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Zero-Stabilization for Some Diffusive Models with State Constraints

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Abstract. We discuss the zero-controllability and the zero-stabilizability for the nonnegative solutions to some Fisher-like models with nonlocal terms describing the dynamics of biological populations with diffusion, logistic term and migration. A necessary and sufficient condition for the nonnegative zero-stabilizabiity for a linear integro-partial differential equation is obtained in terms of the sign of the principal eigenvalue to a certain non-selfadjoint operator. For a related semilinear problem a necessary condition and a sufficient condition for the local nonnegative zero-stabilizability are also derived in terms of the magnitude of the above mentioned principal eigenvalue. The rate of stabilization corresponding to a simple feedback stabilizing control is dictated by the principal eigenvalue. A large principal eigenvalue leads to a fast stabilization to zero. A necessary condition and a sufficient condition for the stabilization to zero of the predator population in a predator-prey system is also investigated. Finally, a method to approximate the above mentioned principal eigenvalues is indicated.

Keywords and phrases: zero-stabilizability, state constraints, comparison result, principal eigenvalue, feedback control, population dynamics, reaction-diffusion system

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1. Null controllability and null stabilizability with state constraints

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ be a bounded domain and $\omega \subset\subset \Omega$ be a nonempty open subset, both with sufficiently smooth boundaries $\partial\Omega$ and $\partial\omega$, respectively, and such that $\Omega \setminus \overline{\omega}$ is a domain. One of the problems we are concerned in is the following

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) - a(x,t)y(x,t) = m(x)u(x,t), & (x,t) \in Q_T \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_T \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, $d \in (0,+\infty)$, $a \in L^\infty(\overline{\Omega} \times [0,T))$, $y_0 \in L^\infty(\Omega)$ and m is the characteristic function of ω. Here $u \in L^\infty_{loc}(\overline{\omega} \times [0,T))$ is a control which acts only in ω.

Problem (1.1) describes the gas diffusion as well as the dynamics of a biological population in the domain Ω; $y(x,t)$ denotes the density (of gas or of the population), $a(x,t)$ represents the growth rate at position $x \in \overline{\Omega}$ and moment $t \geq 0$, and d is the diffusion coefficient. The homogeneous Neumann boundary

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condition states that there is no gas or population transfer through the boundary of the domain (the domain is isolated). In case when the gas or the population must be removed from Ω (for example if represents an alien population which destroys the ecosystem) acting (via the control) in a subset, this can be mathematically formulated as follows.

"Given $y_0 \in L^\infty(\Omega)$ and $T \in (0, +\infty)$, is there any $u \in L^\infty(\omega \times (0,T))$ such that the solution y^u to (1.1) satisfies

$$y^u(x,T) = 0 \qquad \text{a.e. } x \in \Omega \text{ ?''}$$

If such a control u exists we say that (1.1) is null-controllable at moment T.

G. Lebeau and L. Robbiano have shown in 1995 [19] (see also [10]) that (1.1) is null-controllable at any moment $T > 0$. However, since $y(x,t)$ is a density, the next question we have to answer is

"Given $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, and $T \in (0, +\infty)$, is there any $u \in L^\infty(\omega \times (0,T))$ such that the solution y^u to (1.1) satisfies

$$y^u(x,t) \geq 0 \qquad \text{a.e. } (x,t) \in Q_T$$

and

$$y^u(x,T) = 0 \qquad \text{a.e. } x \in \Omega \text{ ?''}$$

The answer to this question is negative if for example y_0 is not 0 a.e. on $\Omega \setminus \overline{\omega}$. We argue by contradiction and assume that there exists a control u such that y^u satisfies the two above mentioned conditions, then there exists $g \in L^\infty(\partial\omega \times (0,T))$, $g(x,t) = y^u(x,t) \geq 0$ a.e. $(x,t) \in \partial\omega \times (0,T)$ and y^u is a solution to

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) - a(x,t)y(x,t) = 0, & (x,t) \in (\Omega \setminus \overline{\omega}) \times (0,T) \\ \partial_\nu y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,t) = g(x,t), & (x,t) \in \partial\omega \times (0,T) \\ y(x,0) = y_0(x), & x \in \Omega \setminus \overline{\omega}. \end{cases}$$

Since there exists $M_1 > 0$ such that $a(x,t) \geq -M_1$ a.e. in $\Omega \times (0,T)$, then the comparison result in Appendix implies that

$$0 \leq z(x,t) \leq y^u(x,t) \qquad \text{a.e. } (x,t) \in (\Omega \setminus \overline{\omega}) \times (0,T), \tag{1.2}$$

where z is the solution to

$$\begin{cases} \partial_t z(x,t) - d\Delta z(x,t) + M_1 z(x,t) = 0, & (x,t) \in (\Omega \setminus \overline{\omega}) \times (0,T) \\ \partial_\nu z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ z(x,t) = 0, & (x,t) \in \partial\omega \times (0,T) \\ z(x,0) = y_0(x), & x \in \Omega \setminus \overline{\omega}. \end{cases}$$

By (1.2) we get that

$$z(x,T) = 0 \qquad \text{a.e. } x \in \Omega \setminus \overline{\omega}.$$

The backward uniqueness result for linear parabolic equations (see [12]) implies that

$$z(x,0) = y_0(x) = 0 \qquad \text{a.e. } x \in \Omega \setminus \overline{\omega},$$

which is absurd.

The same argument may be used to show that we do not have null-controllability for nonnegative solutions (with state constraints) if we consider homogeneous Dirichlet boundary conditions instead. Remind that such boundary condition describes a completely inhospitable boundary $\partial\Omega$.

Remark 1.1. If the control u acts in the whole domain, i.e. $\omega = \Omega$ and if $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, then (1.1) is null-controllable. Moreover, the feedback control $u := -\rho y$, for sufficiently large $\rho > 0$, brings the solution y^u to 0 at the given moment $T > 0$ (see Lemma 3.1, p. 364 in [5]) . Of course ρ depends on T. In addition the state y^u remains nonnegative.

Remark 1.2. The result of G. Lebeau and L. Robbiano shows that for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$ which is not 0 a.e. in $\Omega \setminus \overline{\omega}$, and for any $T > 0$, there exists a control $u \in L^\infty(\omega \times (0,T))$ which brings y^u to 0 at moment T, but y^u is necessarily negative on some subdomain of $\Omega \times (0,T)$.

So, a more appropriate approach for the problem of removing the gas or population from Ω is to

"Find a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (1.1) corresponding to $T := +\infty$ satisfies

$$y^u(x,t) \geq 0 \qquad \text{a.e. } (x,t) \in Q_\infty \tag{1.3}$$

and

$$\lim_{t\to+\infty} y^u(\cdot,t) = 0 \qquad \text{in } L^\infty(\Omega)." \tag{1.4}$$

If for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$ such a control exists, we say that (1.1) is nonnegatively zero-stabilizable.

Remark 1.3. If we are only looking for a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (1.1) (with $T := +\infty$) satisfies (1.4), then it is well known that there exists a control with a simple structure $u := Py$ (a feedback control), where the operator P is the solution to a certain algebraic Riccati equation (see [20]). The solution corresponding to this feedback and to a nonnegative initial datum y_0 is not necessarily nonnegative. In fact we shall prove in the next sections that only for certain ω, there exists a stabilizing control u which preserves the nonnegativity of the state y^u.

Another important task will be to find a stabilizing control u with a simple structure. Such stabilization problems for some integro-partial differential equations as well as for some reaction-diffusion systems with state constraints will be investigated in the next sections.

We mention that important stability results concerning nonnegative compartmental systems can be found in [13]. For controllability of positive continuous-time linear systems with time variable coefficients we refer to [17], [18].

Here is the plan of the paper. Section 2 concerns the nonnegative zero-stabilization for a linear integro-partial differential equation. A necessary and sufficient condition for the nonnegative zero-stabilizability is given in terms of the sign of the principal eigenvalue to a certain non-selfadjoint operator. Actually, the main result in this section has been previously established by the author in [2]. In the next section a necessary condition and a sufficient condition for the local nonnegative zero-stabilizability for a related semilinear problem are derived in terms of the magnitude of the above mentioned principal eigenvalue. A necessary condition and a sufficient condition for the stabilization to zero of the predator population in a predator-prey system are investigated in Section 4. A method to approximate the above mentioned principal eigenvalues is indicated in Section 5. Finally, an important comparison result is given in Appendix.

2. Stabilization for a linear integro-partial differential equation

Let us consider the following general linear model which describes the dynamics of a population dynamics with diffusion and migration, and subject to a control localized on ω:

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) + \int_\Omega k(x,x')y(x',t)dx' + m(x)u(x,t), & (x,t) \in Q_\infty \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_\infty \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Here

$$d \in (0,+\infty),\ a \in L^\infty(\Omega),\ y_0 \in L^\infty(\Omega),\ k \in L^\infty(\Omega \times \Omega),$$

$$y_0(x) \geq 0 \qquad \text{a.e. } x \in \Omega, \quad k(x,x') \geq 0 \qquad \text{a.e. } (x,x') \in \Omega \times \Omega.$$

If we denote by $\tilde{a}(x)$ the natural growth rate of the population at position x and by $\overline{a}(x')$ the proportion of the population at position x' that migrate to other positions, and by $l(x,x') \geq 0$ the kernel that gives the redistribution by migration, then

$$\int_{\Omega} l(x,x')\overline{a}(x')y(x',t)dx'$$

gives the population immigrated to position x from allover Ω. The conservation of the total population gives that

$$\int_{\Omega} l(x,x')dx = 1, \qquad \text{a.e. } x' \in \Omega.$$

In (2.1), $a(x) = \tilde{a}(x) - \overline{a}(x)$ and $k(x,x') = l(x,x')\overline{a}(x')$. For other population models with nonlocal terms we refer to [7], [11].

The question we wish to answer is the following: "Is it true that for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. in Ω, there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (2.1) satisfies (1.3) and (1.4) ?"

If the answer to this question is affirmative we say that (2.1) is nonnegatively zero-stabilizable.

We shall emphasize the deep relationship between the nonnegative zero-stabilizability and the sign of the principal eigenvalue λ_1^ω to

$$\begin{cases} -d\Delta\varphi(x) - a(x)\varphi(x) - \displaystyle\int_{\Omega\setminus\overline{\omega}} k(x,x')\varphi(x')dx' = \lambda\varphi(x), & x \in \Omega\setminus\overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \partial_\nu\varphi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

and the sign of the principal eigenvalue $\lambda_{1\gamma}^\omega$ to

$$\begin{cases} -d\Delta\varphi(x) - a(x)\varphi(x) - \displaystyle\int_{\Omega} k(x,x')\varphi(x')dx' + m(x)\gamma\varphi(x) = \lambda\varphi(x), & x \in \Omega \\ \partial_\nu\varphi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $\gamma \in [0,+\infty)$.

By Krein-Rutman's theorem (see [8]), the basic properties of the principal eigenvalue λ_1^ω to (2.2) and of the principal eigenvalue $\lambda_{1\gamma}^\omega$ to (2.3), and of the corresponding eigenvectors can be obtained. One of the main results we shall use is the following one (see [3], [4])

Lemma 2.1. *The mapping $\gamma \mapsto \lambda_{1\gamma}^\omega$ is increasing on $[0,+\infty)$ and*

$$\lim_{\gamma\to+\infty} \lambda_{1\gamma}^\omega = \lambda_1^\omega.$$

Remark 2.2. If $k(x,x') = k(x',x)$ a.e. in $\Omega\times\Omega$, then Rayleigh's principle may be used instead of Krein-Rutman's theorem in order to derive some additional properties and methods to approximate λ_1^ω and $\lambda_{1\gamma}^\omega$ (see [14], [15]). For methods to approximate them in the nonsymmetric case we refer to [2], [14].

The following zero-stabilizability result concerning (2.1) has been established in [2].

Theorem 2.3. *If (2.1) is nonegatively zero-stabilizable then $\lambda_1^\omega > 0$.*

Conversely, if $\lambda_1^\omega > 0$, then (2.1) is nonnegatively zero-stabilizable and for sufficiently large γ the feedback control $u := -\gamma y$ stabilizes (2.1) and preserves the nonnegativity of y^u.

Proof. If (2.1) is nonnegatively zero-stabilizable then for an arbitrary but fixed $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, there exists a stabilizing control u. The comparison result in Appendix allows us to infer that

$$0 \leq \tilde{y}(x,t) \leq y^u(x,t), \qquad \text{a.e. } x \in \Omega \setminus \overline{\omega}, \forall t \geq 0,$$

where $\tilde{y}$ is the solution to

$$\begin{cases} \partial_t \tilde{y}(x,t) - d\Delta \tilde{y}(x,t) = a(x)\tilde{y}(x,t) + \int_{\Omega\setminus\overline{\omega}} k(x,x')\tilde{y}(x',t)dx', & x \in \Omega \setminus \overline{\omega},\ t > 0 \\ \tilde{y}(x,t) = 0, & x \in \partial\omega,\ t > 0 \\ \partial_\nu \tilde{y}(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ \tilde{y}(x,0) = y_0(x), & x \in \Omega \setminus \overline{\omega}. \end{cases} \tag{2.4}$$

Since $\lim_{t\to+\infty} y^u(\cdot,t) = 0$ in $L^\infty(\Omega)$ we conclude that

$$\lim_{t\to+\infty} \tilde{y}(\cdot,t) = 0 \qquad \text{in } L^\infty(\Omega \setminus \overline{\omega}).$$

If φ_1^* is the eigenfunction corresponding to the adjoint problem for (2.2) and to λ_1^ω, and such that $\varphi_1^*(x) > 0$ a.e. in $\Omega\setminus\overline{\omega}$, $\|\varphi_1^*\|_{L^2(\Omega\setminus\overline{\omega})} = 1$, then multiplying the first equation in (2.4) by φ_1^* and integrating on $\Omega\setminus\overline{\omega}$ we get

$$\frac{d}{dt}(\int_{\Omega\setminus\overline{\omega}} \tilde{y}(x,t)\varphi_1^*(x)dx) = -\lambda_1^\omega \int_{\Omega\setminus\overline{\omega}} \tilde{y}(x,t)\varphi_1^*(x)dx,$$

for any $t > 0$ and so

$$\int_{\Omega\setminus\tilde{\omega}} \tilde{y}(x,t)\varphi_1^*(x)dx = \exp(-\lambda_1^\omega t) \int_{\Omega\setminus\overline{\omega}} y_0(x)\varphi_1^*(x)dx, \quad \forall t \geq 0.$$

If we consider y_0 such that $\int_{\Omega\setminus\overline{\omega}} y_0(x)\varphi_1^*(x)dx > 0$ (it is enough to consider y_0 which is not identically 0 on $\Omega \setminus \overline{\omega}$), then we conclude that $\lambda_1^\omega > 0$.

Conversely, if $\lambda_1^\omega > 0$, then for sufficiently large $\gamma > 0$ we have that $\lambda_{1\gamma}^\omega > 0$. If we consider $u := -\gamma y$, then (2.1) becomes

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) + \int_\Omega k(x,x')y(x',t)dx' - m(x)\gamma y(x,t), & (x,t) \in Q_\infty \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_\infty \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

If $\varphi_{1\gamma}^*$ is the eigenfunction corresponding to the adjoint problem for (2.3) and to $\lambda_{1\gamma}^\omega$, and such that $\varphi_{1\gamma}^*(x) > 0$ a.e. in Ω, $\|\varphi_{1\gamma}^*\|_{L^2(\Omega)} = 1$, then multiplying the first equation in (2.5) by $\varphi_{1\gamma}^*$ we get after an easy calculation that

$$\frac{d}{dt}(\int_\Omega y(x,t)\varphi_{1\gamma}^*(x)dx) = -\lambda_{1\gamma}^\omega \int_\Omega y(x,t)\varphi_{1\gamma}^*(x)dx, \quad t > 0,$$

where y is the solution to (2.5). In conclusion

$$\int_\Omega y(x,t)\varphi_{1\gamma}^*(x)dx = \exp(-\lambda_{1\gamma}^\omega t) \int_\Omega y_0(x)\varphi_{1\gamma}^*(x)dx, \quad \forall t \geq 0.$$

Since there exists $M_0 \in (0,+\infty)$ such that

$$M_0 \leq \varphi_{1\gamma}^*(x), \qquad \text{a.e. } x \in \Omega$$

(see [3]) we may infer that

$$\lim_{t\to+\infty} y(\cdot,t) = 0 \qquad \text{in } L^1(\Omega)$$

at the same rate as $\exp(-\lambda_{1\gamma}^\omega t)$. The convergence $\lim_{t\to+\infty} y(\cdot,t) = 0$ in $L^\infty(\Omega)$ follows as in [3], at the same rate. □

3. Stabilization for a semilinear integro-partial differential equation

Consider now the following semilinear model, which includes in addition a possible logistic term or some other nonlinear term depending on the population density y (extends the Fisher model, see [9]):

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) + \displaystyle\int_\Omega k(x,x')y(x',t)dx' \\ \qquad\qquad +f(y(x,t)) + m(x)u(x,t), & (x,t) \in Q_\infty \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_\infty \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Assume that $d,\ a,\ k,\ y_0$ satisfy the hypotheses in Section 2. In addition, we assume $f \in C^1(\mathbf{R})$ and $f(0) = 0$. Let us remark that applying the Banach fixed point theorem we may infer that for any $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$, (3.1) has a unique solution.

Definition 3.1. We say that (3.1) is nonnegatively zero-stabilizable if for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. in Ω, there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (3.1) satisfies (1.3) and (1.4) .

We say that (3.1) is locally nonnegatively zero-stabilizable if there exists $M_2 \in (0,+\infty)$ such that for any $y_0 \in L^\infty(\Omega)$, $\|y_0\|_{L^\infty(\Omega)} \leq M_2$, $y_0(x) \geq 0$ a.e. in Ω, there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (3.1) satisfies (1.3) and (1.4) .

Actually, the following local stabilizability result holds

Theorem 3.2. *If (3.1) is locally nonnegatively zero-stabilizable then $\lambda_1^\omega - f'(0) \geq 0$.*

Conversely, if $\lambda_1^\omega - f'(0) > 0$, then (3.1) is locally nonnegatively zero-stabilizable and for sufficiently large γ the feedback control $u := -\gamma y$ stabilizes (3.1) and preserves the nonnegativity of y^u.

Proof. Since f' is continuous at 0 it follows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|f'(r) - f'(0)| < \varepsilon$ for any $|r| < \delta(\varepsilon)$. If (3.1) is locally nonnegatively zero-stabilizable then for any $y_0 \in L^\infty(\Omega)$, $\|y_0\|_{L^\infty(\Omega)} \leq M_2$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, there exists a stabilizing control u. So, for any $\varepsilon > 0$, there exists $T_{1\varepsilon} \in [0,+\infty)$ such that

$$\|y^u(\cdot,t)\|_{L^\infty(\Omega)} < \delta(\varepsilon), \qquad \forall t \geq T_{1\varepsilon}.$$

This implies that

$$f(y^u(x,t)) = f'(\xi(x,t))y^u(x,t) \geq (f'(0)-\varepsilon)y^u(x,t), \qquad \text{a.e. } x \in \Omega,\ t \geq T_{1\varepsilon}$$

(we have used the mean value theorem. Here $\xi(x,t) \in [0, y^u(x,t)]$). The comparison result in Appendix allows us to infer that

$$0 \leq \tilde{y}(x,t) \leq y^u(x,t), \qquad \text{a.e. } x \in \Omega \setminus \overline{\omega}, \forall t \geq T_{1\varepsilon},$$

where $\tilde{y}$ is the solution to

$$\begin{cases} \partial_t \tilde{y}(x,t) - d\Delta \tilde{y}(x,t) = a(x)\tilde{y}(x,t) + \displaystyle\int_{\Omega\setminus\overline{\omega}} k(x,x')\tilde{y}(x',t)dx' \\ \qquad\qquad +(f'(0)-\varepsilon)\tilde{y}(x,t), & x \in \Omega \setminus \overline{\omega},\ t > T_{1\varepsilon} \\ \tilde{y}(x,t) = 0, & x \in \partial\omega,\ t > T_{1\varepsilon} \\ \partial_\nu \tilde{y}(x,t) = 0, & x \in \partial\Omega,\ t > T_{1\varepsilon} \\ \tilde{y}(x,T_{1\varepsilon}) = y^u(x,T_{1\varepsilon}), & x \in \Omega \setminus \overline{\omega}. \end{cases} \tag{3.2}$$

If y_0 is not identically 0 on $\Omega \setminus \overline{\omega}$, then $y^u(\cdot,T_{1\varepsilon})$ is not identically 0 on $\Omega \setminus \overline{\omega}$. We argue by contradiction. Assume that $y^u(x,T_{1\varepsilon}) = 0$ a.e. in $\Omega\setminus\overline{\omega}$. Since y^u is bounded in Q_∞, we get that there exists $\theta \in (0,+\infty)$ such that

$$f(y^u(x,t)) \geq -\theta y^u(x,t), \qquad \text{a.e. in } Q_\infty.$$

The comparison result in Appendix implies that

$$0 \le z(x,t) \le y^u(x,t), \qquad \text{a.e. in } (\Omega \setminus \overline{\omega}) \times (0,+\infty),$$

and consequently that $z(x,T_{1\varepsilon}) = 0$ a.e. in $\Omega \setminus \overline{\omega}$. Here z is the solution to

$$\begin{cases} \partial_t z(x,t) - d\Delta z(x,t) = a(x)z(x,t) - \theta z(x,t), & (x,t) \in (\Omega \setminus \overline{\omega}) \times (0,+\infty) \\ z(x,t) = 0, & (x,t) \in \partial\omega \times (0,+\infty) \\ \partial_\nu z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ z(x,0) = y_0(x), & x \in \Omega \setminus \overline{\omega}. \end{cases} \tag{3.3}$$

Applying the backward uniqueness theorem for (3.3) we conclude that $y_0(x) = 0$ a.e. $x \in \Omega \setminus \overline{\omega}$, which is absurd.

Now, since $y^u(\cdot, T_{1\varepsilon})$ is not identically 0 on $\Omega \setminus \overline{\omega}$ and the solution $\tilde{y}$ to (3.2) satisfies $\lim_{t\to+\infty} \tilde{y}(\cdot,t) = 0$ in $L^\infty(\Omega \setminus \overline{\omega})$, we may apply the method in the proof of Theorem 2.3 to conclude that

$$\lambda_1^\omega - f'(0) + \varepsilon > 0,$$

for any $\varepsilon > 0$. Finally, we get $\lambda_1^\omega - f'(0) \ge 0$.

Conversely, assume that $\lambda_1^\omega - f'(0) > 0$. Then for sufficiently small $\varepsilon \in (0,+\infty)$, and for sufficiently large $\gamma > 0$ we have that

$$\lambda_{1\gamma}^\omega - f'(0) > \varepsilon.$$

Consider such ε and such γ, and w the solution to

$$\begin{cases} \partial_t w(x,t) - d\Delta w(x,t) = a(x)w(x,t) + \displaystyle\int_\Omega k(x,x')w(x',t)dx' \\ \qquad\qquad +(f'(0)+\varepsilon)w(x,t) - m(x)\gamma w(x,t), & (x,t) \in Q_\infty \\ \partial_\nu w(x,t) = 0, & (x,t) \in \Sigma_\infty \\ w(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{3.4}$$

The comparison result in Appendix implies that w is nonnegative. Multiplying the first equation in (3.4) by $\varphi_{1\gamma}^*$, defined in Section 2, and integrating on Ω we get after an easy evaluation that there exists a positive constant M_3 such that

$$\|w(\cdot,t)\|_{L^1(\Omega)} \le M_3 \|y_0\|_{L^\infty(\Omega)} \exp(-(\lambda_{1\gamma}^\omega - f'(0) - \varepsilon)t), \qquad \forall t \ge 0.$$

Since

$$0 \le \int_\Omega k(x,x')w(x',t)dx' \le \|k\|_{L^\infty(\Omega\times\Omega)} \|w(\cdot,t)\|_{L^1(\Omega)}, \qquad \forall t \ge 0,$$

then using a so called parabolic $L^1 \to L^\infty$ inequality (see [5], [6]) we conclude that there exists a positive constant M_4 such that

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le M_4 \|y_0\|_{L^\infty(\Omega)} \exp(-(\lambda_{1\gamma}^\omega - f'(0) - \varepsilon)t), \qquad \forall t \ge 0.$$

In conclusion, if $\|y_0\|_{L^\infty(\Omega)} < \frac{\delta(\varepsilon)}{M_4}$, then the comparison result in Appendix implies that

$$0 \le y(x,t) \le w(x,t) \qquad \text{a.e. in } Q_\infty,$$

where y is the solution to

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) + \displaystyle\int_\Omega k(x,x')y(x',t)dx' + f(y(x,t)) \\ \qquad\qquad -m(x)\gamma y(x,t), & (x,t) \in Q_\infty \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_\infty \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases}$$

We also infer that $u := -\gamma y$ is a stabilizing control and that the stabilization rate is that of $\exp(-(\lambda_{1\gamma}^\omega - f'(0) - \varepsilon)t)$. □

Remark 3.3. If in addition $f(r) \leq f'(0)r$ for any $r \geq 0$, then an attentive examination of the proof leads us to conclusion that if $\lambda_1^\omega - f'(0) > 0$, then (3.1) is actually nonnegatively zero-stabilizable.

This happens for example if $f(r) = -\alpha r^2$ $(\alpha > 0)$ and represents a logistic term.

Remark 3.4. The main results in this section remain true if we replace the boundary condition in (3.1) by the homogeneous Dirichlet boundary condition and consider λ_1^ω as the principal eigenvalue to

$$\begin{cases} -d\Delta\varphi(x) - a(x)\varphi(x) - \int_{\Omega\setminus\overline{\omega}} k(x,x')\varphi(x')dx' = \lambda\varphi(x), & x \in \Omega\setminus\overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Moreover, $\lambda_{1\gamma}^\omega$ is now the principal eigenvalue to

$$\begin{cases} -d\Delta\varphi(x) - a(x)\varphi(x) - \int_{\Omega} k(x,x')\varphi(x')dx' + m(x)\gamma\varphi(x) = \lambda\varphi(x), & x \in \Omega \\ \varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Remark 3.5. The main result in this section remains true if we replace the operator $-\Delta$ by a general elliptic operator.

Remark 3.6. The evaluation of the principal eigenvalue λ_1^ω and the optimization of the position of ω in order to maximize λ_1^ω are two extremely important problems related to stabilization. For some results concerning these questions we refer to [1] for the case $k \equiv 0$ and to [2], [3], [4] for some particular cases related to nontrivial k.

4. A stabilization result for a predator-prey model

Consider a reaction-diffusion system that describes the dynamics of two interacting populations that are free to move in the habitat Ω and are subject to a control u that acts in ω:

$$\begin{cases} \partial_t h(x,t) - d_1\Delta h(x,t) = r(x)h(x,t) - \mathcal{M}(h(x,t))h(x,t) + \int_\Omega k_1(x,x')h(x',t)dx' \\ \qquad -F(h(x,t),p(x,t))p(x,t), & (x,t) \in Q_\infty \\ \partial_t p(x,t) - d_2\Delta p(x,t) = -a_2(x)p(x,t) + c_2\int_\Omega k_2(x,x')F(h(x',t),p(x',t))p(x',t)dx' \\ \qquad +m(x)u(x,t), & (x,t) \in Q_\infty \\ \partial_\nu h(x,t) = 0,\ \partial_\nu p(x,t) = 0, & (x,t) \in \Sigma_\infty \\ h(x,0) = h_0(x),\ p(x,0) = p_0(x), & x \in \Omega. \end{cases} \tag{4.1}$$

Here $h(x,t)$ is the density of a prey population species and $p(x,t)$ is the density of a predator population species at position x and moment t; $\mathcal{M}(h)h$ represents a logistic term (where $\mathcal{M}(h)$ is an additional mortality rate due to the overcrowding) and the term $\int_\Omega k_1(x,x')h(x',t)dx'$ describes by the migration of the prey population. We have denoted by $F(h,p)$ the functional response to predation, see [4], [21], [22]. The prey captured at location x' and moment t is transformed into biomass via a conversion factor c_2 giving a numerical response to predation $c_2F(h(x',t),p(x',t))p(x',t)$. This quantity is distributed over Ω via the kernel k_2 which must respect the total biomass conservation.

Assume that $r,\ a_2,\ h_0,\ p_0 \in L^\infty(\Omega)$, $d_1,\ d_2,\ c_2 \in (0,+\infty)$, $k_1,\ k_2 \in L^\infty(\Omega\times\Omega)$

$$k_1(x,x') \geq 0,\ k_1(x,x') = k_1(x',x) \qquad \text{a.e. in } \Omega\times\Omega,$$

$$k_2(x,x') \geq 0 \qquad \text{a.e. in } \Omega\times\Omega, \qquad \int_\Omega k_2(x,x')dx = 1 \qquad \text{a.e. } x' \in \Omega,$$

$$h_0(x) \geq 0,\ p_0(x) \geq 0 \qquad \text{a.e. in } \Omega, \quad \|h_0\|_{L^\infty(\Omega)} > 0,\ \|p_0\|_{L^\infty(\Omega)} > 0.$$

$\mathcal{M} : [0, +\infty) \to \mathbf{R}$ is a C^1-class nondecreasing function such that

$$\mathcal{M}(0) = 0, \quad \lim_{r \to +\infty} \mathcal{M}(r) = +\infty.$$

Last $F : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is locally Lipschitz continuous, $h \mapsto F(h, p)$ is nondecreasing on $[0, +\infty)$ for any $p \geq 0$, $p \mapsto F(h, p)$ is nonincreasing on $[0, +\infty)$ for any $h \geq 0$, $F(0, p) = 0$ for any $p \geq 0$, $F(h, 0) > 0$ for any $h \in (0, +\infty)$.

In applications the functional response to predation may take several forms, as Lotka-Volterra, $F(h, p) = \rho h$, Holling type $k + 1$, $F(h, p) = \frac{\rho h^k}{1+qh^k}$, or Beddington-De Angelis, $F(h, p) = \frac{\rho h}{1+qh+rp}$ $(\rho, q, r > 0)$, all covered by our assumptions on F.

Let $\mu_1(\omega)$ be the principal eigenvalue to

$$\begin{cases} -d_1 \Delta\psi(x) - r(x)\psi(x) - \displaystyle\int_\Omega k_1(x, x')\psi(x')dx' = \mu\psi(x), & x \in \Omega \\ \partial_\nu \psi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Assume that $\mu_1(\omega) < 0$ (this happens for example if $r(x) > 0$ a.e.). Actually, this corresponds to the situation when in the absence of predators, the prey population persist.

Consider H the unique positive solution of

$$\begin{cases} -d_1 \Delta h(x) = r(x)h(x) - \mathcal{M}(h(x))h + \displaystyle\int_\Omega k_1(x, x')h(x')dx', & x \in \Omega \\ \partial_\nu h(x) = 0, & x \in \partial\Omega. \end{cases}$$

Existence of this solution can be proved as in [2] (see also [24]).

Definition 4.1. We say that the predator population is nonnegatively zero-stabilizable if for any h_0, p_0 satisfying the hypotheses there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0, +\infty))$ such that the solution (h^u, p^u) to (4.1) satisfies

$$h^u(x, t) \geq 0,\ p^u(x, t) \geq 0 \qquad \text{a.e. in } Q_\infty,$$

and

$$\lim_{t \to +\infty} p^u(\cdot, t) = 0 \quad \text{in } L^\infty(\Omega).$$

We shall prove that the nonnegative zero-stability of the predator population is deeply related to the sign of $\lambda_1(\omega)$, the principal eigenvalue to

$$\begin{cases} -d_2 \Delta\varphi(x) + a_2(x)\varphi(x) - c_2 \displaystyle\int_{\Omega\setminus\overline{\omega}} k_2(x, x')F(H(x'), 0)\varphi(x')dx' = \lambda\varphi(x), & x \in \Omega \setminus \overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \partial_\nu \varphi(x) = 0, & x \in \partial\Omega, \end{cases}$$

and to the sign of $\lambda_{1\gamma}(\omega)$ (for $\gamma \geq 0$), the principal eigenvalue to

$$\begin{cases} -d_2 \Delta\varphi(x) + a_2(x)\varphi(x) - c_2 \displaystyle\int_\Omega k_2(x, x')F(H(x'), 0)\varphi(x')dx' \\ \qquad\qquad + m(x)\gamma\varphi(x) = \lambda\varphi(x), & x \in \Omega \\ \partial_\nu \varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

The existence of $\lambda_1(\omega)$ and $\lambda_{1\gamma}(\omega)$ and their basic properties follows via Lemma 2.1. Morover, the mapping $\gamma \mapsto \lambda_{1\gamma}(\omega)$ is increasing and continuous on $[0, +\infty)$ and

$$\lim_{\gamma \to +\infty} \lambda_{1\gamma}(\omega) = \lambda_1(\omega).$$

Theorem 4.2. *If the predator population is nonegatively zero-stabilizable then* $\lambda_1(\omega) \geq 0$.

Conversely, if $\lambda_1(\omega) > 0$, *then the predator population is nonnegatively zero-stabilizable and for sufficiently large* $\gamma > 0$ *the feedback control* $u := -\gamma p$ *stabilizes the predator population and preserves the nonnegativity of* h^u *and* p^u.

Proof. Assume that there exists a stabilizing control u. Since $\lim_{t\to+\infty} p^u(\cdot,t) = 0$ in $L^\infty(\Omega)$, it follows as in Proposition 2 in [4] (see also [24] for the method of proof) that

$$\lim_{t\to+\infty} h^u(\cdot,t) = H \qquad \text{in } L^\infty(\Omega).$$

We infer that for any $\varepsilon > 0$, there exists $T_\varepsilon \geq 0$ such that

$$\|F(h^u(x,t), p^u(x,t)) - F(H(x),0)\|_{L^\infty(\Omega)} < \varepsilon, \quad \forall t \geq T_\varepsilon.$$

Let $\tilde{p}$ be the solution to

$$\begin{cases} \partial_t p(x,t) - d_2 \Delta p(x,t) = -a_2(x)p(x,t) & \\ \qquad\qquad +c_2 \displaystyle\int_{\Omega\setminus\overline{\omega}} k_2(x,x')(F(H(x'),0) - \varepsilon)p(x',t)dx', & x \in \Omega\setminus\overline{\omega},\ t > T_\varepsilon \\ p(x,t) = 0, & x \in \partial\omega,\ t > T_\varepsilon \\ \partial_\nu p(x,t) = 0, & x \in \partial\Omega,\ t > T_\varepsilon \\ p(x,T_\varepsilon) = p^u(x,T_\varepsilon), & x \in \Omega\setminus\overline{\omega}. \end{cases}$$

The comparison result in Appendix implies that

$$0 \leq \tilde{p}(x,t) \leq p^u(x,t) \qquad \text{a.e. in } (\Omega\setminus\overline{\omega}) \times (T_\varepsilon, +\infty),$$

and consequently

$$\lim_{t\to+\infty} \tilde{p}(\cdot,t) = 0 \qquad \text{in } L^\infty(\Omega\setminus\overline{\omega}).$$

The backward uniqueness theorem for parabolic equations implies that $\tilde{p}(x,T_\varepsilon)$ is not identically 0 in $\Omega\setminus\overline{\omega}$. We may conclude as in Section 2 that the principal eigenvalue $\lambda_1(\omega,\varepsilon)$ to

$$\begin{cases} -d_2\Delta\varphi(x) + a_2(x)\varphi(x) - c_2 \displaystyle\int_{\Omega\setminus\overline{\omega}} k_2(x,x')(F(H(x'),0) - \varepsilon)\varphi(x')dx' = \lambda\varphi(x), & x \in \Omega\setminus\overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \partial_\nu\varphi(x) = 0, & x \in \partial\Omega, \end{cases}$$

is positive (for any $\varepsilon > 0$). Since

$$\lim_{\varepsilon\to 0} \lambda_1(\omega,\varepsilon) = \lambda_1(\omega)$$

(this follows by using of Krein-Rutman's theorem as in [4]) then we conclude that $\lambda_1(\omega) \geq 0$.

Conversely, assume that $\lambda_1(\omega) > 0$. For arbitrary $\varepsilon > 0$ and $\gamma > 0$, consider $\lambda_{1\gamma}(\omega,\varepsilon)$ the principal eigenvalue to the following eigenvalue problem

$$\begin{cases} -d_2\Delta\varphi(x) + a_2(x)\varphi(x) - c_2 \displaystyle\int_{\Omega} k_2(x,x')(F(H(x'),0) + \varepsilon)\varphi(x')dx' + m(x)\gamma\varphi(x) = \lambda\varphi(x), \ x \in \Omega \\ \partial_\nu\varphi(x) = 0, \qquad x \in \partial\Omega. \end{cases}$$

Since $\lim_{\gamma\to+\infty,\varepsilon\to 0} \lambda_{1\gamma}(\omega,\varepsilon) = \lambda_1(\omega)$ (this follows as in [4]), it follows that for sufficiently small $\varepsilon > 0$ and for sufficiently large $\gamma > 0$

$$\lambda_{1\gamma}(\omega,\varepsilon) > 0.$$

If we denote by (h,p) the solution to (4.1) corresponding to $u := -\gamma p$, then the comparison result in Appendix implies that

$$0 \leq h(x,t) \leq \overline{h}(x,t) \qquad \text{a.e. in } Q_\infty,$$

where $\overline{h}$ is the solution to

$$\begin{cases} \partial_t h(x,t) - d_1 \Delta h(x,t) = r(x)h(x,t) - \mathcal{M}(h(x,t))h(x,t) + \int_\Omega k_1(x,x')h(x',t)dx', & (x,t) \in Q_\infty \\ \partial_\nu h(x,t) = 0, & (x,t) \in \Sigma_\infty \\ h(x,0) = h_0(x), & x \in \Omega. \end{cases} \tag{4.2}$$

Since the solution $\overline{h}$ to (4.2) satisfies

$$\lim_{t\to=\infty} \overline{h}(\cdot,t) = H \qquad \text{in } L^\infty(\Omega)$$

(this follows using the method in [2], [4]) we conclude that for any $\varepsilon > 0$, there exists $T_{2\varepsilon} \geq 0$ such that

$$f(h(x,t),p(x,t)) \leq f(\overline{h}(x,t),0) \leq f(H(x),0) + \varepsilon \qquad \text{a.e. in } \Omega \times (T_{2\varepsilon}, +\infty).$$

Let $\overline{p}$ be the solution to

$$\begin{cases} \partial_t \overline{p}(x,t) - d_2 \Delta \overline{p}(x,t) = -a_2(x)\overline{p}(x,t) \\ \qquad +c_2 \int_\Omega k_2(x,x')(F(H(x'),0)+\varepsilon)\overline{p}(x',t)dx' - m(x)\gamma\overline{p}(x,t), & (x,t) \in \Omega \times (T_{2\varepsilon}, +\infty) \\ \partial_\nu \overline{p}(x,t) = 0, & (x,t) \in \partial\Omega \times (T_{2\varepsilon}, +\infty) \\ \overline{p}(x,T_{2\varepsilon}) = p(x,T_{2\varepsilon}), & x \in \Omega. \end{cases}$$

The comparison result implies the nonnegativity of h, and that

$$0 \leq p(x,t) \leq \overline{p}(x,t) \qquad \text{a.e. in } \Omega \times (T_{2\varepsilon}, +\infty).$$

Condition $\lambda_{1\gamma}(\omega,\varepsilon) > 0$ implies that

$$\lim_{t\to+\infty} \overline{p}(\cdot,t) = 0 \qquad \text{in } L^\infty(\Omega),$$

and consequently

$$\lim_{t\to+\infty} p(\cdot,t) = 0 \qquad \text{in } L^\infty(\Omega).$$

□

Remark 4.3. The stabilization rate of p is the rate of $\exp(-\lambda_{1\gamma}(\omega,\varepsilon)t)$. For sufficiently large $\gamma > 0$ and sufficiently small $\varepsilon > 0$ this rate is close to that of $\exp(-\lambda_1(\omega)t)$

5. Final remarks

The relationship between the nonnegative zero-stabilizability of one of the components of a reaction-diffusion system and the magnitude of the principal eigenvalues to some related problems which have particular forms of the problems investigated in the present paper has been studied in [1], [2], [3], [4]. The rate of stabilization to 0 of the solution corresponding to some simple feedback control is also dictated by λ_1^ω and by $\lambda_{1\gamma}^\omega$. Hence, it is obvious the importance of deriving a method to maximize $\lambda_{1\gamma}^\omega$ on the set of all translations of ω or at least to derive an iterative method to improve the position (by translations) of the support of the feedback stabilizing control in order to get a biger principal eigenvalue, and consequently a faster stabilization. Remark that these are difficult problems due to the fact that the operator involved is not self-adjoint. So, variational techniques using min-max formula cannot be used. Thus, as it was stressed by A. Henrot, El Haj Laamri, D. Schmitt [14], the general methods for maximization problem of the principal eigenvalue, following the lines of [15] cannot be used. A certain method to maximize the principal eigenvalue for the sum of an elliptic and an integral operator is proposed in [14]. An alternative method has been presented in [2]. We also remark that for $k \equiv 0$ and constant rates, the rearrangement techniques (see [16]) can also be useful.

Let us remind the results in [2] which allow to approximate the principal eigenvalue $\lambda_{1\gamma}^{\omega}$ and to derive an iterative algorithm to improve the position of the domain where the control acts (by translations) in order to get a faster stabilization.

Consider the following particular population dynamics model with a special logistic term

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) + \zeta y(x,t) + \int_\Omega k(x,x')y(x',t)dx' & \\ \qquad -m(x)\gamma y(x,t) - y(x,t)\int_\Omega y(x,t)dx, & (x,t)\in Q_\infty \\ \partial_\nu y(x,t) = 0, & (x,t)\in \Sigma_\infty \\ y(x,0) = 1, & x\in\Omega, \end{cases} \tag{5.1}$$

where $\gamma > 0$ and ζ is a constant greater than $\lambda_{1\gamma}^{\omega}$. Assume that the hypotheses in Section 2 hold.

The following result has been established in [2] (Theorem 4.1)

Theorem 5.1. *The solution y^ω to (5.1) satisfies*

$$\lim_{t\to+\infty}\int_\Omega y^\omega(x,t)dx = \zeta - \lambda_{1\gamma}^{\omega}.$$

This result provides an approximating method for $\lambda_{1\gamma}^{\omega}$. Namely, for T large enough, $\zeta - \int_\Omega y^\omega(x,T)dx$ approximates $\lambda_{1\gamma}^{\omega}$.

Since our interest (related to the stabilization problem) will be to find a position for ω which provides a large value of $\lambda_{1\gamma}^{\omega}$, it is obvious the importance of investigating the problem of finding a position for ω which gives a small value for

$$\Phi^\omega = \int_\Omega y^\omega(x,T)dx.$$

Let ω_0 be a nonempty open subset of Ω, with a smooth enough boundary and such that $\Omega\setminus\overline{\omega}_0$ is a domain. Consider $\mathcal{O}$ the set of all translations ω of ω_0, satisfying $\omega\subset\subset\Omega$. For any $\omega\in\mathcal{O}$ and $V\in\mathbf{R}^N$ we define the derivative

$$d\Phi^\omega(V) = \lim_{\varepsilon\to 0}\frac{\Phi^{\varepsilon V+\omega} - \Phi^\omega}{\varepsilon}.$$

The derivative of Φ^ω with respect to translations has been obtained in [2] (Theorem 4.2)

Theorem 5.2. *For any $\omega\in\mathcal{O}$ and $V\in\mathbf{R}^N$ we have that*

$$d\Phi^\omega(V) = \gamma\int_0^T\int_{\partial\omega} y^\omega(x,t)p^\omega(x,t)\nu(x)\cdot V d\sigma\ dt$$

(here $\nu(x)$ is the normal inward versor at $x\in\partial\omega$; inward with respect to ω), where p^ω is the solution to the adjoint problem

$$\begin{cases} \partial_t p + d\Delta p + a(x)p + \int_\Omega k(x',x)p(x',t)dx' - m(x)\gamma p + \zeta p & \\ \quad -(\int_\Omega y^\omega(x,t)dx)p - \int_\Omega y^\omega(x,t)p(x,t)dx = 0, & (x,t)\in Q_T \\ \partial_\nu p(x,t) = 0, & (x,t)\in\Sigma_T \\ p(x,T) = 1, & x\in\Omega. \end{cases}$$

Actually, this Theorem allows to derive an iterative algorithm to improve the position of the domain where the control acts (by translations) in order to get a faster stabilization (for the feedback control $u := -\gamma y$). For details (including numerical tests) see [2].

Note that λ_1^ω can be approximated in the same manner.

A. Appendix

We establish here a comparison result which extends the one in [4]. For comparison principles for parabolic equations (without nonlocal terms) we refer to [23].

Set $Y = \{h \in H^1(\Omega);\ h = 0 \text{ on } \Gamma\}$, where $\Gamma \subset \partial\Omega$ is a measurable subset and $T \in (0, +\infty)$. Let $y_i \in C([0,T]; L^2(\Omega)) \cap C((0,T]; Y) \cap C^1((0,T]; L^2(\Omega))$ $(i \in \{1,2\})$ be solutions to

$$\begin{cases} \partial_t y_i - d\Delta y_i - a_i y_i - \int_\Omega F_i(x,x')y_i(x',t)dx' = h_i(x,t), & (x,t) \in Q_T \\ y_i(x,t) = g_i(x,t), & x \in \Gamma,\ t \in (0,T) \\ \partial_\nu y_i(x,t) = 0, & x \in \partial\Omega \setminus \Gamma,\ t \in (0,T) \\ y_i(x,0) = y_{0i}(x), & x \in \Omega \end{cases} \tag{A.1}$$

(assume such solutions exist).

Lemma A.1. (**A comparison result**) *Assume in addition that* $d \in (0,+\infty)$ *and that for* $i \in \{1,2\}$*:* $y_{0i} \in L^\infty(\Omega)$, $a_i,\ h_i \in L^\infty(Q_T)$, $F_i \in L^\infty(\Omega \times \Omega)$, $g_i \in L^\infty(\Gamma \times (0,T))$, *and*

$$\begin{cases} a_1(x,t) \le a_2(x,t) & a.e.\ (x,t) \in Q_T \\ 0 \le y_{01}(x) \le y_{02}(x) & a.e.\ x \in \Omega \\ 0 \le h_1(x,t) \le h_2(x,t) & a.e.\ (x,t) \in Q_T \\ 0 \le F_1(x,x') \le F_2(x,x') & a.e.\ (x,x') \in \Omega \times \Omega \\ 0 \le g_1(x,t) \le g_2(x,t) & a.e.\ (x,t) \in \Gamma \times (0,T). \end{cases}$$

Then $0 \le y_1(x,t) \le y_2(x,t)$ *a.e.* $x \in \Omega$, $\forall t \in [0,T]$.

Proof. Let us prove first that y_i are both nonnegative. Multiplying the first equation in (A.1) by y_i^- and integrating over $\Omega \times (0,t)$ $(t \in (0,T])$ one gets (using that F_i is a nonnegative function)

$$-\frac{1}{2}\int_\Omega |y_i^-(x,t)|^2 dx - d\int_0^t \int_\Omega |\nabla y_i^-(x,s)|^2 dx\ ds$$

$$+\int_0^t \int_\Omega a_i(x,s)|y_i^-(x,s)|^2 dx\ ds - \int_0^t \int_\Omega \int_\Omega y_i^-(x,s)F_i(x,x')y_i^-(x',s)dx\ dx'\ ds \ge 0.$$

Since a_i and F_i are essentially bounded we may conclude that there exists $M \in (0,+\infty)$ such that

$$\int_\Omega |y_i^-(x,t)|^2 dx \le M\int_0^t \int_\Omega |y_i^-(x,s)|^2 dx\ ds, \qquad \forall t \in [0,T].$$

Using now Gronwall's lemma we get that $y_i^-(x,t) =$ a.e. in Q_T.

Set now $w = y_2 - y_1$. Then w is a solution to

$$\begin{cases} \partial_t w - d\Delta w = a_2 w + (a_2 - a_1)y_1 + \int_\Omega F_1(x,x')w(x',t)dx' \\ \qquad + \int_\Omega (F_2(x,x') - F_1(x,x'))y_2(x',t)dx' + h_2(x,t) - h_1(x,t), & (x,t) \in Q_T \\ w(x,t) = g_2(x,t) - g_1(x,t) \ge 0, & x \in \Gamma,\ t \in (0,T) \\ \partial_\nu w(x,t) = 0, & x \in \partial\Omega \setminus \Gamma,\ t \in (0,T) \\ w(x,0) = y_{02}(x) - y_{01}(x) \ge 0, & x \in \Omega. \end{cases} \tag{A.2}$$

Multiplying the first equation in (A.2) by $-w^-$ and integrating over $\Omega \times (0,t)$ $(t \in (0,T])$ one gets after an easy calculation that

$$\int_\Omega |w^-(x,t)|^2 dx \le L\int_0^t \int_\Omega |w^-(x,s)|^2 dx\ ds, \qquad \forall t \in [0,T],$$

where L is a nonnegative constant (we have taken into account that $(a_2 - a_1)y_1$, $h_2 - h_1$ and $\int_\Omega (F_2(x,x') - F_1(x,x'))y_2(x',t)dx'$ are nonnegative functions). The conclusion follows by Gronwall's Lemma. □

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