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Spreading Speeds and Linear Determinacy for Two Species Competition Systems with Nonlocal Dispersal in Periodic Habitats

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Abstract. The current paper is concerned with the existence of spreading speeds and linear determinacy for two species competition systems with nonlocal dispersal in time and space periodic habitats. The notion of spreading speed intervals for such a system is first introduced via the natural features of spreading speeds. The existence and lower bounds of spreading speed intervals are then established. When the periodic dependence of the habitat is only on the time variable, the existence of a single spreading speed is proved. It also shows that, under certain conditions, the spreading speed interval in any direction is a singleton, and, moreover, the linear determinacy holds.

Keywords and phrases: Competition system, nonlocal dispersal, periodic habitat, spreading speed, linear determinacy.

Mathematics Subject Classification: 45C05, 45G10, 45M20, 92D25

1. Introduction

The current paper is concerned with the spatial spreading speeds of the following two species competition system with nonlocal dispersal,

$$\begin{cases} u_t = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v), & x \in \mathbb{R}^N \\ v_t = \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a_2(t,x) - b_2(t,x)u - c_2(t,x)v), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u(t,x), v(t,x)$ represent the population densities of two species, $\kappa(\cdot)$ is a C^1 convolution kernel supported on a ball centered at 0 (i.e. $\kappa(z) > 0$ if $\|z\| < r_0$ and $\kappa(z) = 0$ if $\|z\| \geq r_0$ for some $r_0 > 0$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^N$), $\int_{\mathbb{R}^N} \kappa(z)dz = 1$, and $a_k(\cdot,\cdot)$, $b_k(\cdot,\cdot)$, $c_k(\cdot,\cdot)$ satisfy the following basic hypothesis,

(HB0) $a_k(t,x)$, $b_k(t,x)$, $c_k(t,x)$ $(k = 1,2)$ *are* C^0 *in* $(t,x) \in \mathbb{R}\times\mathbb{R}^N$*; periodic in* t *with period* T *and in* x_i *with period* p_i*, that is,* $a_k(\cdot+T,\cdot) = a_k(\cdot,\cdot+p_i\mathbf{e_i}) = a_k(\cdot,\cdot)$, $b_k(\cdot+T,\cdot) = b_k(\cdot,\cdot+p_i\mathbf{e_i}) = b_k(\cdot,\cdot)$,

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$c_k(\cdot+T,\cdot)=c_k(\cdot,\cdot+p_i\mathbf{e_i})=c_k(\cdot,\cdot)$, $\mathbf{e_i}=(\delta_{i1},\delta_{i2},\cdots,\delta_{iN})$, $\delta_{ij}=1$ *if* $i=j$ *and* 0 *if* $i\neq j$, $i,j=1,2,\cdots,N$; *and* $b_k(t,x)>0$, $c_k(t,x)>0$ *for* $t\in\mathbb{R}$ *and* $x\in\mathbb{R}^N$.

In (1.1), the functions a_1, a_2 represent the respective growth rates of the two species, b_1, c_2 account for self-regulation of the respective species, and c_1, b_2 account for competition between the two species. The periodicity of a_k, b_k, and c_k reflects the time and space periodicity of the environment.

System (1.1) is a nonlocal dispersal counterpart of the following two species competition system with random dispersal,

$$\begin{cases} u_t=\Delta u+u(a_1(t,x)-b_1(t,x)u-c_1(t,x)v), & x\in\mathbb{R}^N\\ v_t=\Delta v+v(a_2(t,x)-b_2(t,x)u-c_2(t,x)v), & x\in\mathbb{R}^N.\end{cases}\tag{1.2}$$

Systems (1.1) and (1.2) describe the population dynamics of two competing species with internal interaction or dispersal. Classically, one assumes that the range of internal interaction of both species is infinitesimal or the internal dispersal is random, which leads to (1.2) (see [4], [21], [22], [33], [40], [45], [46], etc.). However, in many cases dispersal of a species is affected by long range or nonlocal internal interaction. We refer to [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [13], [14], [23], [24], [25], [30], [37], [38], [39], etc., for the study of evolution models with nonlocal spatial interaction. In particular, (1.1) arises in modeling the population dynamics of two competing species with nonlocal spatial internal interaction (see [23]).

In addition to the coexistence and extinction dynamics, spatial spreading speeds and traveling wave solutions are among the central problems investigated for (1.1) and (1.2). Such problems for (1.2) with space and time independent coefficients have been widely studied. For example, spreading speed/minimal wave speed and linear determinacy for (1.2) with temporally and spatially independent coefficients are studied in [12], [15], [18], [19], [20], [26], [27], [28], [29], [31], [41], [42], [43], [44], etc.. It should be pointed out that the works [12], [26], and [43] can be applied to (1.2) with spatially homogeneous and temporally periodic coefficients. Very recently, we learned that Yu and Zhao have been studying spatial spreading speeds and traveling wave solutions of (1.2) with coefficients that are periodic in time and space ([44]).

Spatial spreading speeds and traveling wave solutions for (1.1) with temporally and spatially independent coefficients have also been studied in several papers. In fact, the works [12], [26], and [43] can be applied to (1.1) with coefficients independent of space. However, there is little study on spatial spreading and traveling wave solutions for (1.1) with periodic coefficients in both time and space.

The objective of the current paper is to study the spatial spreading speeds of (1.1) with both temporally and spatially periodic coefficients. Due to the lack of compactness of the solutions, it appears to be difficult to adopt the construction method for spreading speeds of general competitive or cooperative systems from [12], [26], [43], etc. in dealing with (1.1) with temporally and spatially periodic coefficients. Therefore we will employ the natural properties of spreading speeds to give a definition of this concept following an idea from [36] and [37]. We then investigate the boundedness, lower bounds, and uniqueness of spreading speeds. We also study the linear determinacy for the spreading speeds.

To describe the problems studied and the results obtained in the current paper, let

$$X=\{u\in C(\mathbb{R}^N,\mathbb{R})\,|\,u \text{ is uniformly continuous and bounded}\}\tag{1.3}$$

with the supremum norm and

$$X^+=\{u\in X\,|\,u(x)\geq 0\quad\forall\, x\in\mathbb{R}^N\},\quad X^{++}=\{u\in X^+\,|\,\inf_{x\in\mathbb{R}^N}u(x)>0\}.\tag{1.4}$$

For $u,v\in X$, we define

$$u\leq v\quad(u\ll v)\quad\text{if}\quad v-u\in X^+\quad(v-u\in X^{++}).$$

Let

$$X_p=\{u\in C(\mathbb{R}^N,\mathbb{R})\,|\,u(\cdot+p\mathbf{e_i})=u(\cdot)\quad\text{for } i=1,2,\cdots,N\}\tag{1.5}$$

with the maximum norm and

$$X_p^+ = \{u \in X_p \,|\, u(x) \geq 0 \;\forall\; x \in \mathbb{R}^N\}, \quad X_p^{++} = \{u \in X_p^+ \,|\, u(x) > 0 \quad \forall\; x \in \mathbb{R}^N\}. \tag{1.6}$$

By general semigroup theory (see [34]), for any $u_0, v_0 \in X$, (1.1) has a unique (local) solution $(u(t,x;u_0,v_0),\, v(t,x;u_0,v_0))$ with $u(0,x;u_0,v_0) = u_0(x)$ and $v(0,x;u_0,v_0) = v_0(x)$. By comparison principle for two species competition systems with nonlocal dispersal (see [17, Proposition 3.1]), if $u_0, v_0 \in X^+$, then $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ exists and $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0)) \in X^+ \times X^+$ for all $t \geq 0$. We remark that, if $(u_0,v_0) \in X_p \times X_p$, then $(u(t,\cdot;u_0,v_0),\; v(t,\cdot;u_0,v_0)) \in X_p \times X_p$ for t in the existence interval of $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$.

Observe that (1.1) contains the following two sub-systems,

$$u_t = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + u(a_1(t,x) - b_1(t,x)u), \quad x \in \mathbb{R}^N, \tag{1.7}$$

and

$$v_t = \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + v(a_2(t,x) - c_2(t,x)v), \quad x \in \mathbb{R}^N. \tag{1.8}$$

The asymptotic dynamics of (1.7) (resp. (1.8)) is determined by the stability of its trivial solution $u \equiv 0$ (resp. $v \equiv 0$). More precisely, let

$$\mathcal{X}_p = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) | u(\cdot + T, \cdot) = u(\cdot, \cdot + p_i \mathbf{e_i}) = u(\cdot,\cdot), \quad i = 1, \cdots, N\} \tag{1.9}$$

with the norm $\|u\|_{\mathcal{X}_p} = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} |u(t,x)|$, and

$$\mathcal{X}_p^+ = \{u \in \mathcal{X}_p \,|\, u(t,x) \geq 0 \quad \forall\; (t,x) \in \mathbb{R} \times \mathbb{R}^N\}. \tag{1.10}$$

Let I be the identity map on $\mathcal{X}_p$, and $\mathcal{K}$, $a(\cdot,\cdot)I : \mathcal{X}_p \to \mathcal{X}_p$ be defined by

$$\big(\mathcal{K}u\big)(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy, \tag{1.11}$$

$$(a(\cdot,\cdot)Iu)(t,x) = a(t,x)u(t,x), \tag{1.12}$$

where $\kappa(\cdot)$ is as in (1.1) and $a(\cdot,\cdot) \in \mathcal{X}_p$. Let $\sigma(-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I)$ be the spectrum of $-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I$ acting on $\mathcal{X}_p$ and

$$\lambda_0(a) := \sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I)\}.$$

We call $\lambda_0(a)$ the *principal spectrum point* of $-\partial_t + \mathcal{K} - I + a(\cdot,\cdot)I$ acting on $\mathcal{X}_p$ (see Definition 2.1).

Throughout the paper, we assume that

(HB1) *$u \equiv 0$ is a linearly unstable solution of (1.7) in X_p and $v \equiv 0$ is a linearly unstable solution of (1.8) in X_p, that is, $\lambda_0(a_k) > 0$ $(k = 1,2)$.*

Note that $(0,0)$ is a trivial solution of (1.1) in $X_p^+ \times X_p^+$ and that (HB0) and (HB1) imply that (1.1) has two semi-trivial time periodic solutions in $X_p^+ \times X_p^+$, $(u^*(t,\cdot),0) \in (X_p^+ \setminus \{0\}) \times X_p^+$ and $(0, v^*(t,\cdot)) \in X_p^+ \times (X_p^+ \setminus \{0\})$ (see Proposition 2.11).

We also assume throughout this paper that

(HB2) *$(0,v^*)$ is linearly unstable in $X_p^+ \times X_p^+$ (i.e. $\lambda_0(a_1 - c_1 v^*) > 0$) and $(u^*,0)$ is linearly and globally stable in $X_p^+ \times X_p^+$ (i.e. $\lambda_0(a_2 - b_2 u^*) < 0$ and for any $(u_0,v_0) \in X_p^+ \times X_p^+$ with $u_0 \neq 0$, $u(t,x;u_0,v_0) - u^*(t,x) \to 0$ and $v(t,x;u_0,v_0) \to 0$ as $t \to \infty$ uniformly in $x \in \mathbb{R}^N$).*

The assumption (HB2) implies that the species u can invade the species v and the species v cannot invade the species u. We remark that the following assumption (HB2)$'$ on the coefficients a_i, b_i, and c_i $(i = 1,2)$ implies (HB2) (see Proposition 2.11 for the reasoning).

(HB2)$'$ $\lambda_0(a_k) > 0$ *for* $k = 1, 2$ *and* $a_{1L} > \frac{c_{1M}a_{2M}}{c_{2L}}$, $a_{2M} \le \frac{a_{1L}b_{2L}}{b_{1M}}$, *where* $a_{kL} = \inf_{t\in\mathbb{R},x\in\mathbb{R}^N} a_k(t,x)$, $a_{kM} = \sup_{t\in\mathbb{R},x\in\mathbb{R}^N} a_k(t,x)$, *and* b_{kL}, b_{kM}, c_{kL}, c_{kM} $(k = 1, 2)$ *are defined similarly.*

Under the assumptions (HB0)-(HB2), spatial spreading speeds or invading speeds from $(u^*, 0)$ to $(0, v^*)$ and traveling wave solutions connecting $(u^*, 0)$ and $(0, v^*)$ are among most interesting dynamical problems for (1.1). The objective of this paper is to study the spatial spreading speeds of (1.1) from $(u^*, 0)$ to $(0, v^*)$. In order to do so, we first transform (1.1) to a cooperative system via the following standard change of variables,

$$\tilde{u}(t,x) = u(t,x), \quad \tilde{v}(t,x) = v^*(t,x) - v(t,x). \tag{1.13}$$

Dropping the tilde, (1.1) is transformed into

$$\begin{cases} u_t = \mathcal{K}u - u + u\Big(a_1(t,x) - b_1(t,x)u - c_1(t,x)(v^*(t,x) - v)\Big) \\ v_t = \mathcal{K}v - v + b_2(t,x)\Big(v^*(t,x) - v\Big)u + v\Big(a_2(t,x) - 2c_2(t,x)v^*(t,x) + c_2(t,x)v\Big), \end{cases} \tag{1.14}$$

where $x \in \mathbb{R}^N$, $\mathcal{K}u = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy$ and $\mathcal{K}v = \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy$. Observe that the trivial solution $E_0 := (0,0)$ of (1.1) becomes $\tilde{E}_0 = (0, v^*)$, the semitrivial solution $E_1 := (0, v^*)$ of (1.1) becomes $\tilde{E}_1 = (0,0)$, and the semitrivial solution $E_2 := (u^*, 0)$ of (1.1) becomes $\tilde{E}_2 = (u^*, v^*)$. To study the spreading speeds of (1.1) from E_2 to E_1 is then equivalent to study the spreading speeds of (1.14) from $\tilde{E}_2$ to $\tilde{E}_1$. Observe also that (1.14) is cooperative in the region $u \ge 0$ and $0 \le v \le v^*$.

Throughout this paper, we assume (HB0)-(HB2). We denote $(u(t,x;u_0,v_0),\ v(t,x;u_0,v_0))$ as the solution of (1.14) with $(u(0,\cdot;u_0,v_0),\ v(0,\cdot;u_0,v_0)) = (u_0,v_0) \in X \times X$. Note that if $(u_0,v_0) \in X_p \times X_p$, then $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0)) \in X_p \times X_p$ for t in the existence interval of $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$. Moreover, by (HB2), for any $(u_0,v_0) \in X_p^+ \times X_p^+$ with $u_0 \ne 0$ and $v_0 \le v^*(0,\cdot)$,

$$(u(t,x;u_0,v_0), v(t,x;u_0,v_0)) - (u^*(t,x), v^*(t,x)) \to 0$$

as $t \to \infty$ uniformly in $x \in \mathbb{R}^N$.

Let

$$S^{N-1} = \{\xi \in \mathbb{R}^N \mid \|\xi\| = 1\}.$$

For given $\xi \in S^{N-1}$, let

$$X_1^+(\xi) = \{u \in X^+ \mid u(\cdot) \ll u^*(0,\cdot),\ u(x) = 0 \text{ for } x\cdot\xi \gg 1,\ \liminf_{x\cdot\xi\to-\infty} u(x) > 0\}$$

and

$$X_2^+(\xi) = \{v \in X^+ \mid v(\cdot) \ll v^*(0,\cdot),\ v(x) = 0 \text{ for } x\cdot\xi \gg 1,\ \liminf_{x\cdot\xi\to-\infty} v(x) > 0\}.$$

Definition 1.1. Let

$$C_{\sup}(\xi) = \Big\{c \in \mathbb{R} \mid \limsup_{x\cdot\xi\ge ct, t\to\infty} u^2(t,x;u_0,v_0) + v^2(t,x;u_0,v_0) = 0,\ \forall\ (u_0,v_0) \in X_1^+(\xi) \times X_2^+(\xi)\Big\}$$

and

$$C_{\inf}(\xi) = \Big\{c \in \mathbb{R} \mid \limsup_{x\cdot\xi\le ct, t\to\infty} \Big[|u(t,x;u_0,v_0) - u^*(t,x)| + |v(t,x;u_0,v_0) - v^*(t,x)|\Big] = 0,\ \forall\ (u_0,v_0) \in X_1^+(\xi) \times X_2^+(\xi)\Big\}.$$

Let

$$c^*_{\sup}(\xi) = \begin{cases} \inf\{c \,|\, c \in C_{\sup}(\xi)\} & \text{if} \quad C_{\sup}(\xi) \neq \emptyset \\ \infty & \text{if} \quad C_{\sup}(\xi) = \emptyset \end{cases}$$

and

$$c^*_{\inf}(\xi) = \begin{cases} \sup\{c \,|\, c \in C_{\inf}(\xi)\} & \text{if} \quad C_{\inf}(\xi) \neq \emptyset \\ -\infty & \text{if} \quad C_{\inf}(\xi) = \emptyset. \end{cases}$$

$[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is called the spreading speed interval of (1.14) or (1.1) in the direction of ξ.

Remark 1.2. (1) If $c \in C_{\sup}(\xi)$, then $[c, \infty) \subset C_{\sup}(\xi)$ and $C_{\inf}(\xi) \subset (-\infty, c)$
(2) If $c \in C_{\inf}(\xi)$, then $(-\infty, c] \subset C_{\inf}(\xi)$ and $C_{\sup}(\xi) \subset (c, \infty)$.
(3) $c^*_{\inf}(\xi) \le c^*_{\sup}(\xi)$ and for any $c \in (c^*_{\inf}(\xi), c^*_{\sup}(\xi))$, there is $(u_0, v_0) \in X_1^+(\xi) \times X_2^+(\xi)$ such that

$$\limsup_{x\cdot\xi \ge ct, t\to\infty} u^2(t, x; u_0, v_0) + v^2(t, x; u_0, v_0) > 0$$

and

$$\limsup_{x\cdot\xi \le ct, t\to\infty} \Big[|u(t, x; u_0, v_0) - u^*(t, x)| + |v(t, x; u_0, v_0) - v^*(t, x)|\Big] > 0.$$

Let $\lambda_\xi(\mu)$ be the principal spectrum point of the eigenvalue problem

$$\begin{cases} -u_t + \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y - x)u(t, y)dy - u(t, x) + (a_1(t, x) - c_1(t, x)v^*(t, x))u(t, x) = \lambda u(t, x) \\ u(\cdot, \cdot) \in \mathcal{X}_p \end{cases} \tag{1.15}$$

(see Definition 2.1 for detail).

The first two main theorems of this paper are then stated as follows.

Theorem 1.3 (Finiteness and lower bound)**.** *Assume (HB0)-(HB2). For any $\xi \in S^{N-1}$, $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a finite interval. Moreover,*

$$c^*_{\inf}(\xi) \ge \inf_{\mu>0} \frac{\lambda_\xi(\mu)}{\mu}. \tag{1.16}$$

Theorem 1.4 (Single spreading speed)**.** *Assume (HB0)-(HB2). If $a_k(t, x) \equiv a_k(t)$, $b_k(t, x) \equiv b_k(t)$, and $c_k(t, x) \equiv c_k(t)$ $(k = 1, 2)$, then $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$ for every $\xi \in S^{N-1}$.*

Observe that, by Theorem 1.3, $\inf_{\mu>0} \frac{\lambda_\xi(\mu)}{\mu}$ is a lower bound of the spreading speed interval $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ of (1.14). It is of great interest to explore conditions such that $c^*_{\inf}(\xi) = c^*_{\sup}(\xi) = \inf_{\mu>0} \frac{\lambda_\xi(\mu)}{\mu}$. To this end, we introduce the following standing assumptions.

(HL0) *$b_2(t, x)u^*(t, x) \ge c_2(t, x)v^*(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

(HL1) *$a_1(t, x) - c_1(t, x)v^*(t, x) - a_2(t, x) + 2c_2(t, x)v^*(t, x) - b_2(t, x)v^*(t, x) \ge 0$, $b_1(t, x) \ge c_1(t, x)$, and $b_2(t, x) \ge c_2(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

(HL2) *$a_1(t, x) - c_1(t, x)v^*(t, x) - a_2(t, x) + 2c_2(t, x)v^*(t, x) - b_2(t, x)v^*(t, x)\frac{c_{1M}}{b_{1L}} \ge 0$ and $a_1(t, x) - c_1(t, x)v^*(t, x) - a_2(t, x) + 2c_2(t, x)v^*(t, x) - b_2(t, x)v^*(t, x)\frac{c_{2M}}{b_{2L}} \ge 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

We remark that the following assumptions (HL0)$'$, (HL1)$'$, and (HL2)$'$ on the coefficients a_k, b_k, c_k $(k = 1, 2)$ imply (HL0), (HL1), and (HL2), respectively.

(HL0)$'$ $b_2(t, x) \cdot \frac{a_{1L}}{b_{1M}} \ge c_2(t, x) \cdot \frac{a_{2M}}{c_{2L}}$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

(HL1)$'$ *$a_1(t, x) - c_1(t, x) \cdot \frac{a_{2M}}{c_{2L}} - a_2(t, x) + 2c_2(t, x)\frac{a_{2L}}{c_{2M}} - b_2(t, x) \cdot \frac{a_{2M}}{c_{2L}} \ge 0$, $b_1(t, x) \ge c_1(t, x)$, and $b_2(t, x) \ge c_2(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

(HL2)$'$ $a_1(t,x) - c_1(t,x)\frac{a_{2M}}{c_{2L}} - a_2(t,x) + 2c_2(t,x)\frac{a_{2L}}{c_{2M}} - b_2(t,x)\frac{a_{2M}}{c_{2L}}\frac{c_{1M}}{b_{1L}} \geq 0$ *and* $a_1(t,x) - c_1(t,x)\frac{a_{2M}}{c_{2L}} - a_2(t,x) + 2c_2(t,x)\frac{a_{2L}}{c_{2M}} - b_2(t,x)\frac{a_{2M}}{c_{2L}}\frac{c_{2M}}{b_{2L}} \geq 0$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^N$.

The third main theorem of this paper is stated as follows.

Theorem 1.5. *Assume (HB0)-(HB2), (HL0), and (HL1) or (HL2). For any* $\xi \in S^{N-1}$, $c^*_{\inf}(\xi) = c^*_{\sup}(\xi) = \bar{c}^*_{\inf}(\xi)$.

We point out the followings. First, in the spatially homogeneous case, that is, in the case $a_k(t,x) \equiv a_k(t)$, $b_k(t,x) \equiv b_k(t)$, and $c_k(t,x) \equiv c_k(t)$ $(k=1,2)$, $c^*_{\inf}(\xi)$ and $c^*_{\sup}(\xi)$ can be defined in the same way as in [12] and [43] (see Remark 3.6 in Section 3 for more detail).

Second, when a_k, b_k and c_k $(k=1,2)$ are both spatially and temporally homogeneous, $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$ is proved in [12]. In such case, the existence of traveling wave solutions has also been studied in [12] and [32]. The traveling wave problem in general periodic media will be studied somewhere else. It remains open whether $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$ in the general periodic case.

Third, Theorem 1.5 shows that the spreading speed interval is a singleton and is determined by the spectrum of (1.15), which is therefore referred to as *linear determinacy* for the spreading speed. As mentioned in the above, the linear determinacy for the spreading speeds of (1.2) with temporally and spatially independent coefficients has been widely studied. Our assumptions for the linear determinacy of (1.14) in the case that the coefficients are independent of time and space are the same as those in the literature for the linear determinacy of (1.2) (see the following remark).

Remark 1.6. (1) When a_k, b_k, and c_k $(k=1,2)$ are positive constants,

$$u^* = \frac{a_1}{b_1}, \quad v^* = \frac{a_2}{c_2}.$$

Hence the assumption (HL0) becomes

$$\frac{a_1}{a_2} \geq \frac{b_1}{b_2}, \tag{1.17}$$

the assumption (HL1) becomes

$$\begin{cases} a_1 + a_2 - \frac{a_2 c_1}{c_2} - \frac{a_2 b_2}{c_2} \geq 0 \\ b_1 \geq c_1 \\ b_2 \geq c_2, \end{cases} \tag{1.18}$$

and (HL2) becomes

$$\begin{cases} a_1 + a_2 - \frac{a_2 c_1}{c_2} - \frac{a_2 b_2 c_1}{b_1 c_2} \geq 0 \\ a_1 - \frac{a_2 c_1}{c_2} \geq 0. \end{cases} \tag{1.19}$$

(2) In the case that

$$a_1 = r_1, \quad b_1 = r_1, \quad c_1 = \tilde{a}_1 r_1$$

and

$$a_2 = r_2, \quad b_2 = r_2 \tilde{a}_2, \quad c_2 = r_2$$

with

$$\tilde{a}_1 < 1 \leq \tilde{a}_2,$$

(1.17) always holds, (1.18) becomes

$$\frac{\tilde{a}_2 - 1}{1 - \tilde{a}_1} \leq \frac{r_1}{r_2}, \tag{1.20}$$

and (1.19) become

$$\frac{\tilde{a}_1 \tilde{a}_2 - 1}{1 - \tilde{a}_1} \leq \frac{r_1}{r_2}. \tag{1.21}$$

By (1.21), the assumption (HL2) is the same as the condition in Theorem 2.1 in [26].

Fourth, the techniques and theories developed for (1.1) can be extended to two species competition systems with different nonlocal dispersal rates in periodic habitats. To be specific and to control the length of the paper, we restrict the study to the case with same dispersal rates in this paper.

Finally, the methods developed in this paper can also be applied to two species competition systems with random dispersal or discrete dispersal in periodic habitats (see Section 5 for more detail).

The rest of the paper is organized as follows. In Section 2, we collect some preliminary materials for the use in later sections. We investigate the existence of spreading speed intervals, the lower bounds of spreading speed intervals, and the existence of a single spreading speed in Section 3. Theorems 1.3 and 1.4 are proved in this section. Section 4 is devoted to the investigation of linear determinacy of spreading speeds and to the proof of Theorem 1.5. The paper is concluded with some remarks in Section 5 on the applications of the methods developed in this paper to two species competition systems with random or discrete dispersals in periodic habitats.

2. Preliminary Results

In this section, we collect some preliminary materials for the use in later sections, including principal spectrum point and principal eigenvalue theory for nonlocal dispersal operators with periodic coefficients; positive periodic solutions and spreading speeds of single species models in periodic habitats; and some basic properties of two species competition systems with nonlocal dispersal.

2.1. Principal spectrum points and principal eigenvalues of nonlocal dispersal operators

In this subsection, we present some principal spectrum point and principal eigenvalue theory for time periodic nonlocal dispersal operators.

Let $\mathcal{X}_p$ be as in (1.9). Consider the following eigenvalue problem,

$$-v_t + \big(\mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I\big)v = \lambda v, \quad v \in \mathcal{X}_p, \tag{2.1}$$

where $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a(\cdot,\cdot) \in \mathcal{X}_p$. The operator $a(\cdot,\cdot)I$ is as in (1.12) and $\mathcal{K}_{\xi,\mu} : \mathcal{X}_p \to \mathcal{X}_p$ is defined by

$$(\mathcal{K}_{\xi,\mu}v)(t,x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)v(t,y)dy. \tag{2.2}$$

We point out that, if $u(t,x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\phi(t,x)$ with $\phi \in \mathcal{X}_p \setminus \{0\}$ is a solution of the linear equation,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + a(t,x)u(t,x), \quad x \in \mathbb{R}^N, \tag{2.3}$$

then λ is an eigenvalue of (2.1) or $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ and $v = \phi(t,x)$ is a corresponding eigenfunction.

Let $\sigma(-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I)$ be the spectrum of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ acting on $\mathcal{X}_p$ and

$$\lambda_0(\xi,\mu,a) := \sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I)\}.$$

Observe that if $\mu = 0$, $\lambda_0(\xi,\mu,a)$ is independent of ξ and hence we put

$$\lambda_0(a) := \lambda_0(\xi,0,a) \quad \forall\, \xi \in S^{N-1}. \tag{2.4}$$

Definition 2.1. We call $\lambda_0(\xi,\mu,a)$ the principal spectrum point of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$. $\lambda_0(\xi,\mu,a)$ is called the principal eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ if $\lambda_0(\xi,\mu,a)$ is an isolated eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ with finite algebraic multiplicity and a positive eigenfunction $v \in \mathcal{X}_p^+$, and for every $\lambda \in \sigma(-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I) \setminus \{\lambda_0(\xi,\mu,a)\}$, $\mathrm{Re}\lambda \le \lambda_0(\xi,\mu,a)$.

Observe that $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ may not have a principal eigenvalue (see an example in [37]), which reveals some essential difference between random dispersal operators and nonlocal dispersal operators. Let

$$\hat{a}(x)=\frac{1}{T}\int_0^T a(t,x)dt.$$

Proposition 2.2. *If $\hat{a}(\cdot)$ is C^N and the partial derivatives of $\hat{a}(x)$ up to order $N-1$ at some x_0 are zero* (we refer this to as a vanishing condition), *where x_0 is such that $\hat{a}(x_0)=\max_{x\in\mathbb{R}^N}\hat{a}(x)$, then $\lambda_0(\xi,\mu,a)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ for all $\xi\in S^{N-1}$ and $\mu\in\mathbb{R}$.*

Proof. It follows from the arguments of [35, Theorem B (1)]. □

Proposition 2.2 provides a useful sufficient condition for $\lambda_0(\xi,\mu,a)$ to be the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$. The following proposition shows that $\lambda_0(\xi,\mu,a)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ for a in a dense subset of $\mathcal{X}_p$.

Proposition 2.3. *For any $\epsilon>0$ and $M>0$, there are $a^{\pm}(\cdot,\cdot)$ satisfying the vanishing condition in Proposition 2.2 such that*

$$a(t,x)-\epsilon\le a^-(t,x)\le a(t,x)\le a^+(t,x)\le a(t,x)+\epsilon$$

for $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$, and

$$|\lambda_0(\xi,\mu,a)-\lambda_0(\xi,\mu,a^{\pm})|<\epsilon$$

for $\xi\in S^{N-1}$ and $|\mu|\le M$.

Proof. It follows from [36, Proposition 3.7]. □

The principal spectrum point $\lambda_0(\xi,\mu,a)$ of (2.1) is closely related to the largest growth rate of the solutions of

$$u_t=\int_{\mathbb{R}^N}e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy-u(t,x)+a(t,x)u(t,x),\quad x\in\mathbb{R}^N \tag{2.5}$$

in X_p. For given $u_0\in X_p$, let $u(t,\cdot;s,u_0)$ be the solution of (2.5) with $u(s,\cdot;s,u_0)=u_0(\cdot)$. Let $\Phi(t,s;\xi,\mu,a):X_p\to X_p$ be defined by

$$\Phi(t,s;\xi,\mu,a)u_0=u(t,\cdot;s,u_0).$$

Note that $\Phi(t,s;\xi,\mu,a)$ is strongly monotone in the sense that for any $t>s$ and $u_0\in X_p^+\setminus\{0\}$, $\Phi(t,s;\xi,\mu,a)u_0\in X_p^{++}$ (see the arguments of [37, Proposiion 2.2]). We have

Proposition 2.4. *For any given $\xi\in S^{N-1}$ and $\mu\in\mathbb{R}$,*

$$\lambda_0(\xi,\mu,a)=\lim_{t-s\to\infty}\frac{\ln\|\Phi(t,s;\xi,\mu,a)\|}{t-s}.$$

Proof. It follows from [35, Propositions 3.3 and 3.10]. □

Consider the nonhomogeneous linear equation,

$$\frac{\partial u}{\partial t}=\int_{\mathbb{R}^N}e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy-u(t,x)+a(t,x)u(t,x)+h(t,x),\quad x\in\mathbb{R}^N, \tag{2.6}$$

where $h\in\mathcal{X}_p$. We have

Proposition 2.5. *If $\lambda_0(\xi,\mu,a)<0$, then for any given $h(\cdot,\cdot)\in\mathcal{X}_p$,* (2.6) *has a unique entire solution $u^{**}(\cdot,\cdot)\in\mathcal{X}_p$. Moreover, $u^{**}(\cdot,\cdot)$ is a globally stable solution of* (2.6) *with respect to perturbations in X_p, and if $h(t,x)\ge 0$ and $h(t,x)\not\equiv 0$, then $u^{**}(t,\cdot)\in X_p^{++}$.*

Proof. We first prove the uniqueness. Suppose that $u^{**}(t,x)$ and $v^{**}(t,x)$ are entire solutions of (2.6) in $\mathcal{X}_p$. Let $w(t,x) = u^{**}(t,x) - v^{**}(t,x)$. Then $w(t,x)$ is an entire solution of (2.5) in $\mathcal{X}_p$. We then have

$$w(0,\cdot) = w(nT,\cdot) = \Phi(nT;\xi,\mu,a)w(0,\cdot) \quad \text{for all } n \in \mathbb{Z}.$$

If $w(0,\cdot) \neq 0$, then by Proposition 2.4,

$$0 = \lim_{n\to\infty} \frac{\ln \|w(0,\cdot)\|}{nT} = \lim_{n\to\infty} \frac{\ln \|\Phi(nT,0;\xi,\mu,a)w(0,\cdot)\|}{nT} \leq \lambda_0(\xi,\mu,a) < 0,$$

which is a contradiction. Hence $w(0,\cdot) = 0$. This implies that $w(t,x) \equiv 0$ and then $u^{**}(t,x) \equiv v^{**}(t,x)$.

Next, we prove the existence. Let

$$u^{**}(t,\cdot) = \int_{-\infty}^{t} \Phi(t,s;\xi,\mu,a)h(s,\cdot)ds. \tag{2.7}$$

By Proposition 2.4, for given $0 < \epsilon < -\lambda_0(\xi,\mu,a)$, there is $M > 0$ such that for $t - s \geq M$,

$$\|\Phi(t,s;\xi,\mu,a)\| < e^{(\lambda_0(\xi,\mu,a)+\epsilon)(t-s)}.$$

Hence $u^{**}(t,\cdot)$ is well defined and $u^{**}(t,\cdot) \in X_p$ for all $t \in \mathbb{R}$. Moreover, it is easy to verify that $u^{**}(t,x)$ is an entire solution of (2.5). Note that

$$\begin{aligned} u^{**}(t+T,\cdot) &= \int_{-\infty}^{t+T} \Phi(t+T,s;\xi,\mu,a)h(s,\cdot)ds \\ &= \int_{-\infty}^{t} \Phi(t+T,s+T;\xi,\mu,a)h(s+T,\cdot)ds \\ &= \int_{-\infty}^{t} \Phi(t,s;\xi,\mu,a)h(s,\cdot)ds \\ &= u^{**}(t,\cdot). \end{aligned}$$

Hence $u^{**}(t,x)$ is an entire solution in $\mathcal{X}_p$.

We now prove the global stability of $u^{**}(\cdot,\cdot)$. For any given $u_0 \in X_p$, let $u(t,\cdot;u_0)$ be the solution of (2.6) with $u(0,x;u_0) = u_0(x)$. Let $w(t,x) = u^{**}(t,x) - u(t,x;u_0)$. Then $w(t,x)$ is the solution of (2.5) with $w(0,x) = u^{**}(0,x) - u_0(x)$. If $w(0,x) \not\equiv 0$, then we have

$$\limsup_{t\to\infty} \frac{\ln \|\Phi(t,0;\xi,\mu,a)w(0,\cdot)\|}{t} \leq \lambda_0(\xi,\mu,a) < 0.$$

This implies that $\|w(t,\cdot)\| \to 0$ as $t \to \infty$ exponentially. Therefore, $u^{**}(t,x)$ is globally stable.

Finally, if $h(t,x) \geq 0$ and $h(t,x) \not\equiv 0$, by the strong monotonicity of $\Phi(t,s;\xi,\mu,a)$ and (2.7), $u^{**}(t,\cdot) \in X_p^{++}$ for any $t \in \mathbb{R}$. The lemma is thus proved. □

2.2. Positive periodic solution and spreading speeds for single species equations with nonlocal dispersal

In this subsection, we present some results on positive periodic solution and spreading speed for the single species equation,

$$u_t = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + u(a(t,x) - b(t,x)u), \quad x \in \mathbb{R}^N \tag{2.8}$$

where $a, b \in \mathcal{X}_p$ and $b(t,x) > 0$. Let X be as in (1.3). By general semigroup theory (see [34]), for any $u_0 \in X$, (2.8) has a unique (local) solution $u(t,x)$ with $u(0,x) = u_0(x)$. Throughout this subsection, $u(t,x;u_0)$

denotes the solution of (2.8) with $u(0,\cdot;u_0)=u_0(\cdot)\in X$. Note that if $u_0\in X_p$, then $u(t,\cdot;u_0)\in X_p$ for t in the existence interval of $u(t,\cdot;u_0)$.

Let $\tau>0$. A continuous function $u(t,x)$ on $[0,\tau]\times\mathbb{R}^N$ with $u(t,\cdot)\in X^+$ is called a *super-solution* (*sub-solution*) of (2.8) on $[0,\tau]$ if

$$u_t\geq(\leq)\int_{\mathbb{R}^N}\kappa(y-x)u(t,y)dy-u(t,x)+u(a(t,x)-b(t,x)u),\quad t\in[0,\tau],\ \ x\in\mathbb{R}^N.$$

Proposition 2.6.*(1) If $u^1(t,x)$ and $u^2(t,x)$ are bounded sub- and super-solutions of (2.8) on $[0,\tau]$, respectively, and $u^1(0,\cdot)\leq u^2(0,\cdot)$, then $u^1(t,\cdot)\leq u^2(t,\cdot)$* for $t\in[0,\tau]$.

(2) For every $u_0\in X^+$, $u(t,x;u_0)$ exists for all $t\geq 0$.

Proof. It follows from the arguments in [37, Proposition 2.1]. □

Proposition 2.7. *Suppose that $\lambda_0(a)>0$. Then there is a unique positive periodic solution $u^*(\cdot,\cdot)\in\mathcal{X}_p^+\setminus\{0\}$ of (2.8). Moreover, for any $u_0\in X_p^+\setminus\{0\}$,*

$$u(t,x;u_0)-u^*(t,x)\to 0$$

as $t\to\infty$ uniformly in x.

Proof. It follows from [35, Theorem E]. □

Definition 2.8. For a given vector $\xi\in S^{N-1}$, a real number $c_0^*(\xi;a,b)$ is said to be the spreading speed of (2.8) in the direction of ξ if for any $u_0\in X^+$ satisfying that

$$\liminf_{x\cdot\xi\to-\infty}u_0(x)>0\ \text{ and }\ u_0(x)=0\quad\text{for all } x\in\mathbb{R}^N \text{ such that } x\cdot\xi\gg 1,$$

there holds

$$\limsup_{t\to\infty}\sup_{x\cdot\xi\leq ct}|u(t,x;u_0)-u^*(t,x)|=0\quad\forall\ c<c_0^*(\xi;a,b)$$

and

$$\limsup_{t\to\infty}\sup_{x\cdot\xi\geq ct}u(t,x;u_0)=0\quad\forall\ c>c_0^*(\xi;a,b).$$

Proposition 2.9 (Existence of spreading speeds)**.** *Assume $\lambda_0(a)>0$. For any given $\xi\in S^{N-1}$, the spreading speed $c_0^*(\xi;a,b)$ of (2.8) in the direction of ξ exists. Moreover,*

$$c_0^*(\xi;a,b)=\inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a)}{\mu}.$$

Proof. It follows from [36, Theorem 4.1]. □

Remark 2.10. Observe that $c_0^*(\xi;a,b)$ is independent of b and we may then put

$$c_0^*(\xi,a)=c_0^*(\xi;a,b).\tag{2.9}$$

Suppose that $a,\tilde{a}\in X_p$ satisfy $\lambda_0(a)>0$, $\lambda_0(\tilde{a})>0$, $\tilde{a}(t,x)\geq a(t,x)$ for all $t\in\mathbb{R}$, $x\in\mathbb{R}^N$, and $\tilde{a}(t,x)\not\equiv a(t,x)$. Then

$$c_0^*(\xi,\tilde{a})>c_0^*(\xi,a).$$

2.3. Basic properties of two species competition systems with nonlocal dispersal

In this subsection, we present some basic properties for two species competition systems with nonlocal dispersal.

Consider (1.1). Let X be as in (1.3). By general semigroup theory (see [34]), for any $(u_0, v_0) \in X \times X$, there is a unique (local) solution $(u(t,x), v(t,x))$ of (1.1) with $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$. Throughout this subsection, $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ denotes the solution of (1.1) with $(u(0,\cdot;u_0,v_0), v(0,\cdot;u_0,v_0)) = (u_0(\cdot), v_0(\cdot)) \in X \times X$, unless otherwise specified.

Proposition 2.11 (Semitrivial solutions).

(1) If $\lambda_0(a_1) > 0$, then (1.1) *has a semi-trivial periodic solution $(u^*(t,x), 0)$ and for any $u_0 \in X_p^+ \setminus \{0\}$, $(u(t,x;u_0,0), v(t,x;u_0,0)) - (u^*(t,x), 0) \to 0$ as $t \to \infty$ uniformly in x. If, in addition, $\lambda_0(a_2 - b_2 u^*) < 0$, then $(u^*(t,x), 0)$ is locally stable with respect to the perturbation in $X_p \times X_p$, and if $\lambda_0(a_2 - b_2 u^*) > 0$, then $(u^*(t,x), 0)$ is unstable with respect to perturbation in $X_p \times X_p$.*

(2) If $\lambda_0(a_2) > 0$, then (1.1) *has a semi-trivial periodic solution $(0, v^*(t,x))$ and for any $v_0 \in X_p^+ \setminus \{0\}$, $(u(t,x;0,v_0), v(t,x;0,v_0)) - (0, v^*(t,x)) \to 0$ as $t \to \infty$ uniformly in x. If, in addition, $\lambda_0(a_1 - c_1 v^*) < 0$, then $(0, v^*(t,x))$ is locally stable with respect to perturbation in $X_p \times X_p$, and if $\lambda_0(a_1 - c_1 v^*) > 0$, then $(0, v^*(t,x))$ is unstable with respect to perturbation in $X_p \times X_p$.*

(3) If $\lambda_0(a_i) > 0$ for $i = 1,2$ and $a_{1L} > \frac{c_{1M} a_{2M}}{c_{2L}}$, $a_{2M} \le \frac{a_{1L} b_{2L}}{b_{1M}}$, then $(u^*(t,x), 0)$ is globally stable and $(0, v^*(t,x))$ is unstable with respect to perturbations in $X_p^+ \times X_p^+$, where $a_{kL} = \inf_{t\in\mathbb{R}, x\in\mathbb{R}^N} a_k(t,x)$, $a_{kM} = \sup_{t\in\mathbb{R}, x\in\mathbb{R}^N} a_k(t,x)$, and b_{kL}, b_{kM}, c_{kL}, c_{kM} $(k = 1,2)$ are defined similarly.

Proof. (1) Assume $\lambda_0(a_1) > 0$. Then by Proposition 2.7, (1.1) has a semi-trivial periodic solution $(u^*(t,x), 0)$ and for any $u_0 \in X_p^+ \setminus \{0\}$, $(u(t,x;u_0,0), v(t,x;u_0,0)) - (u^*(t,x), 0) \to 0$ as $t \to \infty$ uniformly in x. Moreover,

$$\lambda_0(a_1 - 2b_1 u^*) < 0.$$

Consider the linearization of (1.1) at $(u^*(\cdot,\cdot), 0)$,

$$\begin{cases} u_t = \int_{\mathbb{R}^N} \kappa(y-x) u(t,y) dy - u(t,x) + (a_1(t,x) - 2b_1(t,x) u^*(t,x)) u(t,x) \\ \qquad\qquad -c_1(t,x) u^*(t,x) v(t,x), \qquad x \in \mathbb{R}^N \\ v_t = \int_{\mathbb{R}^N} \kappa(y-x) v(t,y) dy - v(t,x) + (a_2(t,x) - b_2(t,x) u^*(t,x)) v(t,x), \quad x \in \mathbb{R}^N. \end{cases} \tag{2.10}$$

Let P be the Poincaré map or time T map of (2.10) on $X_p \times X_p$. Observe that

$$r(P) = e^{T \max\{\lambda_0(a_1 - 2b_1 u^*), \lambda_0(a_2 - b_2 u^*)\}},$$

where $r(P)$ is the spectral radius of P. Since $\lambda_0(a_1 - 2b_1 u^*) < 0$, if $\lambda_0(a_2 - b_2 u^*) < 0$, then $r(P) < 1$ and hence $(u^*, 0)$ is locally stable with respect to perturbations in $X_p \times X_p$. If $\lambda_0(a_2 - b_2 u^*) > 0$, then $r(P) > 1$ and hence $(u^*, 0)$ is unstable with respect to perturbations in $X_p \times X_p$.

(2) It can be proved by the same arguments as in (1).

(3) Consider

$$\begin{cases} u_t = \int_{\mathbb{R}^N} \kappa(y-x) u(t,y) dy - u(t,x) + u(a_{1L} - b_{1M} u - c_{1M} v), \qquad x \in \mathbb{R}^N \\ v_t = \int_{\mathbb{R}^N} \kappa(y-x) v(t,y) dy - v(t,x) + v(a_{2M} - b_{2L} u - c_{2L} v), \qquad x \in \mathbb{R}^N. \end{cases} \tag{2.11}$$

Then $(u_-^*, 0) = (\frac{a_{1L}}{b_{1M}}, 0)$ and $(0, v_-^*) = (0, \frac{a_{2M}}{c_{2L}})$ are two semitrivial solutions of (2.11). By $\lambda_0(a_2) > 0$, $a_{2M} > 0$. Then by $a_{1L} > \frac{c_{1M} a_{2M}}{c_{2L}}$ and $a_{2M} \le \frac{a_{1L} b_{2L}}{b_{1M}}$, $(0, v_-^*) \in X_p^+ \times X_p^+$ and is unstable with respect to the perturbations in $X_p \times X_p$ and $(u_-^*, 0) \in X_p^+ \times X_p^+$ and is globally stable with respect to the perturbations in $(X_p^+ \setminus \{0\}) \times X_p^+$. For given $u_0 \in X_p^+ \setminus \{0\}$ and $v_0 \in X_p^+$, let $(u_-(t,x;u_0,v_0), v_-(t,x;u_0,v_0))$ be the solution of (2.11) with $(u_-(0,x;u_0,v_0),\ v_-(0,x;u_0,v_0)) = (u_0(x), v_0(x))$. By comparison principle for two species competition systems with nonlocal dispersal (see [17, Proposition 3.1]),

$$u(t,x;u_0,v_0) \ge u_-(t,x;u_0,v_0), \quad v(t,x;u_0,v_0) \le v_-(t,x;u_0,v_0).$$

It then follows that

$$\lim_{t\to\infty} v(t,x;u_0,v_0) = 0$$

uniformly in $x \in \mathbb{R}^N$ and then by Proposition 2.7,

$$\lim_{t\to\infty} [u(t,x;u_0,u_0) - u^*(t,x)] = 0$$

uniformly in $x \in \mathbb{R}^N$. (3) thus follows. □

3. Spreading Speeds and the Proofs of Theorems 1.3 and 1.3

In this section, we investigate the spreading speeds of the cooperative system (1.14) and prove Theorems 1.3 and 1.4.

To do so, we first prove some lemmas. In the rest of this section, we always assume that $\xi \in S^{N-1}$ is given and fixed. Let

$$\begin{cases} f(t,x,u,v) = u\Big(a_1(t,x) - b_1(t,x)u - c_1(t,x)(v^*(t,x) - v)\Big) \\ g(t,x,u,v) = b_2(t,x)\Big(v^*(t,x) - v\Big)u + v\Big(a_2(t,x) - 2c_2(t,x)v^*(t,x) + c_2(t,x)v\Big). \end{cases}$$

Let $(u(t,x;u_0,v_0,z), v(t,x;u_0,v_0,z))$ be the solution of

$$\begin{cases} u_t = \mathcal{K}u - u + f(t,x+z,u,v) \\ v_t = \mathcal{K}v - v + g(t,x+z,u,v) \end{cases} \tag{3.1}$$

with $(u(0,\cdot;u_0,v_0,z), v(0,\cdot;u_0,v_0,z)) = (u_0(\cdot),v_0(\cdot)) \in X \times X$. Note that $(u(t,x;u_0,v_0,0), v(t,x;u_0,v_0,0)) = (u(t,x;u_0,v_0), v(t,x;u_0,v_0))$.

Lemma 3.1. *For given (u_1,v_1), $(u_2,v_2) \in X^+ \times X^+$, if $0 \le u_1 \le u_2 \le u^*(0,\cdot)$ and $0 \le v_1 \le v_2 \le v^*(0,\cdot)$, then*

$$0 \le u(t,\cdot;u_1,v_1) \le u(t,\cdot;u_2,v_2) \le u^*(t,x), \quad 0 \le v(t,\cdot;u_1,v_1) \le v(t,\cdot;u_2,v_2) \le v^*(t,x).$$

Proof. Observe that $(0,0)$ and $(u^*(t,x), v^*(t,x))$ are solutions of (1.14), and $f_v(t,x,u,v) \ge 0$ for all $u \ge 0$ and $g_u(t,x,u,v) \ge 0$ for $u \ge 0$ and $0 \le v \le v^*(t,x)$. The lemma then follows from comparison principle for cooperative systems. □

Lemma 3.2. *Let $(u_n,v_n) \in X^+ \times X^+$ $(n = 1,2,\cdots)$ and $(u_0,v_0) \in X^+ \times X^+$ be given. Assume that $u_n(\cdot) \le u^*(0,\cdot)$ and $v_n(\cdot) \le v^*(0,\cdot)$ for $n = 1,2,\cdots$.*

(1) If $(u_n,v_n) \to (u_0,v_0)$ in compact open topology, then

$$(u(t,x;u_n,v_n,z), v(t,x;u_n,v_n,z)) \to (u(t,x;u_0,v_0,z), v(t,x;u_0,v_0,z))$$

as $n \to \infty$ uniformly in $z \in \mathbb{R}^N$ and (t,x) in bounded subsets of $\mathbb{R}^+ \times \mathbb{R}^N$.

(2) For given $\xi \in S^{N-1}$, if $(u_n,v_n) \to (u_0,v_0)$ uniformly in bounded strips of the form $E_K = \{x \in \mathbb{R}^N \,|\, |x\cdot\xi| \le K\}$ $(K \ge 0)$, then

$$(u(t,x;u_n,v_n,z), v(t,x;u_n,v_n,z)) \to (u(t,x;u_0,v_0,z), v(t,x;u_0,v_0,z))$$

as $n \to \infty$ uniformly in $z \in \mathbb{R}^N$, t in bounded subsets of $\mathbb{R}^+$, and x in bounded strips of the form $E_K = \{x \in \mathbb{R}^N \,|\, |x\cdot\xi| \le K\}$ $(K \ge 0)$.

Proof. (1) Let

$$\begin{cases} u_n(t,x;z) = u(t,x;u_n,v_n,z) - u(t,x;u_0,v_0,z) \\ v_n(t,x;z) = v(t,x;u_n,v_n,z) - v(t,x;u_0,v_0,z). \end{cases}$$

Then

$$\begin{cases} \partial_t u_n(t,x;z) = \mathcal{K}u_n - u_n + a_n^1(t,x;z)u_n + b_n^1(t,x;z)v_n \\ \partial_t v_n(t,x;z) = \mathcal{K}v_n - v_n + a_n^2(t,x;z)u_n + b_n^2(t,x;z)v_n, \end{cases}$$

where

$$a_n^1(t,x;z) = f_u(t,x+z,u_n^1(t,x;z),v_n^1(t,x;z)),$$

$$b_n^1(t,x;z) = f_v(t,x+z,u_n^1(t,x;z),v_n^1(t,x;z)),$$

$$a_n^2(t,x;z) = g_u(t,x+z,u_n^2(t,x;z),v_n^2(t,x;z)),$$

and

$$b_n^2(t,x;z) = g_v(t,x+z,u_n^2(t,x;z),v_n^2(t,x;z))$$

for some $u_n^1(t,x;z)$, $u_n^2(t,x;z)$ between $u(t,x;u_n,v_n,z)$ and $u(t,x;u_0,v_0,z)$, and some $v_n^1(t,x;z)$, $v_n^2(t,x;z)$ between $v(t,x;u_n,v_n,z)$ and $v(t,x;u_0,v_0,z)$.

For given $\rho > 0$, let

$$Y_\rho = \{(u,v) \in C(\mathbb{R}^n,\mathbb{R}^2) \,|\, (e^{-\rho|\cdot|}u(\cdot), e^{-\rho|\cdot|}v(\cdot)) \in X \times X\}$$

equipped with the norm $\|(u,v)\|_{Y_\rho} = \sup_{x\in\mathbb{R}^N} e^{-\rho|x|}(|u(x)| + |v(x)|)$. Observe that

$$(\mathcal{K}-I,\mathcal{K}-I) : Y_\rho \to Y_\rho,$$

$$(\mathcal{K}-I,\mathcal{K}-I)(u,v) = (\mathcal{K}u - u, \mathcal{K}v - v),$$

is a bounded linear operator, and $a_n^k(t,x;z)$ and $b_n^k(t,x;z)$ are bounded on $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$ $(k=1,2)$. Hence there are $M > 0$ and $\omega > 0$ such that

$$\|e^{(\mathcal{K}-I,\mathcal{K}-I)t}\|_{Y_\rho} \le Me^{\omega t}$$

and

$$|a_n^k(t,x;z)| \le M, \quad |b_n^k(t,x;z)| \le M.$$

Note that $(u_n(0,\cdot;z), v_n(0,\cdot;z)) \in Y_\rho$ and

$$\begin{aligned} (u_n(t,\cdot;z), v_n(t,\cdot;z)) &= e^{(\mathcal{K}-I,\mathcal{K}-I)t}(u_n(0,\cdot;z), v_n(0,\cdot;z)) \\ &\quad + \int_0^t e^{(\mathcal{K}-I,\mathcal{K}-I)(t-s)}\Big[a_n^1(s,\cdot;z)u_n(s,\cdot;z) + b_n^1(s,\cdot;z)v_n(s,\cdot;z), \\ &\qquad a_n^2(s,\cdot;z)u_n(s,\cdot;z) + b_n^2(s,\cdot;z)v_n(s,\cdot;z)\Big]ds. \end{aligned}$$

Hence

$$\begin{aligned} \|(u_n(t,\cdot;z), v_n(t,\cdot;z))\|_{Y_\rho} &\le Me^{\omega t}\|(u_n(0,\cdot;z), v_n(0,\cdot;z))\|_{Y_\rho} \\ &\quad + M^2\int_0^t e^{\omega(t-s)}\|(u_n(s,\cdot;z), v_n(s,\cdot;z))\|_{Y_\rho}ds. \end{aligned}$$

By Gronwall's inequality, we have

$$\|(u_n(t,\cdot;z), v_n(t,\cdot;z))\|_{Y_\rho} \le e^{(\omega+M^2)t}M\|(u_n(0,\cdot;z), v_n(0,\cdot;z))|_{Y_\rho}.$$

Note that

$$\|(u_n(0,\cdot;z),v_n(0,\cdot;z))\|_{Y_\rho}\to 0$$

as $n\to\infty$ uniformly in $z\in\mathbb{R}^N$. It then follows that

$$(u_n(t,x;z),v_n(t,x;z))\to(0,0)$$

as $n\to\infty$ uniformly in $z\in\mathbb{R}^N$ and (t,x) in bounded sets of $\mathbb{R}^+\times\mathbb{R}^N$.

(2) It can be proved by the similar arguments in (1) with Y_ρ being replaced by $Y_{\rho,\xi}$,

$$Y_{\rho,\xi}=\{(u,v)\in C(\mathbb{R}^N,\mathbb{R}^2)\,|\,(e^{-\rho|\cdot\cdot\xi|}u(\cdot),e^{-\rho|\cdot\cdot\xi|}v(\cdot))\in X\times X\},$$

equipped with the norm $\|(u,v)\|_{Y_{\rho,\xi}}=\sup_{x\in\mathbb{R}^N}e^{-\rho|x\cdot\xi|}(|u(x)|+|v(x)|)$. □

Lemma 3.3. *For given $u_0,v_0\in X^+$, if $\inf_{x\in\mathbb{R}^N}u_0(x)>0$, $u_0(\cdot)\le u^*(0,\cdot)$, and $v_0(\cdot)\le v^*(0,\cdot)$, then*

$$\lim_{t\to\infty}[|u(t,x;u_0,v_0)-u^*(t,x)|+|v(t,x;u_0,v_0)-v^*(t,x)|]=0$$

uniformly in $x\in\mathbb{R}^N$.

Proof. Assume that $\inf_{x\in\mathbb{R}^N}u_0(x)>0$. Let $u_0^{\inf}=\inf_{x\in\mathbb{R}^N}u_0(x)$. By Lemma 3.1,

$$u^*(t,x)\ge u(t,x;u_0,v_0)\ge u(t,x;u_0^{\inf},0)$$

and

$$v^*(t,x)\ge v(t,x;u_0,v_0)\ge v(t,x;u_0^{\inf},0).$$

Note that $(u_0^{\inf},0)\in(X_p^+\setminus\{0\})\times X_p^+$. By (HB2), we have

$$\lim_{t\to\infty}[|u(t,x;u_0^{\inf},0)-u^*(t,x)|+|v(t,x;u_0^{\inf},0)-v^*(t,x)|]=0$$

uniformly in $x\in\mathbb{R}^N$ and then

$$\lim_{t\to\infty}[|u(t,x;u_0,v_0)-u^*(t,x)|+|v(t,x;u_0,v_0)-v^*(t,x)|]=0$$

uniformly in $x\in\mathbb{R}^N$. □

Lemma 3.4. *For given $c\in\mathbb{R}$ and $(u_0,v_0)\in X_1^+(\xi)\times X_2^+(\xi)$, if $\liminf_{x\cdot\xi\le ct,t\to\infty}u(t,x;u_0,v_0)>0$, then for any $c^{'}<c$,*

$$\limsup_{x\cdot\xi\le c^{'}t,t\to\infty}[|u(t,x;u_0,v_0)-u^*(t,x)|+|v(t,x;u_0,v_0)-v^*(t,x)|]=0.$$

Proof. First of all, by Lemma 3.1,

$$0\le u(t,x;u_0,v_0)\le u^*(t,x),\quad 0\le v(t,x;u_0,v_0)\le v^*(t,x)\tag{3.2}$$

for all $t\ge0$ and $x\in\mathbb{R}^N$. For given $\delta>0$, let

$$u_\delta(x)=u^*(0,x)-\delta,\quad v_\delta(x)=v^*(0,x)-\delta.$$

Observe that, for any $\epsilon>0$, there is $0<\delta\le\epsilon$ such that

$$\begin{cases}|u(t,x;u_\delta(\cdot+z),v_\delta(\cdot+z),z)-u^*(t,x+z)|<\epsilon\\|u(t,x;u_\delta(\cdot+z),v_\delta(\cdot+z),z)-u^*(t,x+z)|<\epsilon\end{cases}\tag{3.3}$$

for all $t \in [0,T]$ and $x, z \in \mathbb{R}^N$.

Let

$$\sigma = \min\{\liminf_{x\cdot\xi \le ct, t\to\infty} u(t,x;u_0,v_0), \min_{t\in\mathbb{R}, x\in\mathbb{R}^N} u^*(t,x), \min_{t\in\mathbb{R}, x\in\mathbb{R}^N} v^*(t,x)\}.$$

Then there is $N_1 > 0$ such that for $t \ge N_1 T$ and $x \in \mathbb{R}^N$ with $x\cdot\xi \le ct$,

$$u(t,x;u_0,v_0) \ge \sigma/2.$$

Let

$$\tilde{u}_0(x) \equiv \sigma/2, \quad \tilde{v}_0(x) \equiv 0.$$

By Lemma 3.3,

$$(u(t,x;\tilde{u}_0,\tilde{v}_0), v(t,x;\tilde{u}_0,\tilde{v}_0)) - (u^*(t,x), v^*(t,x)) \to (0,0) \tag{3.4}$$

as $t \to \infty$ uniformly in $x \in \mathbb{R}^N$.

For any $c^{'} < c$, choose $\tilde{c}$ such that $c^{'} < \tilde{c} < c$. Let $(\tilde{u}_n, \tilde{v}_n) \in X^+ \times X^+$ be such that

$$\tilde{u}_n(x) \begin{cases} \le \sigma/2 & \forall\, x \in \mathbb{R}^N \\ = \sigma/2 & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \le (c-\tilde{c})nT - 1 \\ = 0 & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \ge (c-\tilde{c})nT, \end{cases}$$

and

$$\tilde{v}_n(x) \equiv 0.$$

Then

$$u(nT, x+y; u_0, v_0) \ge \tilde{u}_n(x), \quad v(nT, x+y; u_0, v_0) \ge \tilde{v}_n(x) \tag{3.5}$$

for $y\cdot\xi \le \tilde{c}nT$ and $n \gg 1$, and

$$(\tilde{u}_n(x), \tilde{v}_n(x)) \to (\tilde{u}_0(x), \tilde{v}_0(x)) \tag{3.6}$$

as $n \to \infty$ uniformly in x in bounded strips of the form $\{x \,|\, |x\cdot\xi| \le K\}$. By (3.4), (3.6), and Lemma 3.2, there is $N_2 \ge N_1$ such that

$$u(N_2T, x; \tilde{u}_n, \tilde{v}_n, z) \ge u^*(0, x+z) - \delta \ge \sigma/2$$

and

$$v(N_2T, x; \tilde{u}_n, \tilde{v}_n, z) \ge v^*(0, x+z) - \delta \ge \sigma/2$$

for $|x\cdot\xi| \le \tilde{c}N_2T$, $z \in \mathbb{R}^N$, and $n \gg 1$. Then by (3.5), for any $y \in \mathbb{R}^N$ with $y\cdot\xi \le \tilde{c}nT$ and $n \gg 1$,

$$\begin{aligned} u((n+N_2)T, x+y; u_0, v_0) &= u(N_2T, x; u(nT, \cdot+y; u_0, v_0), v(nT, \cdot+y; u_0, v_0), y) \\ &\ge u(N_2T, x; \tilde{u}_n, \tilde{v}_n, y) \\ &\ge u^*(0, x+y) - \delta \end{aligned}$$

and

$$\begin{aligned} v((n+N_2)T, x+y; u_0, v_0) &= v(N_2T, x; u(nT, \cdot+y; u_0, v_0), v(nT, \cdot+y; u_0, v_0), y) \\ &\ge v(N_2T, x; \tilde{u}_n, \tilde{v}_n, y) \\ &\ge v^*(0, x+y) - \delta \end{aligned}$$

for $|x\cdot\xi| \le \tilde{c}N_2T$ and $n \gg 1$. Hence

$$u(nT, x; u_0, v_0) \ge u^*(0,x) - \delta, \quad v(nT, x; u_0, v_0) \ge v^*(0,x) - \delta \quad \forall\, x\cdot\xi \le \tilde{c}nT,\ n \gg 1. \tag{3.7}$$

Let $(\bar{u}_n, \bar{v}_n) \in X^+ \times X^+$ be such that

$$\bar{u}_n(x) \begin{cases} \le u^*(0,x) - \delta & \forall\, x \in \mathbb{R}^N \\ = u^*(0,x) - \delta & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \le (\tilde{c} - c^{'})nT - 1 \\ = 0 & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \ge (\tilde{c} - c^{'})nT, \end{cases}$$

and

$$\bar{v}_n(x) \begin{cases} \le v^*(0,x) - \delta & \forall\, x \in \mathbb{R}^N \\ = v^*(0,x) - \delta & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \le (\tilde{c} - c^{'})nT - 1 \\ = 0 & \forall\, x \in \mathbb{R}^N,\ x\cdot\xi \ge (\tilde{c} - c^{'})nT. \end{cases}$$

Then

$$u(nT, x+y; u_0, v_0) \ge \bar{u}_n(x), \quad v(nT, x+y; u_0, v_0) \ge \bar{v}_n(x)$$

for $y\cdot\xi \le c^{'}nT$ and $n \gg 1$, and

$$(\bar{u}_n(x), \bar{v}_n(x)) \to (u_\delta(x), v_\delta(x))$$

as $n \to \infty$ uniformly in bounded strips of the form $|x\cdot\xi| \le K$. By Lemma 3.2 again,

$$u(t,x;\bar{u}_n,\bar{v}_n,z) \ge u(t,x;u_\delta,v_\delta,z) - \epsilon, \quad v(t,x;\bar{u}_n,\bar{v}_n,z) \ge v(t,x;u_\delta,v_\delta,z) - \epsilon \tag{3.8}$$

for $|x\cdot\xi| \le c^{'}T$, $z \in \mathbb{R}^N$, $t \in [0,T]$, and $n \gg 1$. Note that

$$\begin{aligned} u(t+nT, x+y; u_0, v_0) &= u(t,x; u(nT, \cdot + y; u_0, v_0), v(nT, \cdot + y; u_0, v_0), y) \\ &\ge u(t,x;\bar{u}_n,\bar{v}_n,y) \end{aligned}$$

and

$$\begin{aligned} v(t+nT, x+y; u_0, v_0) &= v(t,x; u(nT, \cdot + y; u_0, v_0), v(nT, \cdot + y; u_0, v_0), y) \\ &\ge v(t,x;\bar{u}_n,\bar{v}_n,y) \end{aligned}$$

for $t \in [0,T]$, $|x\cdot\xi| \le c^{'}T$, $y\cdot\xi \le c^{'}nT$, and $n \gg 1$. This together with (3.8) implies that

$$\begin{cases} u(t+nT, x+y; u_0, v_0) \ge u(t,x;u_\delta,v_\delta,y) - \epsilon \\ v(t+nT, x+y; u_0, v_0) \ge v(t,x;u_\delta,v_\delta,y) - \epsilon \end{cases} \tag{3.9}$$

for $t \in [0,T]$, $|x\cdot\xi| \le c^{'}T$, $y\cdot\xi \le c^{'}nT$, and $n \gg 1$. By (3.3) and (3.9), we have that for $n \gg 1$,

$$\begin{cases} u(t+nT, x; u_0, v_0) \ge u^*(t+nT, x) - 2\epsilon, & t \in [0,T],\ x\cdot\xi \le c^{'}(t+nT) \\ v(t+nT, x; u_0, v_0) \ge v^*(t+nT, x) - 2\epsilon, & t \in [0,T],\ x\cdot\xi \le c^{'}(t+nT). \end{cases} \tag{3.10}$$

By (3.2), (3.7), and (3.10), for any $\epsilon > 0$, there is $N > 0$ such that for $t \ge NT$ and $x\cdot\xi \le c^{'}t$,

$$-2\epsilon \le u(t,x;u_0,v_0) - u^*(t,x) \le 0, \quad -2\epsilon \le v(t,x;u_0,v_0) - v^*(t,x) \le 0.$$

The lemma thus follows. □

Let $\eta(s)$ be the function defined by

$$\eta(s) = \frac{1}{2}(1 + \tanh\frac{s}{2}), \quad s \in \mathbb{R}. \tag{3.11}$$

Observe that

$$\eta^{'}(s) = \eta(s)(1 - \eta(s)), \quad s \in \mathbb{R} \tag{3.12}$$

and

$$\eta^{''}(s) = \eta(s)(1 - \eta(s))(1 - 2\eta(s)), \quad s \in \mathbb{R}. \tag{3.13}$$

Lemma 3.5. *There is $C_0 > 0$ such that for every $C \ge C_0$ and every $\xi \in S^{N-1}$, $(u^+(t,x;C),\ v^+(t,x;C))$ is a super-solution of (1.14) on $[0,\infty)$, where*

$$u^+(t,x;C) = u^*(t,x)(1-\eta(x\cdot\xi - Ct)),$$

and

$$v^+(t,x;C) = v^*(t,x)(1-\eta(x\cdot\xi - Ct)).$$

Proof. Recall that

$$f(t,x,u,v) = u\Big(a_1(t,x) - b_1(t,x)u - c_1(t,x)(v^*(t,x)-v)\Big).$$

By a direct calculation, we have

$$\begin{aligned}
&f(t,x,u^*(t,x),v^*(t,x))(1-\eta(x\cdot\xi-Ct))\\
&\quad - f\big(t,x,u^*(t,x)(1-\eta(x\cdot\xi-Ct)),v^*(t,x)(1-\eta(x\cdot\xi-Ct))\big)\\
&= \big(1-\eta(x\cdot\xi-CT)\big)u^*(t,x)\Big\{\big[a_1(t,x)-b_1(t,x)u^*(t,x)-c_1(t,x)(v^*(t,x)-v^*(t,x))\big]\\
&\quad - \big[a_1(t,x)-b_1(t,x)u^*(t,x)(1-\eta(x\cdot\xi-Ct))-c_1(t,x)v^*(t,x)\eta(x\cdot\xi-Ct)\big]\Big\}\\
&= \big(1-\eta(x\cdot\xi-Ct)\big)u^*(t,x)\eta(x\cdot\xi-Ct)\big(c_1(t,x)v^*(t,x)-b_1(t,x)u^*(t,x)\big)\\
&= \eta^{'}(x\cdot\xi-Ct)u^*(t,x)\big(c_1(t,x)v^*(t,x)-b_1(t,x)u^*(t,x)\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&u_t^+(t,x;C) - \Big[\int_{\mathbb{R}^N}\kappa(y-x)u^+(t,y;C)dy - u^+(t,x;C) + f(t,x,u^+(t,x;C),v^+(t,x;C))\Big]\\
&= \eta^{'}(x\cdot\xi-Ct)\Big\{Cu^*(t,x) + \int_{\mathbb{R}^N}\kappa(y-x)u^*(t,y)\frac{\eta(y\cdot\xi-Ct)-\eta(x\cdot\xi-Ct)}{\eta^{'}(x\cdot\xi-Ct)}dy\\
&\quad + u^*(t,x)\big(c_1(t,x)v^*(t,x)-b_1(t,x)u^*(t,x)\big)\Big\}.
\end{aligned}$$

Observe that there are M_0 and $M_1 > 0$ such that

$$u^*(t,x),\quad v^*(t,x) \ge M_0 \quad \text{for all} \quad t\ge 0, \quad x\in\mathbb{R}^N,$$

$$\Big|\frac{\eta(y\cdot\xi-Ct)-\eta(x\cdot\xi-Ct)}{\eta^{'}(x\cdot\xi-Ct)}\Big| \le M_1 \quad \text{for all} \quad t\ge 0, \quad x,y\in\mathbb{R}^N, \quad \|y-x\|\le r_0.$$

Therefore there is $C_1 > 0$ such that for every $C \ge C_1$,

$$u_t^+(t,x;C) - \big[\int_{\mathbb{R}^N}\kappa(y-x)u^+(t,y;C)dy - u^+(t,x;C) + f(t,x,u^+(t,x;C),v^+(t,x;C))\big] \ge 0$$

for $t \ge 0$ and $x \in \mathbb{R}^N$.

Similarly, we can prove that there is $C_2 > 0$ such that for every $C \ge C_2$,

$$v_t^+(t,x;C) - \big[\int_{\mathbb{R}^N}\kappa(y-x)v^+(t,y;C)dy - v^+(t,x;C) + g(t,x,u^+(t,x;C),v^+(t,x;C))\big] \ge 0$$

for $t \ge 0$ and $x \in \mathbb{R}^N$. The lemma then follows with $C_0 = \max\{C_1, C_2\}$. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. First, consider

$$\bar{u}_t = \mathcal{K}\bar{u} - \bar{u} + \bar{u}(a_1(t,x) - c_1(t,x)v^*(t,x) - b_1(t,x)\bar{u}), \quad x \in \mathbb{R}^N. \tag{3.14}$$

For given $u_0 \in X^+$, let $\bar{u}(t,x;u_0)$ be the solution of (3.14) with $u(0,x;u_0) = u_0(x)$. Let

$$\bar{c}^*_{\inf}(\xi) = \inf_{\mu>0} \frac{\lambda_\xi(\mu)}{\mu}.$$

By Proposition 2.6 and Lemma 3.1, for any $(u_0,v_0) \in X_1^+(\xi) \times X_2^+(\xi)$,

$$u(t,x;u_0,v_0) \geq \bar{u}(t,x;u_0).$$

By Proposition 2.9, for any $c < \bar{c}^*_{\inf}(\xi)$,

$$\liminf_{x\cdot\xi \leq ct, t\to\infty} \bar{u}(t,x;u_0) > 0$$

and hence

$$\liminf_{x\cdot\xi \leq ct, t\to\infty} u(t,x;u_0,v_0) > 0.$$

Then by Lemma 3.4, for any $c < \bar{c}^*_{\inf}(\xi)$,

$$\limsup_{x\cdot\xi \leq ct, t\to\infty} \left[|u(t,x;u_0,v_0) - u^*(t,x)| + |v(t,x;u_0,v_0) - v^*(t,x)|\right] = 0$$

and hence

$$c^*_{\inf}(\xi) \geq \bar{c}^*_{\inf}(\xi). \tag{3.15}$$

Next, for any given $(u_0,v_0) \in X_1^+(\xi) \times X_2^+(\xi)$, there is $N > 0$ such that

$$u_0(x) \leq u^+(NT,x;C_0), \quad v_0(x) \leq v^+(NT,x;C_0) \quad \forall\, x \in \mathbb{R}^N,$$

where $u^+(t,x;C_0)$ and $v^+(t,x;C_0)$ are as in Lemma 3.5. Then by comparison principle for cooperative systems,

$$u(t,x;u_0,v_0) \leq u^+(t+NT,x;C_0), \quad v(t,x;u_0,v_0) \leq v^+(t+NT,x;C_0) \quad \forall\, t \geq 0,\ x \in \mathbb{R}^N.$$

This implies that for any $c > C_0$,

$$\limsup_{x\cdot\xi \geq ct, t\to\infty} [u(t,x;u_0,v_0) + v(t,x;u_0,v_0)] = 0$$

and hence

$$c^*_{\sup}(\xi) \leq C_0. \tag{3.16}$$

Therefore $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a bounded interval.
Finally, (1.16) follows from (3.15). □

Before proving Theorem 1.4, we make the following remark.

Remark 3.6.(1) In the spatially homogeneous case, that is, $a_k(t,x) \equiv a_k(t)$, $b_k(t,x) \equiv b_k(t)$, and $c_k(t,x) \equiv c_k(t)$ $(k=1,2)$, as in [12] and [43], we can introduce two spreading speeds, $c^*_+(\xi)$ and $\bar{c}_+(\xi)$, for the time T map or Poincaré map Q of (1.14), as follows. Note that in such case, $u^*(t,x) \equiv u^*(t)$, $v^*(t,x) \equiv v^*(t)$, and $Q(u^*(0),v^*(0)) = (u^*(0),v^*(0))$. Let $\tilde{\phi}_0(\cdot)$ and $\tilde{\psi}_0(\cdot) : \mathbb{R} \to \mathbb{R}^+$ be continuous non-increasing functions with

$$\tilde{\phi}_0(s) = 0, \quad \tilde{\psi}_0(s) = 0 \quad \forall s \geq 0$$

and

$$\tilde{\phi}_0(-\infty) = u^*(0), \quad \tilde{\psi}_0(-\infty) = v^*(0).$$

For any $c \in \mathbb{R}$, let

$$\phi_0(c,x) = \tilde{\phi}_0(x \cdot \xi), \quad \psi_0(c,x) = \tilde{\psi}(x \cdot \xi),$$

$$\phi_n(c,x) = \max\{\phi_0(x), u(T, x + c\xi; \phi_{n-1}(c,\cdot), \psi_{n-1}(c,\cdot))\},$$

$$\psi_n(c,x) = \max\{\psi_0(x), v(T, x + c\xi; \phi_{n-1}(c,\cdot), \psi_{n-1}(c,\cdot))\},$$

and

$$\tilde{\phi}_n(c,s) = \phi_n(c, s\xi), \quad \tilde{\psi}_n(c,s) = \psi_n(c, s\xi)$$

for $n = 1, 2, \cdots$. Thanks to the spatial homogeneity, $\tilde{\phi}_n(c,s)$ and $\tilde{\psi}_n(c,s)$ are non-increasing in c and s. By comparison principle for cooperative systems, $\tilde{\phi}_n(c,s)$ and $\tilde{\psi}_n(c,s)$ are non-decreasing in n. Let

$$\tilde{\phi}(c,x) = \lim_{n\to\infty} \tilde{\phi}_n(c,x), \quad \tilde{\psi}(c,x) = \lim_{n\to\infty} \tilde{\psi}_n(c,x).$$

Define $c_+^*(\xi)$ and $\bar{c}_+(\xi)$ by

$$c_+^*(\xi) = \sup\{c \,|\, (\tilde{\phi}(c,\infty), \tilde{\psi}(c,\infty)) = (u^*(0), v^*(0))\}$$

and

$$\bar{c}_+(\xi) = \sup\{c \,|\, (\tilde{\phi}(c,\infty), \tilde{\psi}(c,\infty)) \neq (0,0)\}.$$

We remark that, by the similar arguments as in [12] and [43], $c_+^*(\xi)$ and $\bar{c}_+(\xi)$ are well defined. We also remark that

$$c^*_{\inf}(\xi) = c_+^*(\xi), \quad c^*_{\sup}(\xi) = \bar{c}_+(\xi).$$

(2) In the case that $a_k(t,x)$, $b_k(t,x)$, and $c_k(t,x)$ $(k = 1,2)$ are independent of both t and x, it is proved in [12, Theorem 5.3] that $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$.

(3) It remains open whether (1.1) has a single spreading speed in each direction in both temporally and spatially periodic media.

Proof of Theorem 1.4. It can be proved by applying the similar arguments as in [12, Theorem 5.3]. For completeness, we give a proof in the following.

We first consider the case that $\xi = (1, 0, \cdots, 0)$.

First of all, note that, for given (u_0, v_0), if $u_0(x_1, x_2, \cdots, x_N) = u_0(x_1, 0, \cdots, 0)$ and $v_0(x_1, x_2, \cdots, x_N) = v_0(x_1, 0, \cdots, 0)$, (i.e. $u_0(x)$ and $v_0(x)$ are independent of $x_2, \cdots, x_N$), then $u(t, x; u_0, v_0)$ and $v(t, x; u_0, v_0)$ are also independent of $x_2, \cdots, x_N$. We then consider the following system in one space dimension induced from (1.14),

$$\begin{cases} u_t(t,x_1) = \int_{\mathbb{R}} \tilde{\kappa}(y_1 - x_1) u(t,y_1) dy_1 - u(t,x_1) \\ \qquad\qquad + u\Big(a_1(t) - b_1(t)u - c_1(t)(v^*(t) - v)\Big), \quad x_1 \in \mathbb{R} \\ v_t(t,x_1) = \int_{\mathbb{R}} \tilde{\kappa}(y_1 - x_1) v(t,y_1) dy_1 - v(t,x_1) + b_2(t)\Big(v^*(t) - v\Big)u \\ \qquad\qquad + v\Big(a_2(t) - 2c_2(t)v^*(t) + c_2(t)v\Big), \quad x_1 \in \mathbb{R}, \end{cases} \tag{3.17}$$

where

$$\tilde{\kappa}(x_1) = \int_{\mathbb{R}^{N-1}} \kappa(x_1, x_2, \cdots, x_N) dx_2 \cdots dx_N.$$

Observe that $u^*(t,x) \equiv u^*(t)$ and $v^*(t,x) \equiv v^*(t)$, and $\tilde{E}_0 = (0, v^*)$, $\tilde{E}_1 = (0,0)$, and $\tilde{E}_2 = (u^*, v^*)$ are time periodic solutions of (3.17). Hence $(0, v^*(0))$, $(0,0)$, and $(u^*(0), v^*(0))$ are the fixed points of the time T map or the Poincaré map Q of (3.17). Observe also that $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is the spreading speed interval of (3.17), and hence as mentioned in Remark 3.6, $c^*_{\inf}(\xi)$ and $c^*_{\sup}(\xi)$ are two spreading speeds of the Poincaré map Q of (3.17).

Next, it is not difficult to see that the Poincaré map Q of (3.17) satisfies (A1)-(A5) in [12] with $\beta = (u^*(0), v^*(0))$,

$$\mathcal{M} = \{(u, v) : \mathbb{R} \to \mathbb{R}^2 \,|\, u, v \text{ are non-increasing and bounded functions}\},$$

and

$$\mathcal{M}_\beta = \{(u(\cdot), v(\cdot)) \in \mathcal{M} \,|\, 0 \le u(x) \le u^*(0), \quad 0 \le v(x) \le v^*(0) \quad \forall\, x \in \mathbb{R}\}.$$

Assume $c^*_{\inf}(\xi) < c^*_{\sup}(\xi)$. Then by [12, Theorem 3.1(1) and (3)], for any $c^*_{\inf}(\xi) \le c < c^*_{\sup}(\xi)$, there are non-increasing functions $\Phi(x_1)$ and $\Psi(x_1)$ such that

$$Q^n(\Phi, \Psi)(x_1) = (\Phi(x_1 - cnT), \Psi(x_1 - cnT)) \tag{3.18}$$

for all but countably many $x \in \mathbb{R}$,

$$\Phi(-\infty) = u^*(0), \quad \Psi(-\infty) = v^*(0) \tag{3.19}$$

and

$$\Phi(\infty) = 0, \quad \Psi(\infty) = v^*(0). \tag{3.20}$$

By (3.19), (3.20) and the monotonicity of $\Psi(\cdot)$, we must have $\Psi(x_1) \equiv v^*(0)$. Hence

$$u(nT, x_1; \Phi(\cdot)) = \Phi(x_1 - cnT) \tag{3.21}$$

for all but countably many $x \in \mathbb{R}$, where $u(t, x_1; \Phi(\cdot))$ is the solution of

$$u_t = \int_{\mathbb{R}} \tilde{\kappa}(y_1 - x_1) u(t, y_1) dy_1 - u(t, x_1) + u\big(a_1(t) - b_1(t)u\big). \tag{3.22}$$

But $c^*_{\inf}(\xi) < c^*_0(\xi, a_1)$, where $c^*_0(\xi, a_1)$ is the spreading speed of (3.22). Choose $c, c^{'}$ such that $c^*_{\inf}(\xi) < c < c^{'} < \min\{c^*_0(a_1, \xi), c^*_{\sup}(\xi)\}$. By Proposition 2.9,

$$\limsup_{x_1 \le c^{'} nT, n \to \infty} |u(nT, x_1; \Phi(\cdot)) - u^*(nT)| = 0. \tag{3.23}$$

By (3.21),

$$\limsup_{x_1 \le c^{'} nT, n \to \infty} |u(nT, x_1; \Phi(\cdot)) - u^*(nT)| = \limsup_{x_1 \le c^{'} nT, n \to \infty} |\Phi(x_1 - cnT) - u^*(nT)| > 0,$$

which contradicts to (3.23). Therefore, $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$.

For a general $\xi \in S^{n-1}$, without loss of generality, we may assume that $\xi_1 \ne 0$. Then make the following change of space variables,

$$\tilde{x}_1 = x \cdot \xi, \quad \tilde{x}_2 = x_2, \cdots, \tilde{x}_n = x_n$$

and consider the following system induced from (1.14),

$$\begin{cases} u_t(t, \tilde{x}_1) = \int_{\mathbb{R}} \tilde{\kappa}(\tilde{y}_1 - \tilde{x}_1) u(t, \tilde{y}_1) d\tilde{y}_1 - u(t, \tilde{x}_1) \\ \qquad\qquad +u\Big(a_1(t) - b_1(t)u - c_1(t)(v^*(t) - v)\Big), \quad \tilde{x}_1 \in \mathbb{R} \\ v_t(t, \tilde{x}_1) = \int_{\mathbb{R}} \tilde{\kappa}(\tilde{y}_1 - \tilde{x}_1) v(t, \tilde{y}_1) d\tilde{y}_1 - v(t, \tilde{x}_1) + b_2(t)\Big(v^*(t) - v\Big)u \\ \qquad\qquad +v\Big(a_2(t) - 2c_2(t)v^*(t) + c_2(t)v\Big), \quad \tilde{x}_1 \in \mathbb{R}, \end{cases}$$

where

$$\tilde{\kappa}(\tilde{x}_1) = \xi_1 \int_{\mathbb{R}^{N-1}} \kappa\Big(\frac{1}{\xi_1}(\tilde{x}_1 - \xi_2 \cdot \tilde{x}_2 - \cdots - \xi_N \cdot \tilde{x}_N, \tilde{x}_2, \cdots, \tilde{x}_N) d\tilde{x}_2 \cdots \tilde{x}_N.$$

Now by the similar arguments as in the above, $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$. □

Remark 3.7. The concept of spreading speed intervals for (1.1) introduced in this section and the results and techniques developed in this section can be extended to two species competition systems with different nonlocal dispersal rates, that is, the following two species competition systems,

$$\begin{cases} u_t = d_1[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v), & x \in \mathbb{R}^N \\ v_t = d_2[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + v(a_2(t,x) - b_2(t,x)u - c_2(t,x)v), & x \in \mathbb{R}^N. \end{cases} \tag{3.24}$$

4. Linear Determinacy for Spreading Speeds and the Proof of Theorem 1.5

In this section, we explore the linear determinacy for the spreading speeds and prove Theorem 1.5. Throughout this section, we assume (HB0)-(HB2). We also assume (HL0) and (HL1) or (HL2). In addition, we assume that $\xi \in S^{N-1}$ is given and fixed. We first prove some lemmas.

Consider the linearization of (1.14) at $(0,0)$,

$$\begin{cases} u_t = \mathcal{K}u - u + (a_1(t,x) - c_1(t,x)v^*(t,x))u, & x \in \mathbb{R}^N \\ v_t = \mathcal{K}v - v + b_2(t,x)v^*(t,x)u + (a_2(t,x) - 2c_2(t,x)v^*(t,x))v, & x \in \mathbb{R}^N. \end{cases} \tag{4.1}$$

Consider also the following associated eigenvalue problem of (4.1),

$$\begin{cases} -u_t + \mathcal{K}_{\xi,\mu}u - u + (a_1(t,x) - c_1(t,x)v^*(t,x))u = \lambda u, & x \in \mathbb{R}^N \\ -v_t + \mathcal{K}_{\xi,\mu}v - v + b_2(t,x)v^*(t,x)u + (a_2(t,x) - 2c_2(t,x)v^*(t,x))v = \lambda v, & x \in \mathbb{R}^N \\ u, v \in \mathcal{X}_P, \end{cases} \tag{4.2}$$

where $\mathcal{K}_{\xi,\mu}$ is as in (2.2). Recall that $\lambda_\xi(\mu)$ is the principal spectrum point of

$$\begin{cases} -u_t + \mathcal{K}_{\xi,\mu}u - u + (a_1(t,x) - c_1(t,x)v^*(t,x))u = \lambda u, & x \in \mathbb{R}^N \\ u \in \mathcal{X}_p. \end{cases} \tag{4.3}$$

Let

$$\bar{c}^*_{\inf}(\xi) = \inf_{\mu>0} \frac{\lambda_\xi(\mu)}{\mu}.$$

Before proving Theorem 1.5, we prove some lemmas.

Note that $\frac{\lambda_\xi(\mu)}{\mu} \to \infty$ as $\mu \to 0+$ or $\mu \to \infty$ (see the arguments in [36, Proposiiton 3.4]). Hence there is $\mu^*(\xi)$ such that

$$\bar{c}^*_{\inf}(\xi) = \frac{\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)}.$$

Lemma 4.1. *Assume (HL1) or (HL2). For any $\epsilon > 0$, there are $\bar{a}_1(t,x) \ge a_1(t,x)$, $\bar{c}_1(t,x) \ge c_1(t,x)$, and $\bar{v}^*(t,x)$ such that*

(1)

$$\bar{a}_1(t,x) - \bar{c}_1(t,x)\bar{v}^*(t,x) \ge a_1(t,x) - c_1(t,x)v^*(t,x);$$

(2) the principal eigenvalue $\bar{\lambda}_\xi(\mu^*(\xi))$ of

$$\begin{cases} -u_t + \mathcal{K}_{\mu^*(\xi),\xi}u - u + (\bar{a}_1(t,x) - \bar{c}_1(t,x)\bar{v}^*(t,x))u = \lambda u \\ u \in \mathcal{X}_p, \end{cases} \tag{4.4}$$

exists;

(3)

$$\frac{\bar{\lambda}_\xi(\mu^*(\xi))}{\mu^*(\xi)} \le \frac{\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)} + \epsilon;$$

(4) the principal spectrum point of

$$\begin{cases} -v_t + \mathcal{K}_{\xi,\mu} v - v + (a_2(t,x) - 2c_2(t,x)v^*(t,x))v - \bar{\lambda}_\xi(\mu^*(\xi))v = \lambda v \\ v \in \mathcal{X}_p \end{cases} \tag{4.5}$$

is negative.

Proof. (1), (2), and (3) follow from Proposition 2.3.

(4) By (1) and (HL1) or (HL2),

$$\bar{a}_1(t,x) - \bar{c}_1(t,x)\bar{v}^*(t,x) \geq a_1(t,x) - c_1(t,x)v^*(t,x) > a_2(t,x) - 2c_2(t,x)v^*(t,x)$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Note that

$$\lambda(\mu, \xi, \bar{a}_1 - \bar{c}_1\bar{v}^* - \bar{\lambda}_\xi(\mu^*(\xi))) = 0.$$

Hence

$$\lambda(\mu, \xi, a_2(t,x) - 2c_2(t,x)v^*(t,x) - \bar{\lambda}_\xi(\mu^*(\xi))) < 0.$$

□

In the following, we put $\mu = \mu^*(\xi)$. Let $u_{\xi,\mu}(\cdot,\cdot) \in \mathcal{X}_p$ be a positive principal eigenfunction of (4.4). By Lemma 4.1 (4) and Proposition 2.5, there is a unique time and space periodic positive solution in $\mathcal{X}_p$, denoted by $v_{\xi,\mu}$, of

$$v_t = \mathcal{K}_{\xi,\mu} v - v + (a_2(t,x) - 2c_2(t,x)v^*(t,x))v - \bar{\lambda}_\xi(\mu)v + b_2(t,x)v^*(t,x)u_{\xi,\mu}(t,x), \quad x \in \mathbb{R}^N \tag{4.6}$$

Moreover, $v_{\xi,\mu}$ is a globally asymptotically stable solution of (4.6) in X_p.

Lemma 4.2. *Assume (HL1) or (HL2). Then*

$$c_1(t,x)v_{\xi,\mu}(t,x) \leq b_1(t,x)u_{\xi,\mu}(t,x)$$

and

$$c_2(t,x)v_{\xi,\mu}(t,x) \leq b_2(t,x)u_{\xi,\mu}(t,x).$$

Proof. We first assume (HL1). We claim that $v = u_{\xi,\mu}(t,x)$ is a super-solution of (4.6). In fact,

$$\begin{aligned}
&\partial_t u_{\xi,\mu}(t,x) - \Big[\mathcal{K}_{\xi,\mu} u_{\xi,\mu} - u_{\xi,\mu} + (a_2(t,x) - 2c_2(t,x)v^*(t,x))u_{\xi,\mu} \\
&\qquad\qquad - \bar{\lambda}_\xi(\mu)u_{\xi,\mu} + b_2(t,x)v^*(t,x)u_{\xi,\mu}(t,x)\Big] \\
&= \mathcal{K}_{\xi,\mu} u_{\xi,\mu} - u_{\xi,\mu} + (\bar{a}_1 - \bar{c}_1\bar{v}^*(t,x))u_{\xi,\mu} - \bar{\lambda}_\xi(\mu)u_{\xi,\mu} \\
&\quad - \Big[\mathcal{K}_{\xi,\mu} u_{\xi,\mu} - u_{\xi,\mu} + (a_2(t,x) - 2c_2(t,x)v^*(t,x))u_{\xi,\mu} \\
&\qquad - \bar{\lambda}_\xi(\mu)u_{\xi,\mu} + b_2(t,x)v^*(t,x)u_{\xi,\mu}(t,x)\Big] \\
&\geq \Big[a_1(t,x) - c_1(t,x)v^*(t,x) - a_2(t,x) + 2c_2(t,x)v^*(t,x) - b_2(t,x)v^*(t,x)\Big]u_{\xi,\mu} \\
&\geq 0 \qquad\qquad \text{(by (HL1))}.
\end{aligned}$$

Hence $v = u_{\xi,\mu}(t,x)$ is a super-solution of (4.6). Therefore, we must have

$$v_{\xi,\mu}(t,x) \leq u_{\xi,\mu}(t,x).$$

By (HL1) again, $b_1(t,x) \ge c_1(t,x)$ and $b_2(t,x) \ge c_2(t,x)$. It then follows that

$$c_1(t,x)v_{\xi,\mu}(t,x) \le b_1(t,x)u_{\xi,\mu}(t,x)$$

and

$$c_2(t,x)v_{\xi,\mu}(t,x) \le b_2(t,x)u_{\xi,\mu}(t,x).$$

Next, we assume (HL2). We claim that both $v = \frac{b_{1L}}{c_{1M}}u_{\xi,\mu}$ and $v = \frac{b_{2L}}{c_{2M}}u_{\xi,\mu}$ are super-solutions of (4.6). In fact,

$$\begin{aligned}
&\partial_t \frac{b_{1L}}{c_{1M}}u_{\xi,\mu}(t,x) - \Big[\mathcal{K}_{\xi,\mu}\frac{b_{1L}}{c_{1M}}u_{\xi,\mu} - \frac{b_{1L}}{c_{1M}}u_{\xi,\mu} + (a_2(t,x) - 2c_2(t,x)v^*(t,x))\frac{b_{1L}}{c_{1M}}u_{\xi,\mu}\\
&\qquad\qquad - \bar\lambda_\xi(\mu)\frac{b_{1L}}{c_{1M}}u_{\xi,\mu} + b_2(t,x)v^*(t,x)u_{\xi,\mu}(t,x)\Big]\\
&\ge \Big[a_1(t,x) - c_1(t,x)v^*(t,x) - a_2(t,x) + 2c_2(t,x)v^*(t,x) - b_2(t,x)v^*(t,x)\frac{c_{1M}}{b_{1L}}\Big]\frac{b_{1L}}{c_{1M}}u_{\xi,\mu}\\
&\ge 0 \qquad\qquad\qquad \text{(by (HL2))}.
\end{aligned}$$

Hence $v = \frac{b_{1L}}{c_{1M}}u_{\xi,\mu}$ is a super-solution of (4.6). Similarly, we can prove that $v = \frac{b_{2L}}{c_{2M}}u_{\xi,\mu}$ is a super-solution of (4.6). Therefore, we also have

$$v_{\xi,\mu}(t,x) \le \frac{b_{1L}}{c_{1M}}u_{\xi,\mu}(t,x)$$

and

$$v_{\xi,\mu}(t,x) \le \frac{b_{2L}}{c_{2M}}u_{\xi,\mu}(t,x).$$

It then follows that

$$c_1(t,x)v_{\xi,\mu}(t,x) \le b_1(t,x)u_{\xi,\mu}(t,x)$$

and

$$c_2(t,x)v_{\xi,\mu}(t,x) \le b_2(t,x)u_{\xi,\mu}(t,x).$$

□

We now prove Theorem 1.5.

Proof of Theorem 1.5. First of all, for any $\epsilon > 0$, let $\bar a_1(t,x)$, $\bar c_1(t,x)$, $\bar v^*(t,x)$, and $\mu^*(\xi)$ be as in Lemma 4.1. For any $M > 0$, let

$$\begin{cases}
\tilde u(t,x;M) = Me^{-\mu^*(\xi)\left(x\cdot\xi - \frac{\bar\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)}t\right)}u_{\xi,\mu^*(\xi)}(t,x)\\
\tilde v(t,x;M) = Me^{-\mu^*(\xi)\left(x\cdot\xi - \frac{\bar\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)}t\right)}v_{\xi,\mu^*(\xi)}(t,x)
\end{cases}$$

and

$$\begin{cases}
u^+(t,x;M) = \min\{u^*(t,x), \tilde u(t,x;M)\}\\
v^+(t,x;M) = \min\{v^*(t,x), \tilde v(t,x;M)\}.
\end{cases}$$

We claim that, for any given $(u_0, v_0) \in X_1^+ \times X_2^+$, if

$$u_0(x) \le u^+(0,x;M), \quad v_0(x) \le v^+(0,x;M) \quad \forall\, x \in \mathbb{R}^N,$$

then

$$u(t,x;u_0,v_0) \le u^+(t,x;M), \quad v(t,x;u_0,v_0) \le v^+(t,x;M) \quad \forall\, 0 \le t \le T, \quad x \in \mathbb{R}^N. \tag{4.7}$$

Assume first that the claim holds. For any $(u_0, v_0) \in X_1^+(\xi) \times X_2^+(\xi)$, let $M_0 > 0$ be such that

$$u_0(x) \le u^+(0, x; M_0), \quad v_0(x) \le v^+(0, x; M_0) \quad \forall\, x \in \mathbb{R}^N.$$

Then by (4.7),

$$u(t, x; u_0, v_0) \le u^+(t, x; M_0), \quad v(t, x; u_0, v_0) \le v^+(t, x; M_0) \quad \forall\, 0 \le t \le T, \quad x \in \mathbb{R}^N.$$

Let

$$M_1 = M_0 e^{\bar\lambda_\xi(\mu^*(\xi))T}.$$

Then

$$u(T, x; u_0, v_0) \le u^+(0, x; M_1), \quad v(T, x; u_0, v_0) \le v^+(0, x; M_1) \quad \forall\, x \in \mathbb{R}^N.$$

By (4.7) again,

$$u(t + T, x; u_0, v_0) = u(t, x; u(T, \cdot; u_0, v_0), v(T, \cdot; u_0, v_0)) \le u^+(t, x; M_1) = u^+(t + T, x; M_0)$$

and

$$v(t + T, x; u_0, v_0) = v(t, x; u(T, \cdot; u_0, v_0), v(T, \cdot; u_0, v_0)) \le v^+(t, x; M_1) = v^+(t + T, x; M_0)$$

for $t \in [0, T]$ and $x \in \mathbb{R}^N$. Hence

$$u(t, x; u_0, v_0) \le u^+(t, x; M_0), \quad v(t, x; u_0, v_0) \le v^+(t, x; M_0) \quad \forall\, 0 \le t \le 2T, \quad x \in \mathbb{R}^N.$$

Continuing the above process, we have

$$u(t, x; u_0, v_0) \le u^+(t, x; M_0), \quad v(t, x; u_0, v_0) \le v^+(t, x; M_0) \quad \forall\, t \ge 0, \quad x \in \mathbb{R}^N.$$

Then for any $c > \frac{\bar\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)}$,

$$\limsup_{x\cdot\xi \ge ct, t\to\infty} [u(t, x; u_0, v_0) + v(t, x; u_0, v_0)] = 0$$

and hence

$$c^*_{\sup}(\xi) \le \frac{\bar\lambda_\xi(\mu^*(\xi))}{\mu^*(\xi)} \le \bar c^*_{\inf}(\xi) + \epsilon.$$

By Theorem 1.3(2),

$$c^*_{\inf}(\xi) \ge \bar c^*_{\inf}(\xi).$$

Letting $\epsilon \to 0$, we have

$$c^*_{\sup}(\xi) = c^*_{\inf}(\xi) = \bar c^*_{\inf}(\xi).$$

Thus the theorem follows.

We now prove that the claim holds. To this end, choose $\bar M > 0$ such that

$$\begin{cases} \bar M - 1 + a_1(t, x) - 2\tilde d \cdot b_1(t, x)u^*(t, x) - c_1(t, x)v^*(t, x) > 0 \\ \bar M - 1 + a_2(t, x) - \tilde d \cdot b_2(t, x)u^*(t, x) - 2c_2(t, x)v^*(t, x) > 0 \end{cases} \tag{4.8}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where

$$\tilde d = \max\Big\{1, \frac{u^*(t, x)}{u^*(\tau, x)} \cdot \frac{u_{\xi,\mu^*(\xi)}(\tau, x)}{u_{\xi,\mu^*(\xi)}(t, x)} \cdot e^{\bar\lambda_\xi(\mu^*(\xi))(\tau - t)}, \frac{v^*(t, x)}{u^*(\tau, x)} \cdot \frac{u_{\xi,\mu^*(\xi)}(\tau, x)}{v_{\xi,\mu^*(\xi)}(t, x)} \cdot e^{\bar\lambda_\xi(\mu^*(\xi))(\tau - t)}$$

$$\Big| t - T \le \tau \le t,\ t, \tau \in \mathbb{R},\ x \in \mathbb{R}^N\Big\}.$$

Let

$$\bar{u}(t,x;M,\bar{M}) = e^{\bar{M}t}\tilde{u}(t,x;M), \quad \bar{v}(t,x;M,\bar{M}) = e^{\bar{M}t}\tilde{u}(t,x;M)$$

and

$$\bar{u}^*(t,x;\bar{M}) = e^{\bar{M}t}u^*(t,x), \quad v^*(t,x;\bar{M}) = e^{\bar{M}t}v^*(t,x).$$

Let

$$\begin{cases} \bar{u}^+(t,x;M,\bar{M}) = \min\{\bar{u}^*(t,x;\bar{M}), \bar{u}(t,x;M,\bar{M})\} \\ \bar{v}^+(t,x;M,\bar{M}) = \min\{\bar{v}^*(t,x;M), \bar{v}(t,x;M,\bar{M})\} \end{cases}$$

and

$$\begin{cases} \bar{u}(t,x;u_0,v_0,\bar{M}) = e^{\bar{M}t}u(t,x;u_0,v_0) \\ \bar{v}(t,x;u_0,v_0,\bar{M}) = e^{\bar{M}t}v(t,x;u_0,v_0). \end{cases}$$

It suffices to prove that, for any $(u_0,v_0) \in X_1^+ \times X_2^+$, if

$$u_0(x) \le \bar{u}^+(0,x;M,\bar{M}), \quad v_0(x) \le \bar{v}^+(0,x;M,\bar{M}) \quad \forall\, x \in \mathbb{R}^N,$$

then

$$\bar{u}(t,x;u_0,v_0,\bar{M}) \le \bar{u}^+(t,x;M,\bar{M}), \quad \bar{v}(t,x;u_0,v_0,\bar{M}) \le \bar{v}^+(t,x;M,\bar{M}) \tag{4.9}$$

for $0 \le t \le T$ and $x \in \mathbb{R}^N$.

In the following, if no confusion occurs, we write $\bar{u}(t,x;M,\bar{M})$, $\bar{u}^*(t,x;M)$, $\bar{u}^+(t,x;M,\bar{M})$, $\bar{v}(t,x;M,\bar{M})$, $\bar{v}^*(t,x;M)$, and $v^+(t,x;M,\bar{M})$ as $\bar{u}(t,x)$, $\bar{u}^*(t,x)$, $\bar{u}^+(t,x)$, $\bar{v}(t,x)$, $\bar{v}^*(t,x)$, and $\bar{v}^+(t,x)$, respectively.

Let

$$\bar{f}(t,x,u,v) = u\Big(\bar{M} - 1 + a_1(t,x) - e^{-\bar{M}t}b_1(t,x)u - c_1(t,x)v^*(t,x) + e^{-\bar{M}t}c_1(t,x)v\Big)$$

and

$$\bar{g}(t,x,u,v) = b_2(t,x)\Big(v^*(t,x) - e^{-\bar{M}t}v\Big)u + v\Big(\bar{M} - 1 + a_2(t,x) - 2c_2(t,x)v^*(t,x) + e^{-\bar{M}t}c_2(t,x)v\Big).$$

Note that

$$\begin{cases} \bar{f}_v(t,x,u,v) \ge 0 \quad \forall\, t \in \mathbb{R},\ x \in \mathbb{R}^N,\ u \ge 0 \\ \bar{g}_u(t,x,u,v) \ge 0 \quad \forall\, 0 \le v \le \bar{v}^*(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^N. \end{cases} \tag{4.10}$$

By (4.8),

$$\bar{f}_u(t,x,u,v) \ge 0, \quad \bar{g}_v(t,x,u,v) \ge 0 \tag{4.11}$$

for $0 \le u \le \tilde{d}\bar{u}^*(t,x)$, $v \ge 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$.

We first show that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, we have

$$b_2(t,x)\bar{u}^+(t,x) \ge c_2(t,x)\bar{v}^+(t,x). \tag{4.12}$$

If (t,x) is such that $\bar{v}^+(t,x) = \bar{v}^*(t,x)$ and $\bar{u}^+(t,x) = \bar{u}^*(t,x)$, then by (HL0),

$$b_2(t,x)\bar{u}^+(t,x) \ge c_2(t,x)\bar{v}^+(t,x).$$

If (t,x) is such that $\bar{v}^+(t,x) = \bar{v}(t,x) \le \bar{v}^*(t,x)$, then by Lemma 4.2,

$$b_2(t,x)\bar{u}^+(t,x) = b_2(t,x)\min\{\bar{u}(t,x), \bar{u}^*(t,x)\} \ge c_2(t,x)\bar{v}^+(t,x).$$

If (t,x) is such that $\bar{u}^+(t,x) = \bar{u}(t,x) \le \bar{u}^*(t,x)$, then by Lemma 4.2 again,

$$b_2(t,x)\bar{u}^+(t,x) \ge c_2(t,x)\bar{v}(t,x) \ge c_2(t,x)\bar{v}^+(t,x).$$

Hence for any $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, (4.12) holds.

By (4.10) and (4.11),

$$\begin{cases} \bar u^*(t,x) & = \bar u^*(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u^*(\tau,y)dy+\bar f(\tau,x,\bar u^*(\tau,x),\bar v^*(\tau,x)\Big]d\tau\\ & \ge \bar u^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u^+(\tau,y)dy+\bar f(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x)\Big]d\tau\\ \bar v^*(t,x) & = \bar v^*(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v^*(\tau,y)dy+\bar g(\tau,x,\bar u^*(\tau,x),\bar v^*(\tau,x)\Big]d\tau\\ & \ge \bar v^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v^+(\tau,y)dy+\bar g(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x)\Big]d\tau \end{cases} \tag{4.13}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Note that if $(t,x)\in[0,T]\times\mathbb{R}^N$ is such that $\bar u(t,x)=\bar u^+(t,x)\le\bar u^*(t,x)$, then

$$\bar u(\tau,x)\le e^{\bar\lambda_\xi(\mu^*(\xi))(\tau-t)}\cdot\frac{u^*(t,x)}{u^*(\tau,x)}\cdot\frac{u_{\xi,\mu^*(\xi)}(\tau,x)}{u_{\xi,\mu^*(\xi)}(t,x)}\cdot\bar u^*(\tau,x)\le\tilde d\cdot\bar u^*(\tau,x)$$

for $0\le\tau\le t$. Hence

$$\begin{aligned} \bar u(t,x)&=\bar u(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u(\tau,y)dy+(\bar M-1+\bar a_1(\tau,x)-\bar c_1(\tau,x)\bar v^*(\tau,x))\bar u\Big]d\tau\\ &\ge\bar u(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u(\tau,y)dy+\bar f(\tau,x,\bar u(\tau,x),\bar v(\tau,x))\Big]d\tau \qquad\text{(by Lemma 4.2)}\\ &\ge\bar u^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u^+(\tau,y)dy+\bar f(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x))\Big]d\tau \quad\text{(by (4.10), (4.11)).} \end{aligned}$$

If $(t,x)\in[0,T]\times\mathbb{R}^N$ is such that $\bar v(t,x)=\bar v^+(t,x)$, then

$$\bar u(\tau,x)\le e^{\bar\lambda_\xi(\mu^*(\xi))(\tau-t)}\cdot\frac{v^*(t,x)}{u^*(\tau,x)}\cdot\frac{u_{\xi,\mu^*(\xi)}(\tau,x)}{v_{\xi,\mu^*(\xi)}(t,x)}\cdot\bar u^*(\tau,x)\le\tilde d\cdot\bar u^*(\tau,x)$$

Hence

$$\begin{aligned} \bar v(t,x)&=\bar v(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v(\tau,y)dy+b_2(\tau,x)v^*(\tau,x)\bar u(\tau,x)\\ &\qquad\qquad+(\bar M-1+a_2(\tau,x)-2c_2(\tau,x)v^*(\tau,x))\bar v\Big]d\tau\\ &\ge\bar v^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v^+(\tau,y)dy+b_2(\tau,x)v^*(\tau,x)\bar u^+(\tau,x)\\ &\qquad\qquad+(\bar M-1+a_2(\tau,x)-2c_2(\tau,x)v^*(\tau,x))\bar v^+(\tau,x)\Big]d\tau \qquad\text{(by (4.8))}\\ &\ge\bar v^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v^+(\tau,y)dy+\bar g(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x))\Big]d\tau \qquad\text{(by (4.12)).} \end{aligned}$$

It then follows that for any $(t,x)\in[0,T]\times\mathbb{R}^N$,

$$\begin{cases} \bar u^+(t,x)\ge\bar u^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar u^+(\tau,y)dy+\bar f(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x)\Big]d\tau\\ \bar v^+(t,x)\ge\bar v^+(0,x)+\int_0^t\Big[\int_{\mathbb{R}^N}\kappa(y-x)\bar v^+(\tau,y)dy+\bar g(\tau,x,\bar u^+(\tau,x),\bar v^+(\tau,x)\Big]d\tau \end{cases} \tag{4.14}$$

For given (u_0,v_0) with $u_0\le u^+(0,\cdot)$ and $v_0\le v^+(0,\cdot)$, put

$$\bar u_0(t,x)=e^{\bar Mt}u(t,x;u_0,v_0),\quad \bar v_0(t,x)=e^{\bar Mt}v(t,x;u_0,v_0).$$

Then for all $t \geq 0$ and $x \in \mathbb{R}^N$,

$$\begin{cases} \bar{u}_0(t,x) = \bar{u}_0(x) + \int_0^t \Big[\int_{\mathbb{R}^N} \kappa(y-x)\bar{u}_0(\tau,y)dy + \bar{f}(\tau,x,\bar{u}_0(\tau,x),\bar{v}_0(\tau,x)) \Big] d\tau \\ \bar{v}_0(t,x) = \bar{v}_0(x) + \int_0^t \Big[\int_{\mathbb{R}^N} \kappa(y-x)\bar{v}_0(\tau,y)dy + \bar{g}(\tau,x,\bar{u}_0(\tau,x),\bar{v}_0(\tau,x)) \Big] d\tau. \end{cases} \tag{4.15}$$

Let

$$\tilde{u}(t,x) = \bar{u}^+(t,x) - \bar{u}_0(t,x), \quad \tilde{v}(t,x) = \bar{v}^+(t,x) - \bar{v}_0(t,x).$$

By (4.14) and (4.15),

$$\begin{cases} \tilde{u}(t,x) \geq \tilde{u}(0,x) + \int_0^t \Big[\int_{\mathbb{R}^N} \kappa(y-x)\tilde{u}(\tau,y)dy + \tilde{a}_1(\tau,x)\tilde{u}(\tau,x) + \tilde{b}_1(\tau,x)\tilde{v}(\tau,x) \Big] d\tau \\ \tilde{v}(t,x) \geq \tilde{v}(0,x) + \int_0^t \Big[\int_{\mathbb{R}^N} \tilde{\kappa}(y-x)\tilde{v}(\tau,y)dy + \tilde{a}_2(\tau,x)\tilde{u}(\tau,x) + \tilde{b}_2(\tau,x)\tilde{v}(\tau,x) \Big] d\tau, \end{cases}$$

where

$$\tilde{a}_1(t,x) = \bar{f}_u(t,x,\tilde{u}_1(t,x),\tilde{v}_1(t,x)), \quad \tilde{b}_1(t,x) = \bar{f}_v(t,x,\tilde{u}_1(t,x),\tilde{v}_1(t,x)),$$

and

$$\tilde{a}_2(t,x) = \bar{g}_u(t,x,\tilde{u}_2(t,x),\tilde{v}_2(t,x)), \quad \tilde{b}_2(t,x) = \bar{g}_v(t,x,\tilde{u}_2(t,x),\tilde{v}_2(t,x))$$

for some $\tilde{u}_1(t,x)$, $\tilde{u}_2(t,x)$ between $\bar{u}^+(t,x)$ and $\bar{u}_0(t,x)$, and some $\tilde{v}_1(t,x)$, $\tilde{v}_2(t,x)$ between $\bar{v}^+(t,x)$ and $\bar{v}_0(t,x)$. By (4.10) and (4.11),

$$\tilde{a}_i(t,x) \geq 0, \quad \tilde{b}_i(t,x) \geq 0 \quad \forall\, t \in [0,T], \quad x \in \mathbb{R}^N, \quad i = 1,2.$$

Then by comparison principle for cooperative systems,

$$\tilde{u}(t,x) = \bar{u}^+(t,x) - \bar{u}_0(t,x) \geq 0, \quad \tilde{v}(t,x) = \bar{v}^+(t,x) - \bar{v}_0(t,x) \geq 0$$

for $t \in [0,T]$ and $x \in \mathbb{R}^N$, which implies (4.9) and then (4.7). This completes the proof of the claim and then the theorem. □

5. Remarks on Spreading Speeds of Two Species Competition Systems with Random and Discrete Dispersals in Periodic Habitats

We remark that the methods developed in this paper can be applied to the study of spreading speeds and linear determinacy for the following two species competition systems,

$$\begin{cases} u_t = d_1 \Delta u + u(a_1(t,x) - b_1(t,x)u - c_1(t,x)v), \quad x \in \mathbb{R}^N \\ v_t = d_2 \Delta v + v(a_2(t,x) - b_2(t,x)u - c_2(t,x)v), \quad x \in \mathbb{R}^N, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \dot{u}_j(t) = d_1 \sum_{k\in K} (u_{j+k}(t) - u_j(t)) + u_j(t)(a_1(t,j) - b_1(t,j)u_j(t) - c_1(t,j)v_j(t)), \quad j \in \mathbb{Z}^N \\ \dot{v}_j(t) = d_2 \sum_{k\in K} (v_{j+k}(t) - v_j(t)) + v_j(t)(a_2(t,j) - b_2(t,j)u_j(t) - c_2(t,j)v_j(t)), \quad j \in \mathbb{Z}^N, \end{cases} \tag{5.2}$$

where K is a bounded subset of $\mathbb{Z}^N$.

In particular, the definition of spreading speed interval for (1.1) (see Definition 1.1) can be applied to (5.1) and (5.2). Similar results to Theorems 1.3, 1.4, and 1.5 can also be obtained for (5.1) and (5.2).

As mentioned in the introduction, the authors of [44] have been studying the spreading speeds and traveling waves of (5.1). The results in [12] and [43] apply to (5.1) and (5.2) with spatial homogeneous and time periodic coefficients. One is also referred to [16] and references therein for the study of traveling wave solutions of (5.2) with space and time independent coefficients.

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