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A Characterization of Compact SG Pseudo-differential Operators on $L^2(\mathbb{R}^n)$

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Abstract. We give a necessary and sufficient condition for pseudo-differential operators with SG symbols to be compact from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

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1. Introduction

For $m_1, m_2 \in \mathbb{R}$, we let S^{m_1,m_2} be the set of all functions $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta} \langle x \rangle^{m_2-|\alpha|} \langle \xi \rangle^{m_1-|\beta|}, \quad x,\xi \in \mathbb{R}^n,$$

where for all $y \in \mathbb{R}^n$,

$$\langle y \rangle = \left(1+|y|^2\right)^{1/2}.$$

Functions in $S^{m_1,m_2}(\mathbb{R}^n \times \mathbb{R}^n)$ are called SG symbols of order (m_1, m_2). It is clear that for $m_2 \le 0$, $S^{m_1,m_2} \subset S^{m_1}$, where S^{m_1} is the class of symbols extensively studied in [8, 15].

Let $\sigma \in S^{m_1,m_2}$. Then the pseudo-differential operator T_σ corresponding to the symbol σ of is defined by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in the Schwartz space $\mathcal{S}$. In this paper, the Fourier transform $\hat{f}$ of a function f in $L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

SG pseudo-differential operators have been studied in [2–6, 11]. They are also called pseudo-differential operators with exit behavior at infinity [6]. The following theorems on the basic calculus of SG pseudo-differential operators can be found in [6]

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Theorem 1.1. *Let $\sigma \in S^{m_1,m_2}$ and $\tau \in S^{\mu_1,\mu_2}$, $-\infty < m_1, m_2, \mu_1, \mu_2 < \infty$. Then $T_\sigma T_\tau = T_\lambda$, where*

$$\lambda \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau),$$

i.e., for every positive integer M, there exists a positive integer N such that

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau) \in S^{m_1+\mu_1-M, m_2+\mu_2-M}.$$

Theorem 1.2. *Let $\sigma \in S^{m_1,m_2}$. The formal adjoint T_σ^* of T_σ is a SG pseudo-differential operator T_τ, where*

$$\tau \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \bar{\sigma}.$$

Here the asymptotic expansion means that for every positive integer M, there exists a positive integer N such that

$$\tau - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \bar{\sigma} \in S^{m_1-M, m_2-M}.$$

Using the formal adjoint, we can extend the definition of a SG pseudo-differential operator from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions. Indeed, let $\sigma \in S^{m_1,m_2}$. Then for all $u \in \mathcal{S}'$, we define $T_\sigma u$ by

$$(T_\sigma u)(\varphi) = u\left(\overline{T_\sigma^* \bar{\varphi}}\right), \quad \varphi \in \mathcal{S}.$$

It is easy to check that T_σ maps $\mathcal{S}'$ into $\mathcal{S}'$ continuously. The following theorem follows from Theorem 10.7 in [15] and the fact that every symbol in $S^{0,0}$ is in S^0.

Theorem 1.3. *Let $\sigma \in S^{0,0}$. Then $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.*

Of particular concern to us in this paper is the case when $p = 2$. The aim of this paper is to use spectral theory, a Calkin algebra and a result of Gohberg to prove the following theorem.

Theorem 1.4. *Let $\sigma \in S^{0,0}$. Then $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact if and only if*

$$\lim_{|(x,\xi)|\to\infty} |\sigma(x,\xi)| = 0.$$

In Section 2, Bessel potentials and L^p-Sobolev spaces for SG pseudo-differential operators are recalled. The Sobolev embedding theorem and the L^p-boundedness of SG pseudo-differential operators obtained in [5] are also stated. Then in Section 3, we use Gohberg's lemma obtained by Grushin in [7] to give a proof of the main theorem on the compactness in $L^2(\mathbb{R}^n)$ of SG pseudo-differential operators with symbols in $S^{0,0}$.

Similar results for pseudo-differential operators on the unit circle centered at the origin can be found in [9, 10].

2. Sobolev Spaces

For $s_1, s_2 \in \mathbb{R}$, we define the Bessel potential J_{s_1,s_2} of order (s_1, s_2) by

$$J_{s_1,s_2} = T_{\sigma_{s_1,s_2}},$$

where

$$\sigma_{s_1,s_2}(x,\xi) = \langle x\rangle^{-s_2} \langle \xi\rangle^{-s_1}, \quad x, \xi \in \mathbb{R}^n.$$

It is clear that $\sigma_{s_1,s_2} \in S^{-s_1,-s_2}$.

For $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, we define the L^p-Sobolev space $H^{s_1,s_2,p}$ of order (s_1, s_2) by

$$H^{s_1,s_2,p} = \{u \in \mathcal{S}' : J_{-s_1,-s_2}u \in L^p(\mathbb{R}^n)\}.$$

Obviously,

$$H^{0,0,p} = L^p(\mathbb{R}^n).$$

The following result on the L^p-boundedness of SG pseudo-differential operators with symbols in S^{m_1,m_2}, $-\infty < m_1, m_2 < \infty$, can be found in [5].

Theorem 2.1. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$. Then $T_\sigma : H^{s_1,s_2,p} \to H^{s_1-m_1,s_2-m_2,p}$ is a bounded linear operator for all $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$.*

We also need the following theorem, which is known as the Sobolev embedding theorem in [5].

Theorem 2.2. *Let $-\infty < s_1, s_2, t_1, t_2 < \infty$ be such that $s_1 < t_1$ and $s_1 < t_2$. Then the inclusion $i : H^{t_1,t_2,p} \hookrightarrow H^{s_1,s_2,p}$ is compact for $1 < p < \infty$.*

3. Compact SG Pseudo-Differential Operators

A closed linear operator A from a complex Banach space X into a complex Banach space Y with dense domain $\mathcal{D}(A)$ is said to be Fredholm if the range $R(A)$ of A is a closed subspace of Y, the null space $N(A)$ of A and the null space $N(A^t)$ of the true adjoint A^t of A are finite dimensional. By Atkinson's theorem [1], a closed linear operator $A : X \to Y$ with dense domain $\mathcal{D}(A)$ is Fredholm if and only if there exists a bounded linear operator $B : Y \to X$ such that $AB - I : Y \to Y$ and $BA - I : X \to X$ are compact operators, where I denotes the identity operator on Y in $AB - I : Y \to Y$ and on X in $BA - I : X \to X$.

Let $A : X \to X$ be a closed linear operator with dense domain $\mathcal{D}(A)$. Then the spectrum $\Sigma(A)$ of A is defined by

$$\Sigma(A) = \mathbb{C} - \rho(A),$$

where $\rho(A)$ is the resolvent set of A given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective}\}.$$

The essential spectrum $\Sigma_e(A)$ of A, which is given in [14] by Wolf, is defined by

$$\Sigma_e(A) = \mathbb{C} - \Phi_e(A),$$

where

$$\Phi_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm}\}.$$

For properties of essential spectra, see [12,13]. The following proposition is a special case of Theorem 4.4 in [5].

Proposition 3.1. *Let $\sigma \in S^{0,0}$ be such that*

$$\lim_{|(x,\xi)|\to\infty} |\sigma(x,\xi)| = 0.$$

Then

$$\Sigma_e(T_\sigma) = \{0\}.$$

A bounded linear operator A on a complex, separable and infinite-dimensional Hilbert space X is said to be essentially normal if $AA^t - A^tA$ is compact.

Proposition 3.2. *Let $\sigma \in S^{0,0}$. Then the bounded linear operator $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is essentially normal.*

Proof. Let $\tau \in S^{0,0}$ be such that $T_\tau = T_\sigma^t$. Then using Theorem 1.1,

$$T_\sigma T_\tau = T_\gamma$$

and

$$T_\tau T_\sigma = T_{\tilde{\gamma}},$$

where γ and $\tilde{\gamma}$ are symbols of order $(0,0)$. Moreover, $\gamma - \sigma\tau \in S^{-1,-1}$ and $\tilde{\gamma} - \sigma\tau \in S^{-1,-1}$. Therefore $\gamma - \tilde{\gamma} \in S^{-1,-1}$. Hence, by Proposition 2.1 and the Sobolev embedding theorem,

$$T_\sigma T_\sigma^t - T_\sigma^t T_\sigma = T_{\gamma-\tilde{\gamma}} : L^2(\mathbb{R}^n) \to H^{1,1,2} \hookrightarrow L^2(\mathbb{R}^n)$$

is compact, which completes the proof. □

The following theorem is known as Gohberg's lemma, which can be found in [7].

Theorem 3.3. *Let $\sigma \in S^{0,0}$. Then for all compact operators K on $L^2(\mathbb{R}^n)$,*

$$\|T_\sigma - K\|_* \geq d, \tag{3.1}$$

where

$$d = \limsup_{|(x,\xi)|\to\infty} |\sigma(x,\xi)|$$

In order to prove our main theorem, we need the notion of the Calkin algebra. Let $B(L^2(\mathbb{R}^n))$ and $K(L^2(\mathbb{R}^n))$ be, respectively, the C^*-algebra of bounded linear operators on $L^2(\mathbb{R}^n)$ and the ideal of compact operators on $L^2(\mathbb{R}^n)$. The Calkin algebra $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$ is a C^*- algebra in which the product and the adjoint are defined, respectively, by

$$[A][B] = [AB]$$

and

$$[A]^* = [A^*]$$

for all A and B in $B(L^2(\mathbb{R}^n))$. Let $[A]$ and $[B]$ be in $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$. Then

$$[A] = [B] \quad \Leftrightarrow \quad A - B \in K(L^2(\mathbb{R}^n)).$$

The norm $\| \|_C$ in $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$ is given by

$$\|[A]\|_C = \inf_{K\in K(L^2(\mathbb{R}^n))} \|A - K\|_*, \quad [A] \in B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n)),$$

where $\| \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. It can be shown that $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$ is a C^*-algebra. By using the Calkin algebra, (3.1) in Gohberg's lemma is the same as

$$\|[T_\sigma]\|_C \geq d.$$

Now, we are ready to prove the main theorem in this paper.

Proof of Theorem 1.4 We first assume that

$$\lim_{|(x,\xi)|\to\infty} |\sigma(x,\xi)| = 0.$$

Then $T_\sigma : L^2(\mathbb{R}^n \to L^2(\mathbb{R}^n)$ is compact if and only if

$$[T_\sigma] = 0$$

in $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$. By Proposition 3.2, T_σ is essentially normal on $L^2(\mathbb{R}^n)$. So, $[T_\sigma]$ is normal in the Calkin algebra $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$. Hence

$$r([T_\sigma]) = \|[T_\sigma]\|_C,$$

where $r([T_\sigma])$ is the spectral radius of $[T_\sigma]$. On the other hand, by Proposition 3.1, $\Sigma_e(T_\sigma) = \{0\}$. Therefore by Atkinson's theorem in [1], the spectrum $\Sigma([T_\sigma])$ of $[T_\sigma]$ in the Calkin algebra $B(L^2(\mathbb{R}^n))/K(L^2(\mathbb{R}^n))$ is given by

$$\Sigma([T_\sigma]) = \{0\}.$$

Thus,

$$\|[T_\sigma]\|_C = r([T_\sigma]) = 0.$$

It follows that $[T_\sigma] = 0$ and hence T_σ is compact. Conversely, suppose that T_σ is compact. If we set $K = T_\sigma$ in (3.1), then we get

$$\lim_{|(x,\xi)\to\infty} |\sigma(x,\xi)| = 0.$$

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