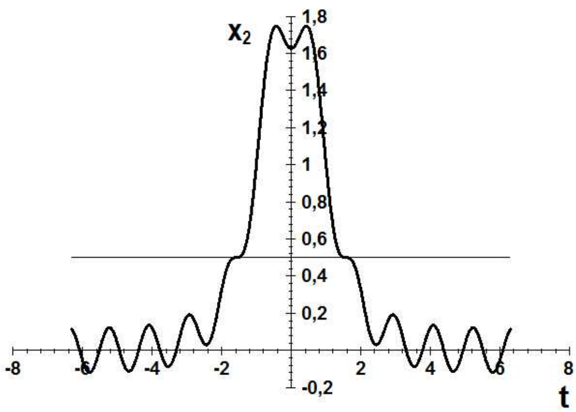
$$\begin{aligned} &\Lambda a = -\Omega c - \omega e, \\ &a = 1 + b + d, \\ &\varepsilon\Lambda^2 a = -\varepsilon\Omega^2 b - \varepsilon\omega^2 d, \\ &-\Lambda(1+\varepsilon^2\Lambda^2)a = \Omega(1-\varepsilon^2\Omega^2)c + \omega(1-\varepsilon^2\omega^2)e. \end{aligned} \tag{7.1}$$



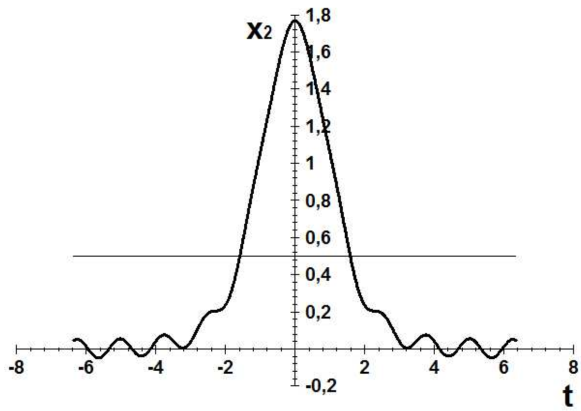
One of two symmetric homoclinic orbits to γ_c for small c, $n = 2$



Another symmetric homoclinic orbit to γ_c for small c, $n = 2$



The shape of a homoclinic orbit at the confluence of two symmetric ones



The continuation of homoclinic orbits in ε

FIGURE 5. Unfoldings of homoclinic orbits to γ_c

The system breaks into two independent ones w.r.t. b, c, d, e, from where we have:

$$b = \frac{(1-a)\omega^2 - a\Lambda^2}{\Omega^2 - \omega^2},\ d = \frac{a\Lambda^2 - (1-a)\Omega^2}{\Omega^2 - \omega^2},$$

$$c = \left(\frac{a\Lambda}{\Omega}\right)\frac{\omega^2 + \Lambda^2}{\Omega^2 - \omega^2},\ e = -\left(\frac{a\Lambda}{\omega}\right)\frac{\Lambda^2 + \Omega^2}{\Omega^2 - \omega^2}.$$

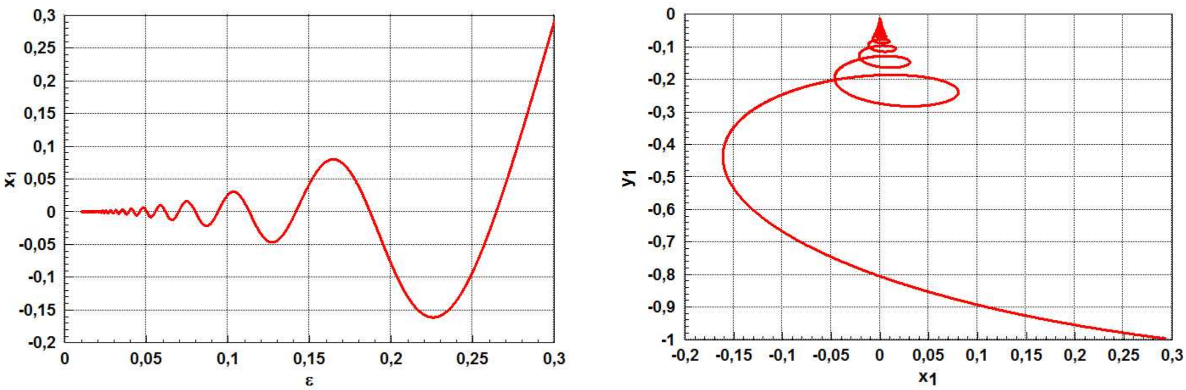
The equation for finding T reads

$$\Omega(1 - \varepsilon^2\Omega^2)[b\sin(\Omega t) + c\cos(\Omega t)] + \omega(1 - \varepsilon^2\omega^2)[d\sin(\omega t) + e\cos(\omega t)] = 0,$$

we are interested in the first positive root.

We present the projection of this curve onto the planes (ε, x_1) and (x_1, y_1). The related curves are presented in Fig. 6

One can see in this Figure that the curve obtained is a spiral with tightly contracted sections around the mean value $x_1 = 0$. For certain values of $\varepsilon = \varepsilon_n$ this curve intersects the plane $x_1 = 0$ that corresponds to the existence of homoclinic loops to the saddle-center.



a) b)

FIGURE 6. a) Trace on the cross-section $y_2 = 0$ of outgoing separatrix as function of ε. b) The same on (x_1, y_1) plane.

7.1. On transitional symmetric periodic orbits

Let us briefly discuss existence of the simplest symmetric periodic orbits that cut the switching plane two times. We will call them the transitional ones. They are obtained as common orbits of two symmetric families, one is (3.7) from the left system and another one from (3.10) of the right system. Thus one needs to find intersection of traces of these families on the switching plane. In fact to do this we have similar equations as (6.1), only in the left hand sides we should fix $c > 0$ and insert parametric representations for the related ellipse (or its part) being the trace of solutions (3.6) on $x_2 = a$. These equations can be solved in a similar way and sometimes their solutions exist. Through the related points symmetric transitional periodic orbits pass. In particular, the calculations confirm their existence for values $c \in (a/2, (a + \varepsilon^2\omega^4(1-a)^2/2)$. As we will see, the related periodic orbits are saddle ones and they play essential role when considering bifurcations in the last section.

8. On border-collision bifurcations

In this section we would like to discuss the type of bifurcations in non-smooth systems that is known as border-collision bifurcations [6, 18]. In the system under study they arise in particular when the family of periodic orbits of one of two linear systems touches the switching plane $\Sigma : x_2 = a$ (a grazing periodic orbit). There are three such families in the system. The first one is the Lyapunov family (3.2) that exists near saddle-center singular point, related periodic orbits fill 2-plane till the extreme orbit intersects Σ: $\lambda(1 - \varepsilon^2\lambda^2)x_1 + \lambda y_2 = 0$, $y_1 + \varepsilon\lambda^2 x_2 = 0$. The orbit that touches first Σ is defined by equality $\sqrt{m^2 + l^2} = a$, it occurs at the point $x_1 = y_2 = 0, x_2 = a, y_1 = -\varepsilon\lambda^2 a$. To construct Poincaré map at N we should fix the level $H = a(1 + a\varepsilon^2\lambda^4)/2$ that is of the order ε^{-2} which is too high for our problems.

There are two more families: one consists of periodic orbits with the finite period $2\pi/\omega$, the related 2-plane is given as $(1-\varepsilon^2\omega^2)x_1+y_2 = 0$, $\varepsilon\omega^2(x_2-1)+y_1 = 0$. Another family contains periodic orbits with the small period $2\pi/\Omega$, the grazing orbit appears here on the level of the order $1/\varepsilon$. Therefore we consider only the family with the finite period. Here an extreme periodic orbit that touches first Σ is given as $x_1 = \omega(1-a)\sin(\omega t)$, $x_2 = 1-(1-a)\cos(\omega t)$, $y_1 = \varepsilon(1-a)\omega^2\cos(\omega t)$, $y_2 = -\omega(1-\varepsilon^2\omega^2)(1-a)\omega^2\sin(\omega t)$, its period is $2\pi/\omega$. The related value of the Hamiltonian is $h_t = (a + \varepsilon^2\omega^4(1-a)^2)/2 \sim a/2$. As a cross-section we take the 3-plane $N = \{y_2 = 0\}$ near the point $Q = (0, 2-a, -\varepsilon\omega^2(1-a), 0)$ which lies in the half-period from the tangency point $(0, a, \varepsilon\omega^2(1-a), 0)$. For points near Q one has $y_2' \sim -1$ and therefore, due to periodicity of the orbit through Q, the nearby orbits return to N in a time close to $2\pi/\omega$. Now we should fix the level $H = h_t$ inside N, it is a quadratic surface $x_1^2 + y_1^2 - (x_2-1)^2/(1-a) = a + \varepsilon^2\omega^4(1-a)^2$, i.e. a hyperboloid of two sheets, one of them contains point Q. Following the ideas of M.I. Feigin [9–11] (see also his book [12] and recent paper [7]), we construct Poincaré map on N near Q and study its properties. It is worth mentioning that if a periodic orbit does not touch Σ then the corresponding Poincaré map is smooth since it is the composition of smooth successive maps from N to Σ, from Σ to Σ and finally from Σ to N. For such orbit the local analysis is performed in a usual way by means of the normal form methods. Another case is when we deal with the grazing orbit. Then one needs to find a separating curve l through Q which divides a small enough neighborhood D of Q on N into two disks D_-, D_+, so that the orbits from D_- do not intersect Σ whereas those from D_+ do intersect Σ. Orbits through this smooth curve l are those of the right system which have tangency with Σ; it occurs along the plane $x_1 = 0$, $x_2 = a$. We study the Poincaré map using simulations shown on the plots below. It is seen from these calculations that below the critical value of c we have an elliptic fixed point (the trace of that from the Lyapunov family of the period $2\pi/\omega$) enclosed by the family of closed curves (traces of invariant tori of the right system on the related level of the Hamiltonian), (see Fig. 7d). There is also a symmetric saddle fixed point corresponding to larger values of y_1. It is worth mentioning that the related saddle periodic orbit of the Hamiltonian system is made up of two pieces lying in both half-spaces. Separatrices of the saddle fixed point intersect and generate homoclinic tangle and stochastic layer (Fig. 7e). When c approaches to the critical value, these two fixed points move closer (Fig. 7c) and coalesce at the critical value (Fig. 7b). Apparently, when they merge, a parabolic fixed point with a cusp of its stable and unstable separatrices does not form, as it would happen in the smooth case. We assume that this degenerate fixed point has two separatrices as for a parabolic fixed point, but separatrices have a finite angle between them at the fixed point, as a consequence of non-smoothness. After their disappearance a new stochastic layer is formed (fig. 7a,b).

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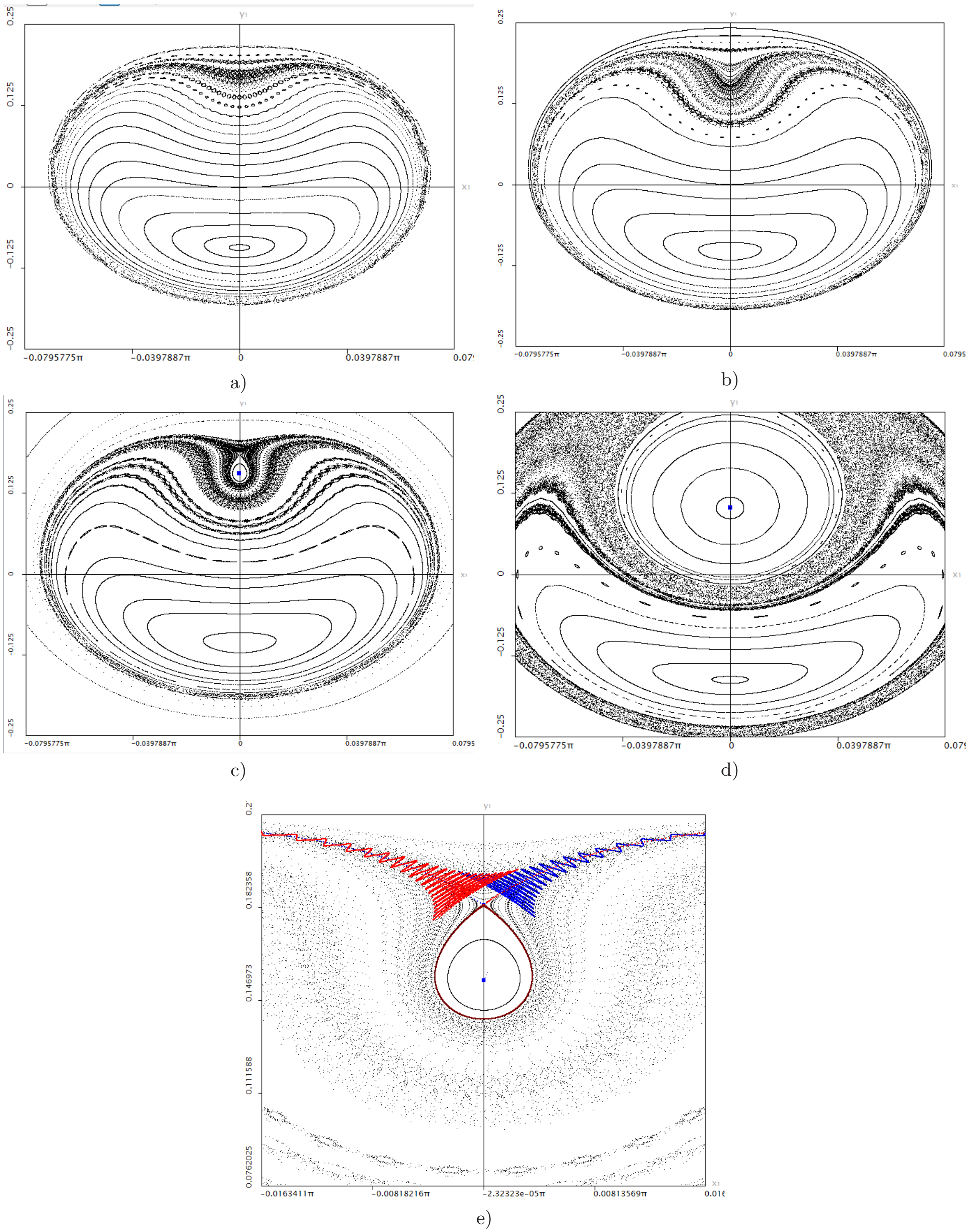

FIGURE 7. Homoclinic tangle for the saddle fixed point of the Poincaré map