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L^p-boundedness of Multilinear Pseudo-differential Operators on $\mathbb{Z}^n$ and $\mathbb{T}^n$

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Abstract. The aim of this paper is to introduce and study multilinear pseudo-differential operators on $\mathbb{Z}^n$ and $\mathbb{T}^n = (\mathbb{R}^n/2\pi\mathbb{Z}^n)$ the n-torus. More precisely, we give sufficient conditions and sometimes necessary conditions for L^p-boundedness of these classes of operators. L^2-boundedness results for multilinear pseudo-differential operators on $\mathbb{Z}^n$ and $\mathbb{T}^n$ with L^2-symbols are stated. The proofs of these results are based on elementary estimates on the multilinear Rihaczek transforms for functions in $L^2(\mathbb{Z}^n)$ respectively $L^2(\mathbb{T}^n)$ which are also introduced.

We study the weak continuity of multilinear operators on the m-fold product of Lebesgue spaces $L^{p_j}(\mathbb{Z}^n)$, $j = 1, \dots, m$ and the link with the continuity of multilinear pseudo-differential operators on $\mathbb{Z}^n$.

Necessary and sufficient conditions for multilinear pseudo-differential operators on $\mathbb{Z}^n$ or $\mathbb{T}^n$ to be a Hilbert-Schmidt operators are also given. We give a necessary condition for a multilinear pseudo-differential operators on $\mathbb{Z}^n$ to be compact. A sufficient condition for compactness is also given.

Keywords and phrases: Fourier series, multilinear pseudo-differential operators, multilinear Rihaczek transforms, L^p-boundedness, L^p-compactness, Young convolution inequality for $\mathbb{Z}^n$, Hausdorff-Young inequality for the torus $\mathbb{T}^n$, Hilbert-Schmidt operators

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1. Introduction

We begin this section with some notations that are convenient for the multilinear Fourier analysis throughout this paper. Points in $(\mathbb{R}^n)^m$ are denoted by $x = (x_1, \dots, x_m)$ where $x_1, \dots, x_m$ are points in $\mathbb{R}^n$. For all points $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in $(\mathbb{R}^n)^m$ we denote by

$$x \cdot y = \sum_{j=1}^{m} x_j \cdot y_j,$$

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the inner product of x and y, where $x_j \cdot y_j$ is the ordinary Euclidean inner product of x_j and y_j in $\mathbb{R}^n$. The sum $|x|$ of x is given by

$$|x| = \sum_{j=1}^{m} x_j.$$

We also denote by dx the Lebesgue measure $dx_1 \dots dx_m$. For the measurable function $f_1, \dots, f_m$ on $\mathbb{R}^n$ we denote by $\bigotimes_{j=1}^{m} f_j$ the tensor product of these which is the function on $(\mathbb{R}^n)^m$ defined by

$$\left(\bigotimes_{j=1}^{m} f_j\right)(x) = f_1(x_1) \dots f_m(x_m),$$

for all $x = (x_1, \dots, x_m)$ in $(\mathbb{R}^n)^m$.

Now, let $f \in L^1(\mathbb{Z}^n)$. Then the Fourier transform of f is the function on the n-torus $\mathbb{T}^n = (\mathbb{R}^n / 2\pi\mathbb{Z}^n)$, defined by

$$(\mathcal{F}_{\mathbb{Z}^n} f)(\theta) = \sum_{y \in \mathbb{Z}^n} f(y) e^{iy\cdot\theta}, \quad \theta \in \mathbb{T}^n.$$

Let $\sigma : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$ be a measurable function. Then for every m sequences $f_1, \dots, f_m$ in $L^2(\mathbb{Z}^n)$ we define the sequence $T_\sigma(f_1, \dots, f_m)$ formally by

$$(T_\sigma f)(x) = \int_{(\mathbb{T}^n)^m} e^{-ix\cdot|\theta|} \sigma(x, \theta) \left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} f_j\right)(\theta) d\mu(\theta), \quad x \in \mathbb{Z}^n,$$

where $d\mu(\theta) = (2\pi)^{-nm} d\theta_1 \dots d\theta_m$, where $d\theta_1 \dots d\theta_n$ is the projection on $\mathbb{T}^n$ of the Lebesgue measure of $\mathbb{R}^n$ (the n-torus $\mathbb{T}^n$ can be identified with $[-\pi, \pi]^n$ if we identify $\pi \approx -\pi$ on the interval $[-\pi, \pi]$).

Therefore we assume throughout the paper that the volume of the n-torus $\mathbb{T}^n$ is equal to 1.

T_σ is called multilinear (or m-linear) pseudo-differential operator on $\mathbb{Z}^n$ corresponding to the symbol σ, whenever the integral exists for all x in $\mathbb{Z}^n$. The operator T_σ is a natural analogous on $\mathbb{Z}^n$ of the standard multilinear pseudo-differential operator on $\mathbb{R}^n$ (see for example [2], [5] and references therein).

Let $\sigma : \mathbb{T}^n \times (\mathbb{Z}^n)^m \to \mathbb{C}$ be a measurable function. Then for all $f = (f_1, \dots, f_m)$ in $L^2(\mathbb{T}^n)^m$ (the m-fold product of Lebesgue space $L^2(\mathbb{T}^n)$ with itself) we define $T_\sigma f$ to be the function on $\mathbb{T}^n$ formally defined by

$$(T_\sigma f)(\theta) = \sum_{x \in (\mathbb{Z}^n)^m} \sigma(\theta, x) e^{i\theta\cdot|x|} \left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{T}^n} f_j\right)(x), \quad \theta \in \mathbb{T}^n,$$

where

$$\left(\mathcal{F}_{\mathbb{T}^n} f_j\right)(x_j) = \int_{\mathbb{T}^n} e^{-ix_j\theta} f_j(\theta) d\mu(\theta), \ x_j \in \mathbb{Z}^n, \ 1 \leq j \leq m$$

denote the Fourier transform of the functions f_j, $1 \leq j \leq m$ and $x = (x_1, \dots, x_m)$.

We call T_σ the multilinear (or m-linear) pseudo-differential operator on $\mathbb{T}^n$ corresponding to the symbol σ, whenever the multiple trigonometric series is convergent for all $\theta \in \mathbb{T}^n$.

This paper is organized as follows. In Sections 2,3 multilinear Rihaczek transforms for functions in $L^2(\mathbb{Z}^n)$ respectively $L^2(\mathbb{T}^n)$ and the corresponding multilinear pseudo-differential operators are given. We also give in these sections the Moyal identity for the multilinear Rihaczek transforms and use this to prove L^2-boundedness of multilinear pseudo-differential operators on $\mathbb{Z}^n$ and $\mathbb{T}^n$ with L^2-symbols.

In Section 4 we give two sufficient conditions for L^p-boundedness of multilinear pseudo-differential operators on $\mathbb{Z}^n$. To this end we use multilinear version of the Riesz-Thorin interpolation theorem and the discrete version of the Young convolution inequality. It is also proved in this section a necessary condition for $(p_1, \ldots, p_m, r)$-boundedness of multilinear pseudo-differential operators on $\mathbb{Z}^n$.

The relation between weak continuity and continuity of multilinear operators defined on the m-fold product of Lebesgue spaces $L^{p_j}(\mathbb{Z}^n)$, $1 < j \leq m$ is studied in Section 5.

In Section 6 we give a sufficient condition in order that multilinear pseudo-differential operators on $\mathbb{Z}^n$ to be compact. It is also given a necessary condition for compactness.

In Section 7 we give a necessary and sufficient condition for multilinear pseudo-differential operators on $\mathbb{Z}^n$ to be Hilbert-Schmidt operators. A sufficient condition for L^p-boundedness of multilinear pseudo-differential operators on $\mathbb{T}^n$ is given in Section 8. We also give in this section a necessary and sufficient condition for multilinear pseudo-differential operators on $\mathbb{T}^n$ to be Hilbert-Schmidt operators.

The linear pseudo-differential operators on $\mathbb{Z}^n$ and $\mathbb{T}^n$ have been studied among others by S. Molahajloo, M.W. Wong ([11], [12], [17]), M. Ruzhansky and V. Turunen ([14],[15], [16]), T. Carlos Andres in ([4]), M. Pirhayati in ([13]). The multilinear Fourier multipliers on $\mathbb{R}^n$, $\mathbb{T}^n$, $\mathbb{Z}^n$ have been studied among others by R. Auscher and M.J. Caro ([1]), L. Grafakos and R.H. Torres ([7]), L. Grafakos and P. Honzik ([8]), D. Bose, S. Madan, P. Mohanty and S. Shrivastava ([3]).

2. Multilinear Rihaczek transforms and multilinear pseudo-differential operators on $\mathbb{Z}^n$

The multilinear Rihaczek transform of $f = (f_1, \ldots, f_m)$ in $L^2(\mathbb{Z}^n)^m$ (the m-fold product of Lebesgue space $L^2(\mathbb{Z}^n)$ with itself) and g in $L^2(\mathbb{Z}^n)$ is the function on $\mathbb{Z}^n \times (\mathbb{T}^n)^m$ defined by

$$R(f,g)(x,\theta) = e^{-ix\cdot|\theta|}\left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} f_j\right)(\theta)\bar{g}(x)$$

for all x in $\mathbb{Z}^n$ and θ in $(\mathbb{T}^n)^m$. Let $\sigma \in L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. Then, for $f \in L^2(\mathbb{Z}^n)^m$ and g in $L^2(\mathbb{Z}^n)$ we get

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{Z}^n)} &= \sum_{x\in\mathbb{Z}^n} (T_\sigma f)(x)\bar{g}(x) \\ &= \sum_{x\in\mathbb{Z}^n}\left(\int_{(\mathbb{T}^n)^m} e^{-ix\cdot|\theta|}\sigma(x,\theta)\left(\bigotimes_{j=1}^{m}\mathcal{F}_{\mathbb{Z}^n} f_j\right)(\theta)d\mu(\theta)\right)\bar{g}(x) \\ &= \sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)R(f,g)(x,\theta)d\mu(\theta).\end{aligned}$$

The following theorem gives the Moyal identity for the discrete Rihaczek transform.

Theorem 2.1. *For all $f = (f_1, \ldots, f_m)$ and $h = (h_1, \ldots, h_m)$ in $L^2(\mathbb{Z}^n)^m$ and all g_1 and g_2 in $L^2(\mathbb{Z}^n)$,*

$$(R(f,g_1), R(h,g_2))_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)} = \prod_{j=1}^{m}(f_j,h_j)_{L^2(\mathbb{Z}^n)}\overline{(g_1,g_2)}_{L^2(\mathbb{Z}^n)}.$$

Proof. By the definition of the Rihaczek transform, the Plancherel formula for Fourier transform on $\mathbb{Z}^n$ and Fubini's theorem we get

$$\begin{aligned}
&(R(f,g_1),R(h,g_2))_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)}\\
&=\sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m} R(f,g_1)(x,\theta)\overline{R(h,g_2)(x,\theta)}d\theta\\
&=\sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m}\bigotimes_{j=1}^m(\mathcal{F}_{\mathbb{Z}^n}f_j)(\theta)\overline{(\mathcal{F}_{\mathbb{Z}^n}h_j)(\theta)g_1(x)}g_2(x)d\theta\\
&=\prod_{j=1}^m(\mathcal{F}_{\mathbb{Z}^n}f_j,\mathcal{F}_{\mathbb{Z}^n}h_j)_{L^2(\mathbb{T}^n)}\overline{(g_1,g_2)}_{L^2(\mathbb{Z}^n)}\\
&=\prod_{j=1}^m(f_j,h_j)_{L^2(\mathbb{Z}^n)}\overline{(g_1,g_2)}_{L^2(\mathbb{Z}^n)}.
\end{aligned}$$

As an immediate Corollary of Theorem 2.1 is the $L^2(\mathbb{Z}^n)$-boundedness of multilinear pseudo-differential operators with L^2-symbols. More precisely we have the following result. □

Theorem 2.2. *Let $\sigma\in L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)$. Then $T_\sigma: L^2(\mathbb{Z}^n)^m\to L^2(\mathbb{Z}^n)$ is a bounded multilinear operator and*

$$||T_\sigma||_{B(L^2(\mathbb{Z}^n)^m,L^2(\mathbb{Z}^n))}\le||\sigma||_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)}.$$

Proof. Let $f=(f_1,\dots,f_m)\in L^1(\mathbb{Z}^n)^m$ and $g\in L^1(\mathbb{Z}^n)$. Then by Schwartz' inequality and the Moyal identity for the discrete Rihaczek transform we get

$$\begin{aligned}
|(T_\sigma f,g)|_{L^2(\mathbb{Z}^n)}&\le\sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m}|\sigma(x,\theta)|\,|R(f,g)(x,\theta)|d\mu(\theta)\\
&\le\sum_{x\in\mathbb{Z}^n}\left(\int_{(\mathbb{T}^n)^m}|\sigma(x,\theta)|^2d\mu(\theta)\right)^{1/2}\left(\int_{(\mathbb{T}^n)^m}|R(f,g)(x,\theta)|^2d\mu(\theta)\right)^{1/2}\\
&\le\left(\sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m}|\sigma(x,\theta)|^2d\mu(\theta)\right)^{1/2}\left(\sum_{x\in\mathbb{Z}^n}\int_{(\mathbb{T}^n)^m}|R(f,g)(x,\theta)|^2d\mu(\theta)\right)^{1/2}\\
&=||\sigma||_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)}||R(f,g)||_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)}\\
&=||\sigma||_{L^2(\mathbb{Z}^n\times(\mathbb{T}^n)^m)}\prod_{j=1}^m||f_j||_{L^2(\mathbb{Z}^n)}||g||_{L^2(\mathbb{Z}^n)}.
\end{aligned}$$

The proof is complete by using a standard density argument. □

Let $f:\mathbb{Z}^n\to\mathbb{C}$. Then we write $f\in S(\mathbb{Z}^n)$ if for any $M<\infty$ there exists positive constant $C_{f,M}$ such that $|f(x)|<C_{f,M}\langle x\rangle^{-M}$ holds for all $x\in\mathbb{Z}$, where we write $\langle x\rangle=(1+|x|^2)^{1/2}$. Then $S(\mathbb{Z}^n)$ denote the space of rapidly decaying functions $\mathbb{Z}^n\to\mathbb{C}$. The topology on $S(\mathbb{Z}^n)$ is given by the seminorms

$$|f|_k=\sup_{x\in\mathbb{Z}^n}\langle x\rangle^k|f(x)|,\quad k\in\mathbb{N}.$$

We say that the function $\varphi:\mathbb{Z}^n\times\mathbb{T}^n\to\mathbb{C}$ is in $C^\infty(\mathbb{Z}^n\times\mathbb{T}^n)$ when φ is smooth on $\mathbb{T}^n$ for all $x\in\mathbb{Z}^n$. Then we write $\varphi\in S(\mathbb{Z}^n\times\mathbb{T}^n)$ if $\varphi\in C^\infty(\mathbb{Z}^n\times\mathbb{T}^n)$ and in addition for any $M<\infty$, there exists a positive constant $C_{\varphi,M}$ such that

$$|\varphi(x,\theta)|\le C_{\varphi,M}\langle x\rangle^{-M}\text{ for all }x\in\mathbb{Z}^n\text{ and }\theta\in\mathbb{T}^n.$$

The topology on $S(\mathbb{Z}^n \times \mathbb{T}^n)$ is given by the seminorms

$$|\varphi|_k = \sup_{(x,\theta)\in\mathbb{Z}^n\times\mathbb{T}^n} \langle x\rangle^k |\varphi(x,\theta)|, \ k \in \mathbb{N}.$$

One can show that the continuous linear functionals on $S(\mathbb{Z}^n \times \mathbb{T}^n)$ are of the form

$$\varphi \mapsto \langle u, \varphi\rangle = \sum_{x\in\mathbb{Z}^n} \int_{\mathbb{T}^n} u(x,\theta)\varphi(x,\theta)d\mu(\theta),$$

where $u : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C}$ has polynomially growth, i.e. there exist constants $M, N < \infty$ and $C_{u,M,N}$ such that

$$|u(x,\theta)| < C_{u,M,N}\langle x\rangle^M \langle\theta\rangle^N$$

for all $x \in \mathbb{Z}^n$, $\theta \in T^n$.

Let $f = (f_1, \ldots, f_m) \in S(\mathbb{Z}^n)^m$, $g \in S(\mathbb{Z}^n)$ and $\sigma \in S(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. Then

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{Z}^n)} &= \sum_{x\in\mathbb{Z}^n} \int_{(\mathbb{T}^n)^m} e^{-ix\cdot|\theta|}\sigma(x,\theta) \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{Z}^n} f_j\right)(\theta)\bar{g}(x)d\mu(\theta) \\ &= \sum_{x\in\mathbb{Z}^n} \int_{(\mathbb{T}^n)^m} \sigma(x,\theta)R(f,g)(x,\theta)d\mu(\theta).\end{aligned}$$

Now, let $\sigma \in S'(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$ be a tempered distribution on $\mathbb{Z}^n \times (\mathbb{T}^n)^m$. Then the multilinear pseudo-differential operator T_σ corresponding to the symbol σ is defined on $S(\mathbb{Z}^n)^m$ by

$$(T_\sigma f)(g) = \sigma(R(f, \bar{g})),$$

for all $f = (f_1, \ldots, f_m) \in S(\mathbb{Z}^n)^m$ and $g \in S(\mathbb{Z}^n)$. It can be proved that $T_\sigma f$ is a tempered distribution on $S(\mathbb{Z}^n)$.

3. Multilinear Rihaczek transforms and multilinear pseudo-differential operators on $\mathbb{T}^n$

The multilinear Rihaczek transform of $f = (f_1, \ldots, f_m)$ in $L^2(\mathbb{T}^n)^m$ and g in $L^2(\mathbb{T}^n)$ is the function on $\mathbb{T}^n \times (\mathbb{Z}^n)^m$ defined by

$$R(f,g)(\theta,x) = e^{i\theta\cdot|x|} \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\bar{g}(\theta),$$

for all $\theta \in \mathbb{T}^n$ and $x \in (\mathbb{Z}^n)^m$.

Let $\sigma \in L^2(\mathbb{T}^n \times (\mathbb{Z}^n)^m)$. Then for all $f = (f_1, \ldots, f_m)$ in $L^2(\mathbb{T}^n)^m$ and g in $L^2(\mathbb{T}^n)$ we get

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{T}^n)} &= \int_{\mathbb{T}^n} (T_\sigma f)(\theta)\bar{g}(\theta)d\theta \\ &= \int_{\mathbb{T}^n} \sum_{x\in(\mathbb{Z}^n)^m} e^{i\theta\cdot|x|}\sigma(\theta,x) \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\bar{g}(\theta)d\theta \\ &= \int_{\mathbb{T}^n} \sum_{x\in(\mathbb{Z}^n)^m} \sigma(\theta,x)R(f,g)(\theta,x)d\theta \\ &= \sum_{x\in(\mathbb{Z}^n)^m} \int_{\mathbb{T}^n} \sigma(\theta,x)R(f,g)(\theta,x)d\theta.\end{aligned}$$

The following theorem gives the Moyal identity for Rihaczek transform for functions in $L^2(\mathbb{T}^n)$,

Theorem 3.1. *For all $f = (f_1, \ldots, f_m)$ and $h = (h_1, \ldots, h_m)$ in $L^2(\mathbb{T}^n)^m$ and $g_1, g_2 \in L^2(\mathbb{T}^n)$,*

$$(R(f, g_1), R(h, g_2))_{L^2(\mathbb{T}^n \times (\mathbb{Z}^n)^m)} = \prod_{j=1}^{m} (f_j, h_j)_{L^2(\mathbb{T}^n)} \overline{(g_1, g_2)}_{L^2(\mathbb{T}^n)}.$$

The proof of this theorem is likewise as that of Theorem 2.1.

As an immediate corollary we can prove the following result concerning $L^2(\mathbb{T}^n)$-boundedness of multilinear pseudo-differential operators on $\mathbb{T}^n$ with L^2-symbols.

Theorem 3.2. *Let $\sigma \in L^2(\mathbb{T}^n \times (\mathbb{Z}^n)^m)$. Then $T_\sigma : L^2(\mathbb{T}^n)^m \to L^2(\mathbb{T}^n)$ is a bounded multilinear operator on $L^2(\mathbb{T}^n)^m$. Moreover*

$$||T_\sigma||_{B(L^2(\mathbb{T}^n)^m, L^2(\mathbb{T}^n))} \leq ||\sigma||_{L^2(\mathbb{T}^n \times (\mathbb{Z}^n)^m)}.$$

The proof of Theorem 3.2 is analogous as that of Theorem 2.2.

4. Sufficient conditions or necessary conditions for $L^p(\mathbb{Z}^n)$-continuity

Theorem 4.1. *Let $\sigma \in L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. Then $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^q(\mathbb{Z}^n)$ is a bounded multilinear pseudo-differential operator, where $1 \leq p \leq 2$ and $1/p + 1/q = 1$. Moreover,*

$$||T_\sigma||_{B(L^p(\mathbb{Z}^n)^m, L^q(\mathbb{Z}^n))} \leq ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}.$$

Proof. Let $\sigma \in L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. By using the Schwartz inequality we remark that

$$\int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)| d\mu(\theta) \leq \left(\int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)|^2 d\mu(\theta) \right)^{1/2} \leq$$

$$\leq \left(\sum_{x \in \mathbb{Z}^n} \int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)|^2 d\mu(\theta) \right)^{1/2} = ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)} = M < +\infty,$$

according to the hypothesis. We claim that $T_\sigma : L^1(\mathbb{Z}^n)^m \to L^\infty(\mathbb{Z}^n)$ is a bounded multilinear operator. In fact, we get according to the above remark and the definition of the tensor product of functions that

$$|T_\sigma f(x)| = \left| \int_{(\mathbb{T}^n)^m} e^{ix \cdot |\theta|} \sigma(x, \theta) \left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} f_j \right) (\theta) d\mu(\theta) \right| \leq$$

$$\leq \int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)| \left| \left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} f_j \right) (\theta) \right| d\mu(\theta) \leq M \prod_{j=1}^{m} ||f_j||_{L^1(\mathbb{Z}^n)},$$

for all $f = (f_1, \ldots, f_m) \in L^1(\mathbb{Z}^n)^m$. We also have used the following inequalities

$$|\mathcal{F}_{\mathbb{Z}^n} f_j(\theta_j)| \leq \sum_{x \in \mathbb{Z}^n} |f_j(x)| = ||f_j||_{L^1(\mathbb{Z}^n)},$$

for all $1 \leq j \leq m$. So,

$$||T_\sigma f||_{L^\infty(\mathbb{Z}^n)} = \sup_{x \in \mathbb{Z}^n} |T_\sigma f(x)| \leq M \prod_{j=1}^{m} ||f_j||_{L^1(\mathbb{Z}^n)}$$

$$= ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)} \prod_{j=1}^{m} ||f_j||_{L^1(\mathbb{Z}^n)},$$

for all f in $L^1(\mathbb{Z}^n)^m$. This inequality asserts that T_σ is a bounded multilinear operator. Moreover,

$$||T_\sigma||_{B(L^1(\mathbb{Z}^n)^m, L^\infty(\mathbb{Z}^n))} \leq ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}.$$

In virtue of Theorem 2.2 it follows that $T_\sigma : L^2(\mathbb{Z}^n)^m \to L^2(\mathbb{Z}^n)$ is a bounded multilinear operator. Moreover,

$$||T_\sigma||_{B(L^2(\mathbb{Z}^n)^m, L^2(\mathbb{Z}^n))} \leq ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}.$$

Now, let us remind in the following the multilinear version of Riesz-Thorin theorem very much as it has stated in Grafakos' book [8].

Theorem 4.2. *Let T be a multilinear map defined on the set of simple functions of the product of m measure spaces $(X_1, \mu_1) \times \ldots \times (X_m, \mu_m)$ and taking values in the set of measurable functions on another measure space (Z, σ). Let $1 \leq p_{jk} \leq \infty$ for $1 \leq k \leq m$ and $j \in \{0, 1\}$ and also let $1 \leq p_j \leq \infty$ for $j \in \{0, 1\}$. Suppose that T satisfies*

$$||T(f_1, \ldots, f_m)||_{L^{p_j}(\mathbb{Z})} \leq M_j ||f_1||_{L^{p_{j1}}(X_1)} \cdots ||f_m||_{L^{p_{jm}}(X_m)}, \quad j = 0, 1$$

for all simple functions f_k on X_k, $1 \leq k \leq m$.

Let $(1/q, 1/q_1, \ldots, 1/q_m)$ lie on the open line segment joining $(1/p_0, 1/p_{01}, \ldots, 1/p_{0m})$ and $(1/p_1, 1/p_{11}, \ldots, 1/p_{1m})$ in $\mathbb{R}^{m+1}$. Then for some $0 < \theta < 1$ we have

$$1/q = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \; \frac{1}{q_k} = \frac{1-\theta}{p_{0k}} + \frac{\theta}{p_{1k}}, \; 1 \leq k \leq m$$

and T has a bounded extensions from $L^{q_1}(X_1) \times \ldots \times L^{q_m}(X_m)$ to $L^q(Z)$ that satisfies

$$||T(f_1, \ldots, f_m)||_{L^q(Z)} \leq M_0^{1-\theta} M_1^\theta ||f_1||_{L^{q_1}(X_1)} \cdots ||f_m||_{L^{q_m}(X_m)}.$$

Now, if $1 \leq p \leq 2$ satisfy $1/p + 1/q = 1$, then $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^q(\mathbb{Z}^n)$ is a continuous multilinear operator by virtue of Theorem 4.2. Thus the proof of Theorem 4.1 is complete.

□

Theorem 4.3. *Let $\sigma : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$ be a measurable function such that we can find a positive constant C and a function w in $L^q((\mathbb{Z}^n)^m)$, $q \geq 1$ for which*

$$\left|\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x, y)\right| \leq C|w(y)|, \; x \in \mathbb{Z}^n, \; y \in (\mathbb{Z}^n)^m.$$

Let us also suppose that

$$\int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)| d\mu(\theta) < \infty$$

for all $x \in \mathbb{Z}^n$ and let $p, r \in (1, \infty)$ satisfy $r^{-1} = p^{-1} + q^{-1} - 1$. Then the multilinear pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ is a bounded multilinear operator. Furthermore

$$||T_\sigma||_{B(L^p(\mathbb{Z}^n)^m, L^r(\mathbb{Z}^n))} \leq C||w||_{L^q((\mathbb{Z}^n)^m)},$$

where $|| \quad ||_{B(L^p(\mathbb{Z}^n)^m, L^r(\mathbb{Z}^n))}$ is the norm in the Banach space of all bounded multilinear operators from $L^p(\mathbb{Z}^n)^m$ into $L^r(\mathbb{Z}^n)$.

Proof. Let $f = (f_1, \ldots, f_m)$ in $L^1(\mathbb{Z}^n)^m$. Then for all x in $\mathbb{Z}^n$, we get

$$\begin{aligned}
(T_\sigma f)(x) &= \int_{(\mathbb{T}^n)^m} e^{-ix\cdot|\theta|}\sigma(x,\theta)\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{Z}^n} f_j\right)(\theta)d\mu(\theta) \\
&= \int_{(\mathbb{T}^n)^m} e^{-ix|\theta|}\sigma(x,\theta)\left(\sum_{y\in(\mathbb{Z}^n)^m} e^{iy\cdot\theta}\left(\bigotimes_{j=1}^m f_j\right)(y)\right)d\mu(\theta) \\
&= \sum_{y\in(\mathbb{Z}^n)^m}\left(\bigotimes_{j=1}^m f_j\right)(y)\int_{(\mathbb{T}^n)^m} e^{-i(\tilde{x}-y)\cdot\theta}\sigma(x,\theta)d\mu(\theta) \\
&= \sum_{y\in(\mathbb{Z}^n)^m}\left(\bigotimes_{j=1}^m f_j\right)(y)\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,\tilde{x}-y) \\
&= \left(\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,\cdot) * \textstyle\bigotimes_{j=1}^m f_j\right)(\tilde{x}),
\end{aligned}$$

where

$$\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,z) = \int_{(\mathbb{T}^n)^m} e^{-iz\cdot\theta}\sigma(x,\theta)d\mu(\theta),\ z\in(\mathbb{Z}^n)^m$$

and $\tilde{x} = (x, \ldots, x) \in (\mathbb{Z}^n)^m$, $x \in \mathbb{Z}^n$. □

Now, let g be a function in $L^{r'}(\mathbb{Z}^n)$, where r' is the conjugate index of r, that is $1/r + 1/r' = 1$. Then by Hölder's inequality, by the hypothesis and by the Young convolution inequality for $(\mathbb{Z}^n)^m$,

$$\begin{aligned}
|(T_\sigma f, g)|_{L^2(\mathbb{Z}^n)} &= \left|\sum_{x\in\mathbb{Z}^n}(T_\sigma f)(x)\bar{g}(x)\right| \\
&= \left|\sum_{x\in\mathbb{Z}^n}\left(\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,\cdot) * \bigotimes_{j=1}^m f_j\right)(\tilde{x})\bar{g}(x)\right| \\
&\le \sum_{x\in\mathbb{Z}^n}\left|\left(\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,\cdot) * \bigotimes_{j=1}^m f_j\right)(\tilde{x})\right||g(x)| \\
&\le \sum_{x\in\mathbb{Z}^n}\left(\left|\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,\cdot)\right| * \left|\bigotimes_{j=1}^m f_j\right|\right)(\tilde{x})|g(x)| \\
&\le C\sum_{x\in\mathbb{Z}^n}\left(|w| * \left|\bigotimes_{j=1}^m f_j\right|\right)(\tilde{x})|g(x)|
\end{aligned}$$

$$\leq C\left\{\sum_{x\in\mathbb{Z}^n}\left[\left(|w|*\left|\bigotimes_{j=1}^m f_j\right|\right)(\tilde{x})\right]^r\right\}^{1/r}\left(\sum_{x\in\mathbb{Z}^n}|g(x)|^{r'}\right)^{1/r'}$$

$$\leq C\left\{\sum_{x\in(\mathbb{Z}^n)^m}\left[\left(|w|*\left|\bigotimes_{j=1}^m f_j\right|\right)(y)\right]^r\right)^{1/r}||g||_{L^{r'}(\mathbb{Z}^n)}$$

$$= C\left\|\,|w|*\left|\bigotimes_{j=1}^m f_j\right|\,\right\|_{L^r((\mathbb{Z}^n)^m)}||g||_{L^{r'}(\mathbb{Z}^n)}$$

$$\leq C||w||_{L^q((\mathbb{Z}^n)^m)}\left\|\bigotimes_{j=1}^m f_j\right\|_{L^p((\mathbb{Z}^n)^m)}||g||_{L^{r'}(\mathbb{Z}^n)}$$

$$= C||w||_{L^q((\mathbb{Z}^n)^m)}\prod_{j=1}^m||f_j||_{L^p(\mathbb{Z}^n)}||g||_{L^{r'}(\mathbb{Z}^n)},$$

where $\tilde{x}=(x,\ldots,x)\in(\mathbb{Z}^n)^m$. The duality between the spaces $L^r(\mathbb{Z}^n), L^{r'}(\mathbb{Z}^n)$ and a standard density argument ($L^1(\mathbb{Z}^n)^m$ is dense in $L^p(\mathbb{Z}^n)^m$) allow us to conclude the proof.

Remark 4.4. Let us first mention that if $q=1$ then the second condition about the symbol σ is a consequence of the first one. So, if we take $n=m=1$ and $q=1$ in the preceding theorem then we recover Theorem 23.3 in Wong's book [17].

Remark 4.5. M. Charalambides and M. Christ have proved (see Theorem 1.2 in [6]) a version of the Young convolution inequality for torsion-free discrete groups.

A group is said to be torsion-free if it has not finite subgroups except the trivial subgroup of cardinality equal to one. Their result can be read as follows.

Theorem 4.6. *Let $p_1,p_2,q\in(1,\infty)$ satisfy $q^{-1}=p_1^{-1}+p_2^{-1}-1$. There exist a continuous nondecreasing function $\Lambda:(0,1]\to(0,1]$, an exponent γ and a positive constant $c>0$ satisfying $\Lambda(t)\leq 1-c(1-t)^\gamma$ as $t\to 1^{-1}$ such that for any torsion free discrete group G and for any functions $f_i\in L^{p_i}(G)$, $i=1,2$*

$$||f_1*f_2||_{L^q(G)}\leq||f_1||_{L^{p_1}(G)}||f_2||_{L^{p_2}(G)}\Lambda\left(\frac{||f_1||_{L^\infty(G)}}{||f_2||_{L^{p_1}(G)}}\right)\times$$

$$\times\Lambda\left(\frac{||f_2||_{L^\infty(G)}}{||f_2||_{L^{p_2}(G)}}\right).$$

Then, by using Theorem 4.4 and the first part of the proof of Theorem 4.3 we obtain the following estimate

$$||T_\sigma f||_{L^r(\mathbb{Z}^n)}\leq C||w||_{L^q((\mathbb{Z}^n)^m)}\prod_{j=1}^m||f_j||_{L^p(\mathbb{Z}^n)}\Lambda\left(\frac{||w||_{L^\infty}}{||w||_{L^q}}\right)\times$$

$$\times\Lambda\left(\frac{||f_1||_{L^\infty}}{||f_1||_{L^p}}\right)^{1/m}\ldots\Lambda\left(\frac{||f_m||_{L^\infty}}{||f_m||_{L^p}}\right)^{1/m},$$

for all $f=(f_1,\ldots,f_m)$ in $L^p(\mathbb{Z}^n)^m$.

Thus we get once more the conclusion of Theorem 4.3.

Theorem 4.7. *Let $\sigma : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$, be a measurable function and let $T_\sigma : L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ be the multilinear pseudo-differential operator associated with this one, where $1 \leq p_1, \ldots, p_m, r < \infty$. If T_σ is bounded then*

$$\left(\int_{(\mathbb{T}^n)^m} \sigma(x,\theta) e^{-i(\tilde{x}-y)\cdot|\theta|} d\mu(\theta) \right)_{x \in \mathbb{Z}^n} \in L^r(\mathbb{Z}^n)$$

for all $y \in (\mathbb{Z}^n)^m$.

Proof. For $y \in (\mathbb{Z}^n)^m$ let us consider $f_y = (f_{y_1}, \ldots, f_{y_m}) \in L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n)$ defined by $f_{y_j}(x) = 1$ if $x = y_j$ and $f_{y_j}(x) = 0$ if $x \neq y_j$, $1 \leq j \leq m$, $x \in \mathbb{Z}^n$. Then

$$\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{Z}^n} f_{y_j} \right)(\theta) = \prod_{j=1}^m \mathcal{F}_{\mathbb{Z}^n} f_{y_j}(\theta_j) = \prod_{j=1}^m e^{iy_j\theta_j} = e^{iy\cdot\theta},$$

for all $\theta = (\theta_1, \ldots, \theta_n)$ in $(\mathbb{T}^n)^m$. So,

$$|T_\sigma f_y(x)|^r = \left| \int_{(\mathbb{T}^n)^m} \sigma(x,\theta) e^{-ix\cdot|\theta|} e^{iy\cdot\theta} d\mu(\theta) \right|^r =$$

$$= \left| \int_{(\mathbb{T}^n)^m} \sigma(x,\theta) e^{-i(\tilde{x}-y)\cdot|\theta|} d\mu(\theta) \right|^r,$$

for all x in $\mathbb{Z}^n$, where $\tilde{x} = (x, \ldots, x) \in (\mathbb{Z}^n)^m$. Hence

$$||T_\sigma f_y||_{L^r(\mathbb{Z}^n)} = \left(\sum_{x \in \mathbb{Z}^n} |T_\sigma f_y(x)|^r \right)^{1/r} =$$

$$= \left(\sum_{x \in \mathbb{Z}^n} \left| \int_{(\mathbb{T}^n)^m} \sigma(x,\theta) e^{-i(\tilde{x}-y)\cdot\theta} d\mu(\theta) \right|^r \right)^{1/r}.$$

The boundedness hypothesis concerning the operator T_σ lead to the conclusion of the theorem. □

5. Weak type multilinear operators on $\mathbb{Z}^n$

In the sequel we study the relation between weak continuity and continuity of a multilinear operator defined on the m-fold product of Lebesgue spaces $L^p(\mathbb{Z}^n)$. In this connection we begin with the definition of $(p_1, \ldots, p_m, q)$-weak continuity for general measure spaces.

Let $T : L^{p_1}(X_1, \mu_1) \times \ldots \to L^{p_m}(X_m, \mu_m) \to L^{q,\infty}(X, \mu)$ be a multilinear operator defined on the m-fold product of spaces of measurable functions on measure spaces (X_j, μ_j), $1 \leq j \leq m$. We assume that T takes values in the set of measurable functions on another measure space (X, μ):

Let $1 \leq p_j \leq \infty$, $q > 0$, $1 \leq j \leq m$. Then the operator T is said to be of weak type $(p_1, \ldots, p_m, q)$ or $(p_1, \ldots, p_m, q)$-weak multilinear operator if there exists a constant $C > 0$ such that the following inequality is satisfied for all functions $f_j \in L^{p_j}(X_j, \mu_j)$, $1 \leq j \leq m$.

$$\mu\left(\{x \in X; |T(f_1, \ldots, f_m)(x)| > \lambda\}\right) \leq \left(\frac{C}{\lambda} \prod_{j=1}^m ||f_j||_{L^{p_j}(X_j,\mu_j)} \right)^q$$

i.e. T maps the product of Lebesgue spaces $L^{p_j}(X_j, \mu_j)$, $1 \leq j \leq m$, boundedly into $L^{q,\infty}(X, \mu)$.

Let us also remind that if (X, μ) is a measure space and $f : X \to C$ is a measurable function on X, then

$$||f||^p_{L^p(X)} = p\int_0^\infty \lambda^{p-1} m(\lambda) d\lambda,$$

for all $p > 0$, where $m(\lambda) = \mu(\{x \in X; |f(x)| > \lambda\})$ is the distribution function of f.

From now on, we will denote by ϑ the counting measure on $\mathbb{Z}^n$.

Theorem 5.1. *Let $T : L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n) \to \mathcal{M}(\mathbb{Z}^n, \vartheta)$ be a multilinear operator from the m-fold product of Lebesgue spaces $L^{p_j}(\mathbb{Z}^n)$, $1 \le j \le m$ into the space of measurable functions on $(\mathbb{Z}^n, \vartheta)$. If T is a $(p_1, \ldots, p_m, q)$-weak multilinear operator for $q \ge 1$ and $1 \le p_1, \ldots, p_m < \infty$, then*

$$T : L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n) \to L^r(\mathbb{Z}^n)$$

is a bounded multilinear operator for all $r > q + 1$. Moreover

$$||T||_{B\left(\prod_{j=1}^m L^{p_j}(\mathbb{Z}^n), L^r(\mathbb{Z}^n)\right)} \le C^{q/r}\left(\sum_{k\ge 2}\frac{k^q}{(k-1)^r} + (sC)^r\right)^{1/r},$$

for some suitable positive constant C and $s > 1$.

Proof. Let $(f_1, \ldots, f_m)$ be an arbitrary element in $\prod_{j=1}^m L^{p_j}(\mathbb{Z}^n)$ such that $\prod_{j=1}^m ||f_j||_{L^{p_j}(\mathbb{Z}^n)} = 1$ and $r > q+1$. By the hypothesis T is $(p_1, \ldots, p_m, q)$-weak multilinear operator, so

$$\vartheta(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > \lambda\}) \le \left(\frac{C}{\lambda}\right)^q,$$

for some $C > 0$ and for all $\lambda > 0$. Thus

$$\sup_{\lambda>0} \lambda\vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)|\}\right)^{1/q} \le C,$$

which is the same as

$$||T(f_1, \ldots, f_m)||_{L^{q,\infty}(\mathbb{Z}^n)} \le C,$$

where we have denoted by

$$||T(f_1, \ldots, f_m)||_{L^{q,\infty}(\mathbb{Z}^n)} = \sup_{\lambda>0} \lambda\vartheta(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > \lambda\})^{1/q}.$$

Then

$$||T(f_1, \ldots, f_m)||_{L^{q,\infty}(\mathbb{Z}^n)} \le C\prod_{j=1}^m ||f_j||_{L^{p_j}(\mathbb{Z}^n)},$$

for all $(f_1, \ldots, f_m)$ in $\prod_{j=1}^m L^{p_j}(\mathbb{Z}^n)$.

Now, let us remark that we have

$$\begin{aligned}\mathbb{Z}^n &= \{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| \le 1\} \cup \\ &\cup \{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > 1\} \\ &= \bigcup_{k=2}^\infty \left\{x \in \mathbb{Z}^n; \frac{1}{k} < |T(f_1, \ldots, f_m)(x)| \le \frac{1}{k-1}\right\} \\ &\cup \{x \in \mathbb{Z}^n; 1 < |T(f_1, \ldots, f_m)(x)| \le sC\},\end{aligned}$$

for some $s > 1$ and for all $(f_1, \ldots, f_m)$ in $\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n)$ such that

$$\prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{Z}^n)} = 1.$$

Indeed, let us observe that

$$\vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| \leq sC\}\right) \leq \frac{1}{s^q} < 1,$$

so

$$\vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| \leq sC\}\right) = 0,$$

which imply $|T(f_1, \ldots, f_m)(x)| \leq sC$, for all $x \in \mathbb{Z}^n$. On the other hand we remark that

$$\vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > 1\}\right) = 0 \text{ if } C < 1.$$

Now, let us denote by

$$A_k = \left\{x \in \mathbb{Z}^n; \frac{1}{k} < |T(f_1, \ldots, f_m)(x)| \leq \frac{1}{k-1}\right\},\ k \geq 2$$

and

$$B = \{x \in \mathbb{Z}^n; 1 < |T(f_1, \ldots, f_m)(x)| \leq sC\}.$$

Then,

$$\sum_{x \in \mathbb{Z}^n} |T(f_1, \ldots, f_m)(x)|^r = \sum_{x \in \cup A_k} |T(f_1, \ldots, f_m)(x)|^r +$$
$$+ \sum_{x \in B} |T(f_1, \ldots, f_m)(x)|^r$$
$$= \sum_{k \geq 2} \sum_{x \in A_k} |T(f_1, \ldots, f_m)(x)|^r + \sum_{x \in B} |T(f_1, \ldots, f_m)(x)|^r$$
$$\leq \sum_{k \geq 2} \frac{1}{(k-1)^r} \vartheta(A_k) + (sC)^r \vartheta(B)$$
$$\leq C^q \left(\sum_{k \geq 2} \frac{k^q}{(k-1)^r} + (sC)^r \right) < \infty$$

(the series inside the brackets is convergent, because $r > q + 1$ by hypothesis). Thus,

$$||T(f_1, \ldots, f_m)(x)||_{L^r(\mathbb{Z}^n)} \leq$$

$$\leq C^{q/r} \left(\sum_{k \geq 2} \frac{k^q}{(k-1)^r} + (sC)^r \right)^{1/q} \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{Z}^n)}$$

for all $(f_1, \ldots, f_m)$ in $\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n)$, completing the proof of the theorem. □

Theorem 5.2. *Let $T : L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n) \to \mathcal{M}(\mathbb{Z}^n, \vartheta)$ be a multilinear operator of weak-type $(p_1, \ldots, p_m, q)$ for $1 \leq p_1, \ldots, p_m, q < \infty$ and let $p_0 > q \geq 1$. Then $T : L^{p_1}(\mathbb{Z}^n) \times \ldots \times L^{p_m}(\mathbb{Z}^n) \to L^{p_0}(\mathbb{Z}^n)$ is a bounded multilinear operator. Moreover*

$$||T||_{B\left(\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n), L^{p_0}(\mathbb{Z}^n)\right)} \leq s^{1-\frac{q}{p_0}} C \left(\frac{p_0}{p_0 - q}\right)^{1/p_0},$$

for some suitable constant $s > 1$.

Proof. Let $p_0 > q \geq 1$. We assume that T is of weak type $(p_1, \ldots, p_m, q)$. Then

$$\vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > \lambda\}\right) \leq \left(\frac{C}{\lambda} \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{Z}^n)}\right)^q,$$

for all $(f_1, \ldots, f_m)$ in $\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n)$, $\lambda > 0$ and some $C > 0$.

Let $(f_1, \ldots, f_m)$ in $\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n)$ such that $\prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{Z}^n)} = 1$. Hence

$$\begin{aligned} &||T(f_1, \ldots, f_m)||^{p_0}_{L^{p_0}(\mathbb{Z}^n)} = \\ &= p_0 \int_0^\infty \vartheta\left(\{x \in \mathbb{Z}^n; |T(f_1, \ldots, f_m)(x)| > \lambda\}\right) \lambda^{p_0 - 1} d\lambda \\ &\leq p_0 C^q \int_0^{sC} \lambda^{p_0 - q - 1} d\lambda = p_0 C^q \frac{(sC)^{p_0 - q}}{p_0 - q} = \frac{p_0}{p_0 - q} s^{p_0 - q} C^{p_0}. \end{aligned}$$

But we can equivalently write

$$||T(f_1, \ldots, f_m)||_{L^{p_0}(\mathbb{Z}^n)} \leq C s^{1 - q/p_0} \left(\frac{p_0}{p_0 - q}\right)^{1/p_0},$$

where $s > 1$ a suitable constant.

Now, for an arbitrary $(f_1, \ldots, f_m)$ in $\prod_{j=1}^{m} L^{p_j}(\mathbb{Z}^n)$ we get

$$||T(f_1, \ldots, f_m)||_{L^{p_0}(\mathbb{Z}^n)} \leq C s^{1 - q/p_0} \left(\frac{p_0}{p_0 - q}\right)^{1/p_0} ||f_j||_{L^{p_j}(\mathbb{Z}^n)}.$$

This completes the proof of the theorem. □

Remark 5.3. Theorem 5.1 is a special case of Theorem 5.2, so it can be state as a corollary of the last one. However we have given a straight proof for this theorem in order to emphasize the definition of weak continuity and to put in evidence an interesting partition of $\mathbb{Z}^n$ which may be usefully in similar situation.

Theorem 5.4. *Let $1 \leq p_0 < p < \infty$ such that $1/p + 1/p_0 = 1$. If $\sigma : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$ is a measurable function satisfying*

$$(j) \qquad \int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)| d\mu(\theta) < \infty \quad \text{for all} \quad x \in \mathbb{Z}^n$$

$$(jj)\ \left(\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-i(\tilde{x}-y)\cdot\theta}d\mu(\theta)\right)_{(x,y)\in\mathbb{Z}^n\times(\mathbb{Z}^n)^m}\quad \textit{is in } L^{p_0}(\mathbb{Z}^n\times(\mathbb{Z}^n)^m),$$

then $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^p(\mathbb{Z}^n)$ *is a bounded multilinear operator. Moreover*

$$||T||_{B(L^p(\mathbb{Z}^n)^m,L^p(\mathbb{Z}^n))} \le ||\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(\cdot,\cdot)||_{L^{p_0}(\mathbb{Z}^n\times(\mathbb{Z}^n)^m)}.$$

Proof. Let $1 \le p_0 < p < \infty$ such that $1/p + 1/p_0 = 1$ and $(f_1,\dots,f_m) \in L^1(\mathbb{Z}^n)^m$. Then by Fubini's theorem and Hölder's inequality we get

$$|T(f_1,\dots,f_m)(x)|^{p_0} =$$

$$= \left|\sum_{y_1\in\mathbb{Z}^n}\cdots\sum_{y_m\in\mathbb{Z}^n}\left(\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-i(\tilde{x}-y)\cdot\theta}d\mu(\theta)\right)\left(\bigotimes_{j=1}^m f_j\right)(y_1,\dots,y_m)\right|^{p_0}$$

$$\le \left(\sum_{y\in(\mathbb{Z}^n)^m}\left|\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-i(\tilde{x}-y)\cdot\theta}d\mu(\theta)\right|\left|\left(\bigotimes_{j=1}^m f_j\right)(y)\right|\right)^{p_0}$$

$$\le \sum_{y\in(\mathbb{Z}^n)^m}\left|\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-i(\tilde{x}-y)\cdot\theta}d\mu(\theta)\right|^{p_0}\prod_{j=1}^m ||f_j||^{p_0}_{L^p(\mathbb{Z}^n)}$$

for all $x \in \mathbb{Z}^n$. So,

$$||T_\sigma(f_1,\dots,f_m)||^{p_0}_{L^{p_0}(\mathbb{Z}^n)} = \sum_{x\in\mathbb{Z}^n}|T_\sigma(f_1,\dots,f_m)(x)|^{p_0}$$

$$\le \left(\sum_{x\in\mathbb{Z}^n}\sum_{y\in(\mathbb{Z}^n)^m}\left|\int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-i(\tilde{x}-y)\cdot\theta}d\mu(\theta)\right|^{p_0}\right)\prod_{j=1}^m ||f_j||^{p_0}_{L^p(\mathbb{Z}^n)}.$$

Hence $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^{p_0}(\mathbb{Z}^n)$ is a bounded multilinear operator. Moreover

$$||T_\sigma||_{B(L^p(\mathbb{Z}^n)^m,L^{p_0}(\mathbb{Z}^n))} \le ||\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(\cdot,\cdot)||_{L^{p_0}(\mathbb{Z}^n\times(\mathbb{Z}^n)^m)},$$

where

$$\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,y) = \int_{(\mathbb{T}^n)^m}\sigma(x,\theta)e^{-iy\cdot\theta}d\mu(\theta), x\in\mathbb{Z}^n,\ y\in(\mathbb{Z}^n)^m.$$

Since T_σ is $(p,\dots,p,p_0)$-bounded multilinear operator it follows that T_σ is $(p,\dots,p,p_0)$-weak multilinear operator. Hence by Theorem 5.2 and a standard density argument the proof is complete. □

Remark 5.5. When we take $m = 1$ in Theorems 5.1, 5.2 and 5.4 we recover Theorems 3.1, 3.2 respectively Theorem 4.3 in [4].

6. Sufficient conditions and necessary conditions for compactness

Theorem 6.1. *Let* $\sigma : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$ *be a measurable function such that we can find a positive function* $C : \mathbb{Z}^n \to \mathbb{R}_+$ *and a function* w *in* $L^q((\mathbb{Z}^n)^m)$, $q \ge 1$ *for which*

$$\left|\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x,y)\right| \le C(x)|w(y)|,\ x\in\mathbb{Z}^n,\ y\in(\mathbb{Z}^n)^m$$

and

$$\lim_{|x|\to\infty} C(x) = 0$$

In addition let us suppose that

$$\int_{(\mathbb{T}^n)^m} |\sigma(x,\theta)| d\mu(\theta) < \infty$$

for all $x \in \mathbb{Z}^n$ *(i.e.* $\sigma(x,\cdot) \in L^1((\mathbb{T}^n)^m)$ *for every* x *in* $\mathbb{Z}^n$*). Let* $p, r \in (1,\infty)$ *satisfy* $1/r = 1/p + 1/q - 1$*. Then the multilinear pseudo-differential operator* $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ *is a compact operator.*

Proof. By virtue of Theorem 4.3, $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ is a bounded multilinear operator. For every positive integer N, we define the symbol $\sigma_N : \mathbb{Z}^n \times (\mathbb{T}^n)^m \to \mathbb{C}$ by

$$\sigma_N(x,\theta) = \begin{cases} \sigma(x,\theta), \ |x| \le N \\ \quad 0, \qquad |x| > N, \end{cases} \theta \in (\mathbb{T}^n)^m.$$

Then, by using the same reasoning as in the beginning of the proof of Theorem 4.3, we get

$$(T_{\sigma_N} f)(x) = \begin{cases} \left(\mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x,\cdot) * \bigotimes_{j=1}^m f_j \right)(\tilde{x}), & |x| \le N \\ \qquad\qquad 0, & |x| > N, \end{cases}$$

for all $x \in \mathbb{Z}^n$, where $\tilde{x} = (x,\ldots,x) \in (\mathbb{Z}^n)^m$.

Let us remark that the range of $T_{\sigma_N} : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ is finite-dimensional, i.e. T_{σ_N} is a finite rank operator. So, T_σ is a compact operator.

By hypothesis for every positive number ε there exists a positive integer $N_0 = N_\varepsilon$ such that

$$|C(x)| < \varepsilon,$$

for all x in $\mathbb{Z}^n$, $|x| > N_0$. So, we get for $N > N_0$

$$\begin{aligned}
\|(T_\sigma - T_{\sigma_N}) f\|^r_{L^r(\mathbb{Z}^n)} &= \sum_{x \in \mathbb{Z}^n} \left| \left(\mathcal{F}_{(\mathbb{T}^n)^m} (\sigma - \sigma_N)(x,\cdot) * \bigotimes_{j=1}^m f_j \right)(\tilde{x}) \right|^r \\
&= \sum_{|x|>N} \left| \left(\mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x,\cdot) * \bigotimes_{j=1}^m f_j \right)(\tilde{x}) \right|^r \\
&\le \sum_{|x|>N} \left| \left(\left| \mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x,\cdot) \right| * \left| \bigotimes_{j=1}^m f_j \right| \right)(\tilde{x}) \right|^r \\
&\le \sum_{|x|>N} C(x)^r \left| \left(|w| * \left| \bigotimes_{j=1}^m f_j \right| \right)(\tilde{x}) \right|^r \\
&\le \varepsilon^r \sum_{\substack{y \in (\mathbb{Z}^n)^m \\ |y|>N}} \left| \left(|w| * \left| \bigotimes_{j=1}^m f_j \right| \right)(y) \right|^r \\
&\le \varepsilon^r \left\| |w| * \left| \bigotimes_{j=1}^m f_j \right| \right\|^r_{L^r(\mathbb{Z}^n)} .
\end{aligned}$$

By Young's convolution inequality we get for $N > N_0$

$$\|(T_\sigma - T_{\sigma_N})f\|^r_{L^r(\mathbb{Z}^n)} \leq \varepsilon^r ||w||^r_{L^q((\mathbb{Z}^n)^m)} \left\| \bigotimes_{j=1}^m f_j \right\|^r_{L^p(\mathbb{Z}^n)} =$$

$$= \varepsilon^r ||w||^r_{L^q((\mathbb{Z}^n)^m)} \prod_{j=1}^m ||f_j||^r_{L^p(\mathbb{Z}^n)}.$$

Thus, for $N > N_0$

$$||T_\sigma - T_{\sigma_N}||_{B(L^p(\mathbb{Z}^n)^m, L^r(\mathbb{Z}^n))} \leq \varepsilon ||w||_{L^q((\mathbb{Z}^n)^m)}.$$

So, $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ is the limit in the operatorial norm of a sequence of compact operators on $L^p(\mathbb{Z}^n)^m$ and hence it must be compact. □

Remark 6.2. Let us first observe that if $q = 1$ then the hypothesis $\sigma(x, \cdot) \in L^1((\mathbb{T}^n)^m)$ for all x in $\mathbb{Z}^n$ is a consequence of the first condition about the symbol σ in the statement of Theorem 6.1. So, if we take $m = n = 1$ and $q = 1$ in Theorem 6.1 then we recover Theorem 23.4 in Wong's book [17].

We now give a necessary condition for compactness on $L^p(\mathbb{Z}^n)^m$ spaces of multilinear pseudo-differential operators.

Theorem 6.3. *Let $\varepsilon > 0$ and $s \in \mathbb{N}$. If $T_\sigma : L^p(\mathbb{Z}^n)^m \to L^r(\mathbb{Z}^n)$ is a compact operator, where $p, r \in (1, \infty)$, then there exists a function $f_s = (f_{1s}, \dots, f_{ms}) \in L^p(\mathbb{Z}^n)^m$ such that $||T_\sigma f_s||_{L^r(\mathbb{Z}^n)} < \varepsilon$ and $||f_s||_{L^p(\mathbb{Z}^n)^m} = 2ms$, where the norm on $L^p(\mathbb{Z}^n)^m$ is defined by $||f||_{L^p(\mathbb{Z}^n)^m} = ||f_1||_{L^p(\mathbb{Z}^n)} + \dots + ||f_m||_{L^p(\mathbb{Z}^n)}$, for $f = (f_1, \dots, f_m) \in L^p(\mathbb{Z}^n)^m$.*

Proof. Let $\varepsilon > 0$ and $s \in \mathbb{N}$ be given. For $k \in \mathbb{N}$ let us denote

$$M_s^k = \{x \in \mathbb{N},\ s(k-1) \leq x < sk\}.$$

Then we define $g^k : \mathbb{Z} \to \mathbb{C}$ by

$$g_s^k(x) = \begin{cases} 1, \text{ if } x \in M_s^k \\ 0, \text{ otherwise.} \end{cases}$$

Now, we consider the sequence $(f_s^k)_{k \in \mathbb{N}}$ in $L^p(\mathbb{Z}^n)^m$ for which entries $f_{js}^k : \mathbb{Z}^n \to \mathbb{C}$, $1 \leq j \leq m$, are defined by

$$f_{js}^k(x_1, \dots, x_n) = \begin{cases} g_s^k(x_j), & \text{whenever} (x_1, \dots, x_n) \in \\ & \in \{0\} \times \dots \times M_s^k \times \dots \times \{0\} \\ 0, & \text{otherwise,} \end{cases}$$

$1 \leq j \leq m$.

Let us remark that the sequence $(f_s^k)_{k \in \mathbb{N}}$ in $L^p(\mathbb{Z}^n)^m$ is bounded, because

$$||f_s^k||_{L^p(\mathbb{Z}^n)^m} = \sum_{j=1}^m ||f_s^k||_{L^p(\mathbb{Z}^n)^m} = sm,$$

for all $k \in \mathbb{N}$. So, by the compactness hypothesis there exists a subsequence $(k_q)_{q \in \mathbb{N}} \subset \mathbb{N}$ and N in $\mathbb{N}$ such that

$$||T_\sigma f_s^{k_u} - T_\sigma f_\sigma^{k_v}||_{L^r(\mathbb{Z}^n)} < \varepsilon,$$

where $u, v \geq N$.

If we choose $f_s = f_s^{k_N} - f_s^{k_{N+1}}$, then $||f_s||_{L^p(\mathbb{Z}^n)^m} = 2sm$ and $||T_\sigma f_s||_{L^r(\mathbb{Z}^n)} < \varepsilon$. Thus the proof is complete. □

7. Multilinear Hilbert-Schmidt operators

Our approach in the following is based on the theory of the tensor product of Hilbert spaces. For related references on this topic see the work of Kadison and Ringrose [10]. We need in the following some more terminology.

Definition 7.1. Suppose that $H_1, \ldots, H_m$ are Hilbert spaces and $\psi : H_1 \times \ldots \times H_m \to \mathbb{C}$ is a bounded multilinear functional. Then we say that ψ is a Hilbert-Schmidt functional if for some (hence for any) choice of a Hilbert basis Y_1 in $H_1, \ldots, Y_m$ in H_m respectively, we have

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_m \in Y_m} |\psi(y_1, \ldots, y_m)|^2 < \infty.$$

Definition 7.2. Let $H_1, \ldots, H_m$, K be Hilbert spaces and $L : H_1 \times \ldots \times H_m \to K$ a bounded multilinear mapping. Then we say that L is a weak Hilbert-Schmidt mapping if the following statements are valid:

(i) for each u in K, the mapping $L_u : H_1 \times \ldots \times H_m \to \mathbb{C}$ defined by

$$L_u(x_1, \ldots, x_n) = \langle L(x_1, \ldots, x_n), u\rangle$$

is a Hilbert-Schmidt functional on $H_1 \times \ldots \times H_m$;

(ii) there is a real positive number d such that $||L_u||_2 \leq d||u||$ for each u in K. When these conditions are satisfied the least possible value of the constant d in (ii) is denoted by $||L||_2$. We need the following theorem

Theorem 7.3. *(see [10]). Suppose $H_1, \ldots, H_m$ are Hilbert spaces. Then the following statements are valid.*

(i) There is a Hilbert space $H = H_1 \otimes \ldots \otimes H_m$ and a weak Hilbert-Schmidt mapping $p : H_1 \times \ldots \times H_m \to H$ such that given any weak Hilbert-Schmidt mapping $L : H_1 \times \ldots \times H_m \to K$ from $H_1 \times \ldots \times H_m$ into a Hilbert space K there is a unique bounded linear mapping $T : H \to K$ such that $L = Tp$. Moreover $||T|| = ||L||_2$.

(ii) If the couple (H', p') have the same properties as the couple (H, p) in (i), then there is a unitary transformation $U : H \to H'$ such that $p' = Up$.

(iii) If $x_j, y_j \in H_j$ and Y_j is an orthogonal basis of H_j, $(j = 1, \ldots, m)$ then

$$\langle p(x_1, \ldots, x_m), p(y_1, \ldots, y_m)\rangle = \langle x_1, y_1\rangle \ldots \langle x_m, y_m\rangle,$$

the set $\{p(y_1, \ldots, y_m), y_1 \in Y_1, \ldots, y_m \in Y_m\}$ is an orthonormal basis of H and $||p||_2 = 1$.

The above theorem describes the universal property (see (i)) and two other properties of the tensor product of Hilbert spaces.

Now we give the following definition which will be used in the proof of Theorem 7.5.

Definition 7.4. Let $H_1, \ldots, H_m$, K be Hilbert spaces and $L : H_1 \times \ldots \times H_m \to K$ be a multilinear mapping. Then we assert that L is a Hilbert-Schmidt mapping if there is a unique bounded linear mapping $T : H_1 \otimes \ldots \otimes H_m \to K$ such that $T\left(\bigotimes_{j=1}^{m} x_j\right) = L(x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_m)$ in $H_1 \times \ldots \times H_m$ and T is a Hilbert-Schmidt mapping. In such a case we define $||L||_s = ||T||_s$, where $||T||_s$ is the Hilbert-Schmidt norm of the mapping T.

Let us remark that if L is a weak Hilbert-Schmidt mapping then the existence of the operator T in the previous definition is a consequence of the last theorem.

Now, we give a necessary and sufficient condition for a multilinear pseudo-differential operator on $\mathbb{Z}^n$ to be a Hilbert-Schmidt operator.

Theorem 7.5. *Let $\sigma \in L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. Then the multilinear pseudo-differential operator T_σ : $L^2(\mathbb{Z}^n)^m \to L^2(\mathbb{Z}^n)$ is a Hilbert-Schmidt operator if and only if $\sigma \in L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)$. Moreover, if $T_\sigma : L^2(\mathbb{Z}^n)^m \to L^2(\mathbb{Z}^n)$ is a Hilbert-Schmidt operator, then*

$$||T_\sigma||_{S_2} = ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}.$$

Proof. First we remark that $(\varepsilon_l)_{l\in\mathbb{Z}^n}$, where ε_l is given by

$$\varepsilon_l(x) = \begin{cases} 1, \, x = l \\ 0, \, x \neq l \end{cases}, \text{for all } l \text{ in } \mathbb{Z}^n$$

is an orthonormal basis for $L^2(\mathbb{Z}^n)$. Then by virtue of Theorem 7.3 $(\varepsilon_{k_1,\ldots,k_m} = \varepsilon_k)_{k\in(\mathbb{Z}^n)^m}$, where $\varepsilon_k = \varepsilon_{k_1} \otimes \ldots \otimes \varepsilon_{k_m}$ for $k = (k_1, \ldots, k_m)$ in $(\mathbb{Z}^n)^m$ is an orthonormal basis for $L^2(\mathbb{Z}^n) \otimes \ldots \otimes L^2(\mathbb{Z}^n)$ which is the m-fold tensor product of $L^2(\mathbb{Z}^n)$ with itself. For $k = (k_1, \ldots, k_m)$ in $(\mathbb{Z}^n)^m$ we get

$$\left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} \varepsilon_{k_j} \right)(\theta) = \prod_{j=1}^{m} \left(\mathcal{F}_{\mathbb{Z}^n} \varepsilon_{k_j}\right)(\theta_j) = \prod_{j=1}^{m} e^{ik_j\theta_j} = e^{ik\cdot\theta},$$

for all $\theta = (\theta_1, \ldots, \theta_m) \in (\mathbb{T}^m)^m$, where $k \cdot \theta = \sum_{j=1}^{m} k_j \cdot \theta_j$, because

$$\left(\mathcal{F}_{\mathbb{Z}^n} \varepsilon_{k_j}\right)(\theta_j) = \sum_{x\in\mathbb{Z}^n} \varepsilon_{k_j}(x) e^{ix\cdot\theta_j} = e^{ik_j\cdot\theta_j}, \; \theta_j \in \mathbb{T}^n, \; 1 \leq j \leq m.$$

Thus,

$$\begin{aligned}(T_\sigma \varepsilon_k)(x) &= \int_{(\mathbb{T}^n)^m} e^{-ix\cdot|\theta|} \sigma(x,\theta) \left(\bigotimes_{j=1}^{m} \mathcal{F}_{\mathbb{Z}^n} \varepsilon_{k_j} \right)(\theta) d\mu(\theta) \\ &= \int_{(\mathbb{T}^n)^m} e^{-i(\tilde{x}-k)\cdot\theta} \sigma(x,\theta) d\mu(\theta) = \mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x, \tilde{x} - k),\end{aligned}$$

for all x in $\mathbb{Z}^n$, where $\tilde{x} = (x, \ldots, x) \in (\mathbb{Z}^n)^m$. We write $\mathcal{F}_{(\mathbb{T}^n)^m}\sigma(x, \cdot)$ for the Fourier transform of the function $\sigma(x, \cdot)$ on $(\mathbb{T}^n)^m$. It is given by

$$\mathcal{F}_{(\mathbb{T}^n)^m}(x,y) = \int_{(\mathbb{T}^n)^m} e^{-iy\theta} \sigma(x,\theta) d\mu(\theta), \; x \in \mathbb{Z}^n, \; y \in (\mathbb{Z}^n)^m.$$

In the second place we observe that T_σ is a bounded multilinear operator by Theorem 2.2 and it is easy to prove that T_σ is in fact a weak Hilbert-Schmidt mapping. So, by the universal property of tensor product of Hilbert spaces (see (i) in Theorem 7.3) there is a unique bounded operator $T : L^2(\mathbb{Z}^n) \otimes \ldots \otimes L^2(\mathbb{Z}^n) \to L^2(\mathbb{Z}^n)$ such that

$$T(f_1 \otimes \ldots \otimes f_m) = T_\sigma(f_1, \ldots, f_m)$$

for all $(f_1, \ldots, f_m)$ in $L^2(\mathbb{Z}^n)^m$. By using Fubini's theorem and the Plancherel formula for Fourier series we get

$$\begin{aligned}
||T||_{S_2}^2 &= \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} ||T(\varepsilon_{k_1} \otimes \ldots \otimes \varepsilon_{k_m})||_{L^2(\mathbb{Z}^n)}^2 \\
&= \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} ||T_\sigma(\varepsilon_{k_1}, \ldots, \varepsilon_{k_m})||_{L^2(\mathbb{Z}^n)}^2 \\
&= \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} \sum_{x \in \mathbb{Z}^n} |T_\sigma(\varepsilon_{k_1}, \ldots, \varepsilon_{k_m})|^2 \\
&= \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} \sum_{x \in \mathbb{Z}^n} |\mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x, \tilde{x} - k)|^2 \\
&= \sum_{x \in \mathbb{Z}^n} \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} |\mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x, \tilde{x} - k)|^2 \\
&= \sum_{x \in \mathbb{Z}^n} \sum_{k_1 \in \mathbb{Z}^n} \cdots \sum_{k_m \in \mathbb{Z}^n} |\mathcal{F}_{(\mathbb{T}^n)^m} \sigma(x, k)|^2 \\
&= \sum_{x \in \mathbb{Z}^n} \int_{(\mathbb{T}^n)^m} |\sigma(x, \theta)|^2 d\mu(\theta) \\
&= ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}^2.
\end{aligned}$$

From the last equality and by the hypothesis it follows that $T : L^2(\mathbb{Z}^n) \otimes \ldots \otimes L^2(\mathbb{Z}^n) \to L^2(\mathbb{Z}^n)$, is a Hilbert-Schmidt operator if and only if $\sigma \in L^2(\mathbb{Z}^n \times L^2(\mathbb{T}^n)^m)$. Thus by Definition 7.4 the proof is complete. □

8. L^p- continuity of multilinear pseudo-differential operators on $\mathbb{T}^n$

Let $\sigma : \mathbb{T}^n \times (\mathbb{Z}^n)^m \to \mathbb{C}$ be a measurable function. Then for all $f = (f_1, \ldots, f_m)$ in $L^2(\mathbb{T}^n)^m$ (the m-fold product of Lebesgue space $L^2(\mathbb{T}^n)$ with itself), $m, n \geq 1$ we define $T_\sigma f$ to be the function on $\mathbb{T}^n$ defined formally by

$$(T_\sigma f)(\theta) = \sum_{x \in (\mathbb{Z}^n)^m} \sigma(\theta, x) e^{i\theta \cdot |x|} \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j \right)(x), \ \theta \in \mathbb{T}^n,$$

where

$$\left(\mathcal{F}_{\mathbb{T}^n} f_j\right)(x_j) = \int_{\mathbb{T}^n} e^{-ix_j \cdot \theta} f_j(\theta) d\mu(\theta), \ x_j \in \mathbb{Z}^n, \ 1 \leq j \leq n$$

$$d\mu(\theta) = (2\pi)^{-n} d\theta, \ d\theta = d\theta_1 \ldots d\theta_n, \ \theta \in \mathbb{T}^n,$$

and

$$\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j \right)(x) = \prod_{j=1}^m \left(\mathcal{F}_{\mathbb{T}^n} f_j\right)(x_j), \ x = (x_1, \ldots, x_m) \in (\mathbb{Z}^n)^m$$

denote the tensor product of the Fourier transforms of the function f_j, $1 \leq j \leq m$.

We call T_σ the multilinear (or m-linear) pseudo-differential operator on $\mathbb{T}^n$ corresponding to the symbol σ.

Theorem 8.1. *Let $\sigma \in L^p(\mathbb{T}^n \times (\mathbb{Z}^n)^m)$, $1 \leq p \leq 2$. Then $T_\sigma : L^p(\mathbb{T}^n)^m \to L^p(\mathbb{T}^n)$ is a bounded multilinear operator. Moreover,*

$$||T_\sigma||_{B(L^p(\mathbb{T}^n), L^p(\mathbb{T}^n))} \leq ||\sigma||_{L^p(\mathbb{T}^n \times (\mathbb{Z}^n)^m)}.$$

Proof. Let $f = (f_1, \ldots, f_m) \in L^1(\mathbb{T}^n)^m$. Then by Minkowski's inequality in integral form, by the Hölder inequality and by Hausdorff-Young's inequality for the Fourier transform on the torus $\mathbb{T}^n$ we get

$$\left(\int_{\mathbb{T}^n} |(T_\sigma f)(\theta)|^p d\mu(\theta)\right)^{1/p} =$$

$$= \left(\int_{\mathbb{T}^n} \left| \sum_{x\in(\mathbb{Z}^n)^m} e^{i\theta\cdot|x|}\sigma(\theta, x) \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\right|^p d\mu(\theta)\right)^{1/p}$$

$$\leq \sum_{x\in(\mathbb{Z}^n)^m} \left(\int_{\mathbb{T}^n} |\sigma(\theta, x)|^p \left|\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\right|^p d\mu(\theta)\right)^{1/p}$$

$$= \sum_{x\in(\mathbb{Z}^n)^m} \left(\int_{\mathbb{T}^n} |\sigma(\theta, x)|^p d\mu(\theta)\right)^{1/p} \left|\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\right|$$

$$\leq \left(\sum_{x\in(\mathbb{Z}^n)^m} \int_{\mathbb{T}^n} |\sigma(\theta, x)|^p d\mu(\theta)\right)^{1/p} \left(\sum_{x\in(\mathbb{Z}^n)^m} \left|\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} f_j\right)(x)\right|^{p'}\right)^{1/p'}$$

$$\leq \left(\sum_{x\in(\mathbb{Z}^n)^m} \int_{\mathbb{T}^n} |\sigma(\theta, x)|^p d\mu(\theta)\right)^{1/p} \left(\sum_{x\in(\mathbb{Z}^n)^m} \prod_{j=1}^m |(\mathcal{F}_{\mathbb{T}^n} f_j)(x_j)|^{p'}\right)^{1/p'}$$

$$\leq ||\sigma||_{L^p(\mathbb{T}^n\times(\mathbb{Z}^n)^m)} \prod_{j=1}^m ||\mathcal{F}_{\mathbb{T}^n} f_j||_{L^{p'}(\mathbb{Z}^n)}$$

$$\leq ||\sigma||_{L^p(\mathbb{T}^n\times(\mathbb{Z}^n)^m)} \prod_{j=1}^m ||f_j||_{L^p(\mathbb{T}^n)},$$

where p' is the conjugate index of p (i.e. $1/p + 1/p' = 1$). Thus

$$||T_\sigma f||_{B(L^p(\mathbb{T}^n)^m, L^p(\mathbb{T}^n))} \leq ||\sigma||_{L^p(\mathbb{T}^n\times(\mathbb{Z}^n)^m)} \prod_{j=1}^m ||f_j||_{L^p(\mathbb{T}^n)},$$

for all $f = (f_1, \ldots, f_m)$ in $L^p(\mathbb{T}^n)^m$. Hence by using a standard density argument ($L^1(\mathbb{T}^n)^m$ is dense in $L^p(\mathbb{T}^n)^m$) the proof is complete. □

Remark 8.2. If we take $m = n = 1$, $p = 2$ in the above theorem we recover Theorem 22.1 in Wong's book [17].

For $x \in \mathbb{Z}^n$ we define the function $e_x : \mathbb{T}^n \to \mathbb{C}$ by

$$e_x(\theta) = e^{ix\cdot\theta}, \quad \theta \in \mathbb{T}^n.$$

Lemma 8.3. *$\{e_x\}_{x\in\mathbb{Z}^n}$ is an orthogonal set on $L^2(\mathbb{T}^n)$. Moreover $\{e_x\}_{x\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$.*

For the proof of this Lemma and related references on the basic classical Fourier analysis see Grafakos' book [9] and Wong's book [17].

Theorem 8.4. *Let $\sigma \in L^p(\mathbb{T}^n \times (\mathbb{Z}^n)^m)$, $1 \le p < \infty$. Then*

$$\sum_{x \in (\mathbb{Z}^n)^m} ||T_\sigma(e_{x_1}, \dots, e_{x_m})||_p^p = ||\sigma||^p_{L^p(\mathbb{T}^n \times (\mathbb{Z}^n)^m)}.$$

Proof. First we remark that

$$\mathcal{F}_{\mathbb{T}^n} e_{y_j}(x_j) = \begin{cases} 1, \ x_j = y_j \\ 0, \ x_j \neq y_j, \quad 1 \le j \le m \end{cases}$$

and

$$\left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} e_{y_j} \right)(x_1, \dots, x_m) =$$

$$= \begin{cases} 1, \ (x_1, \dots, x_m) = (y_1, \dots, y_m) \\ 0, \ (x_1, \dots, x_m) = (y_1, \dots, y_m). \end{cases}$$

Hence

$$T_\sigma(e_{x_1}, \dots, e_{x_m})(\theta) = \sum_{y \in (\mathbb{Z}^n)^m} e^{i\theta \cdot |y|} \sigma(\theta, y) \left(\bigotimes_{j=1}^m \mathcal{F}_{\mathbb{T}^n} e_{x_j} \right)(y)$$
$$= \sigma(\theta, x) e_{x_1}(\theta) \dots e_{x_m}(\theta).$$

So,

$$\sum_{y \in (\mathbb{Z}^n)^m} ||T_\sigma(e_{x_1}, \dots, e_{x_m})||^p_{L^p(\mathbb{T}^n)} =$$

$$= \sum_{y \in (\mathbb{Z}^n)^m} \int_{T^n} |\sigma(\theta, x)|^p d\mu(\theta) = ||\sigma||_{L^p(\mathbb{T}^n \times (\mathbb{Z}^n)^m)}$$

as asserted. □

Using Theorem 8.4 and Lemma 8.3 we state the following result.

Theorem 8.5. *Let $\sigma : \mathbb{T}^n \times (\mathbb{Z}^n)^m \to \mathbb{C}$ be a measurable function. Then the multilinear pseudo-differential operator $T_\sigma : L^2(\mathbb{T}^n)^m \to L^2(\mathbb{T}^n)$ is a Hilbert-Schmidt operator if and only if $\sigma \in L^2(\mathbb{T}^n \times (\mathbb{Z}^n)^m)$. Moreover, if $T_\sigma : L^2(\mathbb{T}^n)^m \to L^2(\mathbb{T}^n)$ is a Hilbert-Schmidt operator, then*

$$||T_\sigma||_{S_2} = ||\sigma||_{L^2(\mathbb{Z}^n \times (\mathbb{T}^n)^m)}.$$

The proof of Theorem 8.5 follows in the same way as well as the proof of Theorem 7.5.

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