Evolution of Weyl Functions and Initial-Boundary Value Problems

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Abstract. This review is dedicated to some recent results on Weyl theory, inverse problems, evolution of the Weyl functions and applications to integrable wave equations in a semistrip and quarter-plane. For overdetermined initial-boundary value problems, we consider some approaches, which help to reduce the number of the initial-boundary conditions. The interconnections between dynamical and spectral Dirac systems, between response and Weyl functions are studied as well.

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1. Introduction

Initial-boundary value problems for linear and nonlinear wave equations are interesting, difficult and have many applications since initial-boundary conditions should be taken into account in the study of various applied problems connected with wave processes. Weyl theory could be useful in this study (and, vice versa, initial-boundary value problems are of interest in Weyl theory). A number of related results and references was presented in our book [60, Ch. 6]. Here we mainly review the results that appeared after the publication of [60] although some results from [60] and some earlier results that are not contained in [60] are also discussed for the sake of completeness.

In order to make the review reader-friendly and self-sufficient, we provide short schemes of the proofs for some of the main statements and start the review with the section "Preliminaries: Weyl functions, inverse problems and zero curvature representation". However, even this section contains some quite recent important results.

Zero curvature representation [3, 26, 46, 65] is in a certain sense an analogue of the famous Lax pairs. It is a compatibility condition for linear systems, which are so called auxiliary systems for nonlinear integrable wave equations. This compatibility condition is discussed in Subsection 2.3 of the section

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"Preliminaries ...". Here already, an initial-boundary value problem, which is basic for the study of the evolution of Weyl (Weyl-Titchmarsh) functions, appears.

Section 3 is dedicated to generalized Weyl functions and corresponding inverse problems. In Section 4 and in Subsection 5.2 we consider some applications of evolution formulas for Weyl (and generalized Weyl) functions to the problems of uniqueness and existence and to the unbounded solutions of wave equations.

We note that Cauchy problems (in particular, Cauchy problems for integrable nonlinear wave equations) are studied much more thoroughly than initial-boundary value problems. In fact, too many initial-boundary conditions are usually required for the study of initial-boundary value problems (see, e.g., [5, 19, 28, 37, 60] and references therein), and the problem becomes overdetermined. Thus, a crucial step here is the reduction of the initial-boundary conditions. Some important approaches to this reduction are discussed in Section 5.

Finally, Section 6 is dedicated to the initial-boundary value problem for dynamical Dirac system and its connections with Weyl theory.

Appendix A contains some definitions and references from the theory of quasi-analytic functions.

Notations. As usual, $\mathbb{R}$ stands for the real axis, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{C}$ stands for the complex plain, and $\mathbb{C}_+$ for the open upper semi-plane. We use notations $\mathbb{C}_M = \{z : \Im(z) > M\}$ and $\mathbb{C}_M^- = \{z : \Im(z) < -M\}$. The equality $D = \mathrm{diag}\{d_1, ...\}$ means that D is a diagonal matrix with the entries $d_1, \ldots$ on the main diagonal and I_m denotes the $m \times m$ identity matrix. "Locally" (e.g., locally summable) with respect to the semiaxis $[0, \infty)$ means: on all the finite intervals $[0, l]$ (e.g., summable on all $[0, l]$). The class of linear bounded operators acting from the normed space $\mathcal{H}_1$ into the normed space $\mathcal{H}_2$ is denoted by $B(\mathcal{H}_1, \mathcal{H}_2)$ and we write simply $B(\mathcal{H})$ when $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$. We say that a matrix function is boundedly differentiable when its derivative is bounded in the matrix norm, and an operator is boundedly invertible when the inverse operator exists and is bounded. The notation l.i.m. stands for the entrywise limit of a matrix function in the norm of $L^2(0, \ell)$, $0 < \ell \leq \infty$. The notation $\mathfrak{M}$ stands for the operator mapping a Weyl function (or generalized Weyl function, depending on the context) $\varphi(z)$ on the corresponding potential (e.g., $\mathfrak{M}(\varphi) = v$ for Dirac systems (2.1) and (3.1) where V has the form (2.2)). That is, $\mathfrak{M}(\varphi)$ is the solution of the inverse problem to recover the potential from the Weyl function.

2. Preliminaries: Weyl functions, inverse problems and zero curvature representation

2.1. Main scheme

One of the most important auxiliary linear systems in soliton theory is selfadjoint Dirac (also called ZS or AKNS) system

$$\frac{d}{dx} y(x, z) = \mathrm{i}(zj + jV(x))y(x, z), \qquad x \geq 0, \tag{2.1}$$

where

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 =: m, \tag{2.2}$$

I_{m_i} is the $m_i \times m_i$ identity matrix and $v(x)$ is an $m_1 \times m_2$ matrix function.

For instance, the well-known matrix defocusing nonlinear Schrödinger (defocusing NLS or dNLS) equation

$$2v_t = \mathrm{i}(v_{xx} - 2vv^*v) \qquad \left(v_t := \frac{\partial}{\partial t} v\right) \tag{2.3}$$

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admits [66, 67] (i.e., is equivalent to) zero curvature representation

$$G_t - F_x + [G, F] = 0 \qquad ([G, F] := GF - FG), \tag{2.4}$$
$$G = \mathrm{i}(zj + jV), \quad F = -\mathrm{i}\big(z^2 j + zjV - (\mathrm{i}V_x - jV^2)/2\big), \tag{2.5}$$

where V has the form (2.2) and $v = v(x, t)$. For this reason, systems $y_x = Gy$ and $y_t = Fy$ are called auxiliary systems for dNLS, and it is easy to see that $y_x = Gy$ has the form (2.1) for each fixed t. We note that V is called the potential of system (2.1). (Sometimes, for convenience, v is also called the potential.)

Weyl solutions are squarely summable on $(0, \infty)$ solutions of Dirac system (2.1). Clearly, these solutions may be presented as linear combinations of the columns of the normalized by the condition

$$u(0, z) = I_m \tag{2.6}$$

fundamental solution u of (2.1). Thus, Weyl-Titchmarsh (or simply Weyl) function of (2.1) is introduced in terms of u.

Definition 2.1. Let Dirac system (2.1) on $[0, \infty)$ be given and assume that v is locally summable. Then Weyl function is an $m_2 \times m_1$ holomorphic matrix function, which satisfies the inequality

$$\int_0^\infty \begin{bmatrix} I_{m_1} \ \varphi(z)^* \end{bmatrix} u(x, z)^* u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}_+. \tag{2.7}$$

The following proposition is proved in [30] (and in [60, Section 2.2]).

Proposition 2.2. *The Weyl function always exists and it is unique. Moreover, the Weyl function is contractive, that is, $\varphi(z)^* \varphi(z) \leq I_{m_1}$.*

Inverse problems for the classical selfadjoint Dirac systems had been actively studied since 1950s [39, 41] and various interesting results were published last years (see, e.g., [4, 21–24, 31, 35, 45, 60]). In particular, the inverse problem to recover the $m \times m$ locally square summable potential V of the Dirac system (2.1) from its Weyl function $\varphi(z)$ was dealt with in [56]. The case of rectangular (not necessarily square) matrix functions v was studied there and, moreover, only local square-summability of v was required, which was essentially less than in the preceding works. Thus a problem formulated by F. Gesztesy was solved and interesting applications to the case of Schrödinger-type operators with distributional matrix-valued potentials followed [25].

Theorem 2.3. *[56] Let Dirac system (2.1) be given on $[0, \infty)$, let its potential V be locally square-summable and let φ be the Weyl function of this system. Then V is uniquely recovered from φ.*

It is essential that a procedure to solve inverse problem is given in [56] (and will be formulated in the next subsection). Therefore, we see that if we know the evolution $\varphi(t, z)$ of the Weyl function of the system $y_x = G(x, t, z)y$, where G is given by (2.5), we may recover the solution $v(x, t)$ of dNLS (2.3). The same scheme works for many other integrable equations.

2.2. Recovery of the potential from the Weyl function

In this subsection we describe a procedure to solve inverse problem for system (2.1) with locally square summable potentials, which is summed up in [56, Theorem 4.4]. The potential v of (2.1) is easily expressed in terms of the block rows β and γ of the fundamental solution $u(x, z)$ (normalized by (2.6)) at $z = 0$:

$$\beta(x) = \begin{bmatrix} I_{m_1} \ 0 \end{bmatrix} u(x, 0), \quad \gamma(x) = \begin{bmatrix} 0 \ I_{m_2} \end{bmatrix} u(x, 0). \tag{2.8}$$

Indeed, (2.1) yields $u(x, z)^* j u(x, \overline{z}) = j$. Hence, $u(x, 0) j u(x, 0)^* = j$, and so

$$\beta j \beta^* \equiv I_{m_1}, \quad \gamma j \gamma^* \equiv -I_{m_2}, \quad \beta j \gamma^* \equiv 0. \tag{2.9}$$

Using (2.1) and (2.9) we derive

$$v(x) = \mathrm{i}\beta'(x)j\gamma(x)^* \qquad \left(\beta'(x) := \left(\frac{d}{dx}\beta\right)(x)\right). \tag{2.10}$$

Moreover, relations (2.9) and initial condition $\beta(0) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix}$ (which follows from (2.6), (2.8)) provide a procedure to recover β from γ. Namely, we have

$$\beta(x) = \beta_1(x)\widetilde{\beta}(x), \quad \widetilde{\beta}(x) := \begin{bmatrix} I_{m_1} & \gamma_1(x)^*(\gamma_2(x)^*)^{-1} \end{bmatrix}; \tag{2.11}$$

$$\beta_1' = -\beta_1(\widetilde{\beta}'j\widetilde{\beta}^*)(\widetilde{\beta}j\widetilde{\beta}^*)^{-1}, \quad \beta_1(0) = I_{m_1}, \tag{2.12}$$

where β_1 is the left $m_1 \times m_1$ block of β and γ_1 and γ_2 are the $m_2 \times m_1$ and $m_2 \times m_2$, respectively, blocks of γ. We note that (2.9) yields $\det \gamma_2 \neq 0$. It remains to recover γ from φ. Now, we formulate the corresponding theorem.

Theorem 2.4. *Let Dirac system (2.1) be given on $[0, \infty)$, let its potential V be locally square-summable and let φ be the Weyl function of this system.*

Then V is uniquely recovered from γ using formulas (2.2) and (2.10), where β is given by (2.11) and β_1 (in (2.11)) is the unique solution of the first order linear differential system (with initial condition) in (2.12). In turn, the matrix function γ is uniquely recovered from φ in three steps which are described below.

(i) Introduce a matrix function $\Phi_1(x)$ taking a Fourier transformation

$$\Phi_1\left(\frac{x}{2}\right) = \frac{1}{\pi}\mathrm{e}^{\eta x}\mathrm{l.i.m.}_{a\to\infty}\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\xi x}\frac{\varphi(\xi + \mathrm{i}\eta)}{2\mathrm{i}(\xi + \mathrm{i}\eta)}d\xi, \quad \eta > 0, \tag{2.13}$$

where l.i.m. stands for the entrywise limit in the norm of $L^2(0,\ell)$, $0 < \ell \leq \infty$. Here $\Phi_1(x)$ $(0 \leq x < \infty)$ is a well-defined differentiable matrix function (with a locally square-summable derivative), which does not depend on $\eta > 0$.

(ii) Introduce a family of pairs of operators $\Pi_l \in B\left(\mathbb{C}^m, L^2_{m_2}(0,l)\right)$ and S_l acting in $L^2_{m_2}(0,l)$ $(l \in \mathbb{R}_+)$:

$$\Pi_l g = \Phi_1(x)g_1 + g_2 \quad \text{for} \quad g_k \in \mathbb{C}^{m_k}, \ g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \tag{2.14}$$

$$S_l = I - \frac{1}{2}\int_0^l\int_{|x-t|}^{x+t}\Phi_1'\left(\frac{r+x-t}{2}\right)\Phi_1'\left(\frac{r+t-x}{2}\right)^* dr \cdot dt. \tag{2.15}$$

The defined above operator S_l belongs $B\left(L^2_{m_2}(0,l)\right)$, it is positive-definite (i.e., $S_l > 0$) and boundedly invertible, and the matrix function $\Pi_l^* S_l^{-1} \Pi_l$ is absolutely continuous with respect to l.

(iii) The matrix function (Hamiltonian) $H(l) = \gamma(l)^*\gamma(l)$ is recovered via the formula

$$H(l) = \left(\Pi_l^* S_l^{-1} \Pi_l\right)'. \tag{2.16}$$

From H we recover first

$$\gamma_2^{-1}\gamma_1 = \left(\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} H \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} H \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix}. \tag{2.17}$$

Using $\gamma_2^{-1}\gamma_1$, we recover γ_2 from the equation

$$\gamma_2' = \gamma_2(\gamma_2^{-1}\gamma_1)'(\gamma_2^{-1}\gamma_1)^*\left(I_{m_2} - (\gamma_2^{-1}\gamma_1)(\gamma_2^{-1}\gamma_1)^*\right)^{-1}, \quad \gamma_2(0) = I_{m_2}. \tag{2.18}$$

Finally, from $\gamma_2^{-1}\gamma_1$ and γ_2 the expression for γ is immediate.

We note that indices in the notations of operators (e.g., in the notations of operators Π_l and S_l above as well as in the notations S_x and S_l in Theorem 3.5) indicate the spaces, in which the operators act, and should be distinguished from the indices meaning partial differentiation like in (2.3).

2.3. Zero curvature representation as a compatibility condition

In this subsection we do not require any special structure of G and F (like the structure in (2.5)) and consider a general-type zero curvature representation (2.4). The term "compatibility condition" is very often used with respect to the representation (2.4) and it is easy to derive (2.4) from the existence of common fundamental solutions w (i.e., from the compatibility) of systems

$$w_x = Gw, \quad w_t = Fw. \tag{2.19}$$

In other words, it is easy to show that (2.4) is a necessary compatibility condition for systems (2.19). However, the proof of sufficiency is more complicated. It appeared first in [61, 62] and in greater detail and generality in [55, 60].

More precisely, assuming that (2.4) holds in the semistrip

$$\Omega_a = \{(x, t) : 0 \le x < \infty, \ 0 \le t < a\} \tag{2.20}$$

we introduce solutions of systems $W_x = GW$ with fixed values of t and of systems $R_t = FR$ with fixed values of x:

$$W_x(x, t, z) = G(x, t, z)W(x, t, z), \quad W(0, t, z) = I_m; \tag{2.21}$$
$$R_t(x, t, z) = F(x, t, z)R(x, t, z), \quad R(x, 0, z) = I_m. \tag{2.22}$$

Next, we formulate [60, Theorem 6.1].

Theorem 2.5. *Let $m \times m$ matrix functions G and F and their derivatives G_t and F_x exist on the semistrip Ω_a, let G, G_t, and F be continuous with respect to x and t on Ω_a, and let (2.4) hold. Then we have the equality*

$$W(x, t, z)R(0, t, z) = R(x, t, z)W(x, 0, z). \tag{2.23}$$

Thus, (2.4) implies (2.23), and so

$$w(x, t, z) := W(x, t, z)R(0, t, z) = R(x, t, z)W(x, 0, z) \tag{2.24}$$

is the fundamental solution of systems (2.19) normalized by the initial condition

$$w(0, 0, z) = I_m. \tag{2.25}$$

The problem of compatibility of the systems (2.19) may be formulated as an initial-boundary value problem. Indeed, normalizing w in (2.19) via (2.25) and taking into account (2.21) and (2.22) we obtain initial-boundary conditions on w:

$$w(x, 0, z) = W(x, 0, z), \quad w(0, t, z) = R(0, t, z). \tag{2.26}$$

From this point of view, formula (2.24) gives a solution of the initial-boundary value problem (2.19), (2.26). This solution plays a crucial role in the study of the evolution of Weyl functions.

2.4. Evolution of Weyl functions

Inverse Spectral Transform (instead of the Inverse Scattering Transform) was first used for initial-boundary value problems in [15, 36]. More precisely, a special kind of initial-boundary value problem for Toda lattice with a linear law of evolution of the spectral and Weyl functions was studied in [15, 36]. In the papers [61, 62], an essentially more general case of the initial-boundary value problem for Toda lattice as well as some initial-boundary value problems for continuous integrable systems (including square matrix dNLS) were dealt with, and the law of evolution of the Weyl function was presented in the form of Möbius transformation. See further results and references, for instance, in [48, 50, 60].

In this section we demonstrate Inverse Spectral Transform approach on the important case of dNLS (2.3), where v is an $m_1 \times m_2$ matrix function. For that purpose, we use W and R given by (2.21) and (2.22) assuming that G and F have the form (2.5). First, we omit t in the notations and introduce the set (Weyl circle) $\mathcal{N}(b, z)$ of functions of the form

$$\phi(b, z, \mathcal{P}) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} W(b, z)^{-1} \mathcal{P}(z) \Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W(b, z)^{-1} \mathcal{P}(z) \Big)^{-1}, \tag{2.27}$$

where $\mathcal{P}(z)$ are $m \times m_1$ nonsingular meromorphic matrix functions with property-j, that is,

$$\mathcal{P}(z)^* \mathcal{P}(z) > 0, \quad \mathcal{P}(z)^* j \mathcal{P}(z) \geq 0 \quad (z \in \mathbb{C}_+). \tag{2.28}$$

The next formula gives an important property of Weyl functions $\varphi(z)$:

$$\varphi(z) = \lim_{b \to \infty} \phi(b, z) \tag{2.29}$$

for any set of functions $\phi(b, z) \in \mathcal{N}(b, z)$. In order to derive an expression for the Weyl function $\varphi(t, z)$ of the system $y_x = G(x, t, z)y$, we rewrite (2.23) in the form of the equality

$$W(b, t, z)^{-1} = R(0, t, z)W(b, 0, z)^{-1} R(b, t, z)^{-1}$$

and substitute this equality into (2.27). Now, passing to the limit and using (2.29) we obtain the required expression for $\varphi(t, z)$. That is, we obtain the dependence of the Weyl functions on the parameter t (i.e., evolution of the Weyl function) in the case of dNLS.

Theorem 2.6. [58] Let an $m_1 \times m_2$ matrix function $v(x, t)$ be continuously differentiable on the semistrip Ω_a and let v_{xx} exist. Assume that v satisfies the dNLS equation (2.3) as well as the following inequalities (for all $0 \leq t < a$ and some values $M(t) \in \mathbb{R}_+$) :

$$\sup_{x \in \mathbb{R}_+, \, 0 \leq s \leq t} \| v(x, s) \| \leq M(t). \tag{2.30}$$

Then the evolution $\varphi(t, z)$ of the Weyl functions of Dirac systems $y_x(x, t, z) = G(x, t, z)y(x, t, z)$ is given (for $z \in \mathbb{C}_+$) by the equality

$$\varphi(t, z) = \big(R_{21}(t, z) + R_{22}(t, z)\varphi(0, z) \big) \big(R_{11}(t, z) + R_{12}(t, z)\varphi(0, z) \big)^{-1}, \tag{2.31}$$

where $R_{ik}(t, z)$ are the $m_i \times m_k$ blocks of $R(0, t, z)$.

3. Generalized Weyl functions

3.1. Skew-selfadjoint Dirac system

The system

$$\frac{d}{dx} y(x, z) = (\mathrm{i} z j + j V(x)) y(x, z) \qquad (x \geq 0, \quad z \in \mathbb{C}), \tag{3.1}$$

where j and V have the same form (2.2) as in the Dirac system (2.1), is called skew-selfadjoint Dirac. Weyl theory of skew-selfadjoint Dirac systems was studied in the papers [22,29,48,50]. Some further references as well as the results of this subsection are contained in [60, Ch.3]. Like in the case of selfadjoint Dirac system, the fundamental solution of system (3.1) is denoted by $u(x, z)$, and this solution is normalized by the condition (2.6). The notation $\mathbb{C}_M$ stands for the open half-plane $\{z : \Im(z) > M > 0\}$.

Definition 3.1. An $m_2 \times m_1$ matrix function φ, which is holomorphic in $\mathbb{C}_M$ (for some $M > 0$) and satisfies the inequality

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x, z)^* u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}_M, \tag{3.2}$$

is called a Weyl function of the skew-selfadjoint Dirac system (3.1).

First, assume that v is bounded, that is,

$$\|v(x)\| \le M \quad \text{for } x \in [0, \infty). \tag{3.3}$$

For $z \in \mathbb{C}_M$ (with M in $\mathbb{C}_M$ given in (3.3)), we have the following proposition.

Proposition 3.2. *Let Dirac system* (3.1) *be given on* $[0, \infty)$ *and assume that* (3.3) *holds. Then there is a unique Weyl function* $\varphi(z)$ *(for* $z \in \mathbb{C}_M$*) of this system. Moreover,* $\varphi(z)$ *is contractive in* $\mathbb{C}_M$ *and the inequality*

$$\sup_{x \le l,\, z \in \mathbb{C}_M} \left\| e^{-izx} u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \right\| < \infty \tag{3.4}$$

holds on any finite interval $[0, l]$.

Definition 3.3. A generalized Weyl function (GW-function) of the system (3.1), where v is locally bounded on $[0, \infty)$, is an $m_2 \times m_1$ matrix function φ such that for some $M > 0$ it is analytic in $\mathbb{C}_M$ and the inequality (3.4) holds for each $l < \infty$.

Our next proposition (together with Proposition 3.2) shows that, in the case of v bounded on the semiaxis, the definitions of Weyl function and GW-function are equivalent, that is, GW-function coincides with Weyl function.

Proposition 3.4. *For any system* (3.1)*, where* v *is locally bounded on* $[0, \infty)$*, there is no more than one GW-function.*

The following theorem (see [60, Theorem 3.30]) is necessary in the process of solving Goursat problem for the sine-Gordon equation in Section 4. The procedure to construct $\mathfrak{M}(\varphi)$ (solve inverse problem) given in this theorem is in many respects similar to the corresponding procedure for the selfadjoint Dirac system but we do not require here apriori that φ is a GW-function. We note that β and γ stand again for the block rows of u, that is, (2.8) holds.

Theorem 3.5. *Let an* $m_2 \times m_1$ *matrix function* $\varphi(z)$ *be holomorphic in* $\mathbb{C}_M$ *(for some* $M > 0$*) and satisfy condition*

$$\sup \|z^2(\varphi(z) - \phi_0/z)\| < \infty \quad (z \in \mathbb{C}_M), \tag{3.5}$$

where ϕ_0 *is an* $m_2 \times m_1$ *matrix. Then* φ *is a GW-function of a skew–selfadjoint Dirac system. This Dirac system is uniquely recovered from* φ *using the following procedure.*

(i) First recover the matrix function Φ_1 :

$$\Phi_1\left(\frac{x}{2}\right) = \frac{1}{\pi} e^{\eta x} \text{l.i.m.}_{a \to \infty} \int_{-a}^{a} e^{-i\xi x} \frac{\varphi(\xi + i\eta)}{2i(\xi + i\eta)} d\xi, \quad \eta > M, \tag{3.6}$$

where l.i.m. *stands for the entrywise limit in the norm* $L^2(0, \ell)$ $(0 < \ell \le \infty)$.
(ii) Next, for the values $l \in \mathbb{R}_+$*, introduce operators* $S_l \in B\left(L^2(0, l)\right)$:

$$S_l = I + \int_0^l s(x, t) \cdot dt, \quad s(x, t) = \int_0^{\min(x, t)} \Phi_1'(x - r)\Phi_1'(t - r)^* dr. \tag{3.7}$$

These operators are well-defined, positive definite and boundedly invertible.
(iii) We recover $\beta(x)$ *via the formula* :

$$\beta(x) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} - \int_0^x \left(S_x^{-1}\Phi_1'\right)(t)^* \begin{bmatrix} \Phi_1(t) & I_{m_2} \end{bmatrix} dt, \tag{3.8}$$

where S_x^{-1} is applied to Φ_1' columnwise. Since $\beta\beta^* \equiv I_{m_1}$, we can construct a differentiable matrix function $\widetilde{\gamma}$ (with locally bounded derivative) such that

$$\beta\widetilde{\gamma}^* \equiv 0, \quad \widetilde{\gamma}\widetilde{\gamma}^* > 0, \quad \widetilde{\gamma}(0) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}. \tag{3.9}$$

Then $\gamma = \widetilde{\vartheta}\widetilde{\gamma}$, where $\widetilde{\vartheta}$ is determined by the equation and initial condition below:

$$\widetilde{\vartheta}' = -\widetilde{\vartheta}\widetilde{\gamma}'\widetilde{\gamma}^*(\widetilde{\gamma}\widetilde{\gamma}^*)^{-1}, \quad \widetilde{\vartheta}(0) = I_{m_2}. \tag{3.10}$$

$(iiii)$ Finally, v is given by

$$v(x) = \beta'(x)\gamma(x)^*, \tag{3.11}$$

V has the form (2.2), and both v and V are locally bounded.

3.2. Linear system auxiliary to the N-wave equation

Nonlinear optics (N-wave) equation

$$\big[D, \varrho_t\big] - \big[\widehat{D}, \varrho_x\big] = [D, \varrho]\,[\widehat{D}, \varrho] - [\widehat{D}, \varrho]\,[D, \varrho], \tag{3.12}$$

where $\varrho(x,t) = \varrho(x,t)^*$ is an $m \times m$ matrix function, $[D, \varrho] = D\varrho - \varrho D$ and

$$D = \mathrm{diag}\{d_1, d_2, \ldots, d_m\}, \quad d_1 > d_2 > \ldots > d_m > 0, \tag{3.13}$$
$$\widehat{D} = \widehat{D}^* = \mathrm{diag}\{\widehat{d}_1, \widehat{d}_2, \ldots, \widehat{d}_m\}, \tag{3.14}$$

admits zero curvature representation (2.4), where G and F have the form

$$G(x,t,z) = \mathrm{i}zD - \zeta(x,t), \quad F(x,t,z) = \mathrm{i}z\widehat{D} - \widehat{\zeta}(x,t); \tag{3.15}$$
$$\zeta := [D, \varrho], \quad \widehat{\zeta} := [\widehat{D}, \varrho]; \tag{3.16}$$

see [64] for the case $N = 3$ and [1] for $N > 3$.

We shall need some preliminary results on the Weyl theory of the auxiliary system $y_x = Gy$ ($x \geq 0$) from [60, Ch. 4], see also [49,53]. The normalized fundamental solution u of such system is defined by the formula

$$\frac{d}{dx}u(x,z) = \big(\mathrm{i}zD - \zeta(x)\big)u(x,z), \quad u(0,z) = I_m \quad (\zeta = -\zeta^*). \tag{3.17}$$

Here and further we assume that D is a fixed matrix satisfying (3.13).

Definition 3.6. A generalized Weyl function (GW-function) of system (3.17), where ζ is locally bounded, is an $m \times m$ matrix function φ such that for some $M > 0$ it is analytic in the domain $\mathbb{C}_M^- = \{z : \Im(z) < -M\}$ and the inequality

$$\sup_{x \leq l,\ \Im(z) < -M} \big\| u(x,z)\varphi(z)\exp\{-\mathrm{i}zxD\} \big\| < \infty \tag{3.18}$$

holds for each $l < \infty$.

The inverse spectral problem for system (3.17) is the problem to recover (from an analytic matrix function $\varphi(z)$) a locally bounded potential $\zeta(x) = -\zeta(x)^*$ ($\zeta_{kk} \equiv 0$) such that φ is a GW-function of the corresponding system (3.17), that is, (3.18) is valid. The notation $\mathfrak{M}$ stands for the solution of this inverse problem, that is, for an operator mapping the pair D and φ into the corresponding potential ζ (i.e., $\mathfrak{M}(D, \varphi) = \zeta$).

Theorem 3.7. *For any matrix function $\varphi(z)$ which is analytic and bounded in $\mathbb{C}_M^- = \{z : \Im(z) < -M\}$ and has the property*

$$\int_{-\infty}^{\infty} \left(\varphi(\xi + \mathrm{i}\eta) - I_m\right)^* \left(\varphi(\xi + \mathrm{i}\eta) - I_m\right) d\xi < \infty \qquad (\eta < -M), \tag{3.19}$$

there is at most one solution of the inverse spectral problem.

An existence condition (for the solution of the inverse spectral problem) is given in the next theorem.

Theorem 3.8. *Let $\varphi(z)$ be analytic in $\mathbb{C}_M^-$ for some $M > 0$ and satisfy in $\mathbb{C}_M^-$ the inequalities*

$$\sup_{z \in \mathbb{C}_M^-} \left\| z(\varphi(z) - I_m) \right\| < \infty, \quad \det \varphi(z) \neq 0. \tag{3.20}$$

Assume also that for some matrix ϕ_0 and for all fixed $\eta < -M$ we have

$$(\xi + \mathrm{i}\eta)\left(\varphi(\xi + \mathrm{i}\eta) - I_m - \phi_0/(\xi + \mathrm{i}\eta)\right) \in L_{m\times m}^2(-\infty, \infty). \tag{3.21}$$

Then the solution of the inverse problem exists and is unique.

Remark 3.9. The procedure to construct $\mathfrak{M}(D, \varphi)$ under conditions of Theorem 3.8 is given in [60, Theorem 4.10]. This procedure is similar to the procedures in Theorems 2.4 and 3.5.

Further in this paper, we deal with the case of ζ bounded on $[0, \infty)$:

$$\sup_{0 < x < \infty} \left\| \zeta(x) \right\| < \infty, \tag{3.22}$$

and GW-functions $\varphi(z) = \{\varphi_{ij}(z)\}_{i,j=1}^m$ normalized by

$$\varphi_{ij}(z) \equiv 1 \quad \text{for} \quad i = j, \qquad \varphi_{ij}(z) \equiv 0 \quad \text{for} \quad i > j. \tag{3.23}$$

The next proposition follows from [60, Subsections 4.1.1 and 4.1.3].

Proposition 3.10. *When (3.22) holds, a normalized (by (3.23)) GW-function $\varphi(z)$ of (3.17) exists and is unique.*

4. Sine-Gordon theory in a semistrip

1. Sine–Gordon equation in the light cone coordinates (SGE) has the form

$$\frac{\partial^2}{\partial t \partial x}\psi = 2\sin(2\psi). \tag{4.1}$$

Local solutions of the Goursat problem for SGE were studied in [40, 42] and global solutions were constructed first in [50]. The results of this section are obtained mostly in [50] with some modifications and developments in [60, Section 6.2] (see also further references in [60]). SGE admits zero curvature representation (2.4), where G and F have the form [2]:

$$G(x, t, z) = \mathrm{i}zj + jV(x, t), \quad V = \begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix}, \quad v = -\frac{\partial}{\partial x}\psi, \quad \psi = \overline{\psi}, \tag{4.2}$$

$$F(x, t, z) = \frac{1}{\mathrm{i}z} \begin{bmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{bmatrix}, \tag{4.3}$$

and j has the form (2.2) with $m_1 = m_2 = 1$. Hence, system $y_x = Gy$ is a skew-selfadjoint Dirac system (3.1), where $m_1 = m_2 = 1$. The evolution $\varphi(t, z)$ of its GW-function is given by the formula

$$\varphi(t, z) = \frac{R_{21}(t, z) + R_{22}(t, z)\varphi(0, z)}{R_{11}(t, z) + R_{12}(t, z)\varphi(0, z)}, \tag{4.4}$$

where, as usual, $R_{ik}(t, z)$ are the blocks of $R(0, t, z)$ determined by (2.22). If the boundary value $\psi(0, t)$ is given, using (2.22) and (4.3) we recover $R_{ik}(t, z)$. Let us formulate the evolution result rigorously.

Theorem 4.1. *Let $\psi(0, t)$ and $\psi_x(x, t)$ be continuous functions on $[0, a)$ and Ω_a, respectively, and let $\psi_{xt}(x, t)$ exist. Let also SGE (4.1) hold on Ω_a, and assume that $\varphi(0, z)$ is the GW-function of the system $y_x = G(x, 0, z)y$, where G is given by (4.2).*

Then, the function $\varphi(t, z)$ of the form (4.4) is the GW-function of the system $y_x = G(x, t, z)y$.

If the initial–boundary conditions

$$\psi(x, 0) = h_1(x), \quad \psi(0, t) = h_2(t), \quad h_1(0) = h_2(0) \tag{4.5}$$

($h_k = \overline{h_k}$ for $(k = 1, 2)$), are given, we recover $\varphi(0, z)$ from the first condition, recover $R_{ik}(t, z)$ from the second condition, and construct $\varphi(t, z)$ using (4.4). Then we recover $v(x, t)$ using the procedure to construct the solution $\mathfrak{M}(\varphi)$ of the inverse problem, which is described in Theorem 3.5. From v we immediately recover ψ and show that ψ satisfies (4.1) and (4.5) (see the proof of [60, Theorem 6.19]). Thus, an existence theorem follows.

Theorem 4.2. *Assume that h_1 is boundedly differentiable on all the finite intervals on $[0, \infty)$ and that h_2 is continuous on $[0, a)$.*

Moreover, assume that the GW-function $\varphi(0, z)$ of the system (3.1), where

$$m_1 = m_2 = 1, \quad V(x) = - \begin{bmatrix} 0 & h_1'(x) \\ h_1'(x) & 0 \end{bmatrix}, \tag{4.6}$$

exists and satisfies (3.5). Then a solution of the initial-boundary value problem (4.1), (4.5) exists and is given by the equality

$$\psi(x, t) = h_2(t) - \int_0^x \Big(\mathfrak{M}\big(\varphi(t, z)\big) \Big)(\xi)d\xi, \tag{4.7}$$

where $\varphi(t, z)$ is obtained from (4.4).

Remark 4.3. Under conditions of Theorem 4.2, for each $0 < c < a$ there is $M(c) > 0$ such that all functions $\varphi(t, z)$ $(0 \le t \le c)$ satisfy GW-function requirement (3.4) and asymptotic inequality (3.5) in the same half-plane $\mathbb{C}_{M(c)}$.

Some sufficient conditions on h_1, under which the requirements on $\varphi(0, z)$ in Theorem 4.2 hold, were derived using important paper [11] (see [60, Corollary 6.21]). We formulate these conditions.

Corollary 4.4. *If $v \in L^1(\mathbb{R}_+)$, then there is a GW-function φ of the system (3.1) (where V has the form (2.2) and $m_1 = m_2 = 1$). Moreover, if v is two times differentiable and v, v', $v'' \in L^1(\mathbb{R}_+)$, then this GW-function φ satisfies the asymptotic condition (3.5).*

2. Complex sine-Gordon equation was introduced (and its integrability was treated) [43, 47] only several years after the seminal paper [2] on the integrability of SGE was published. Complex sine-Gordon equation is more general than SGE and has the form

$$\psi_{xt} + 4\cos\psi(\sin\psi)^{-3}\chi_x\chi_t = 2\sin(2\psi), \quad \chi_{xt} = (2/\sin(2\psi))(\psi_x\chi_t + \psi_t\chi_x), \tag{4.8}$$

where $\psi = \overline{\psi}$ and $\chi = \overline{\chi}$. There are also two constraint equations

$$2(\cos\psi)^2\chi_x - (\sin\psi)^2\omega_x = 2c(\sin\psi)^2, \quad 2(\cos\psi)^2\chi_t + (\sin\psi)^2\omega_t = 0, \tag{4.9}$$

where c is a constant ($c = \bar{c} \equiv$ const) and $\omega = \bar{\omega}$. For further developments and applications of the results on (4.8), (4.9) see, for instance, some discussions and references in [10, 20, 60]. Below we formulate several results from [60, Subsection 6.2.1].

If $\sin(2\psi) \neq 0$ and (4.8) and (4.9) hold, then zero curvature equation (2.4), where

$$G(x,t,z) := \mathrm{i}zj + jV(x,t), \quad F(x,t,z) := -\frac{\mathrm{i}}{z+c}\vartheta(x,t)^* j\vartheta(x,t), \tag{4.10}$$

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ \bar{v} & 0 \end{bmatrix}, \quad v = \left(-\mathrm{i}\frac{\partial \psi}{\partial x} + 2(\cot \psi)\frac{\partial \chi}{\partial x}\right)\mathrm{e}^{\mathrm{i}(\omega - 2\chi)}, \tag{4.11}$$

$$\vartheta(x,t) := D_1(x,t)\begin{bmatrix} \cos \psi & \mathrm{i}\sin \psi \\ \mathrm{i}\sin \psi & \cos \psi \end{bmatrix} D_2(x,t), \tag{4.12}$$

$$D_1 = \exp\left\{\mathrm{i}\big(\chi + (\omega/2)\big)j\right\}, \quad D_2 = \exp\left\{\mathrm{i}\big(\chi - (\omega/2)\big)j\right\}, \tag{4.13}$$

also holds. Now, we introduce initial-boundary conditions:

$$v(x,0) = h_1(x), \quad \psi(0,t) = h_2(t), \quad \chi(0,t) = h_3(t), \quad \omega(0,0) = h_4. \tag{4.14}$$

Evolution of the GW-function φ is given in the next theorem.

Theorem 4.5. *Let $\{\psi(x,t),\, \chi(x,t),\, \omega(x,t)\}$ be a triple of real-valued and twice continuously differentiable functions on Ω_a. Assume that $\sin(2\psi) \neq 0$, that v given by (4.11) is bounded on Ω_a, and that complex sine-Gordon (4.8), (4.9) and conditions (4.14) hold.*

Then the GW-functions $\varphi(t,z)$ of the auxiliary skew-selfadjoint Dirac systems $y_x = Gy$, where $G(x,t,z)$ is defined via (4.10) and (4.11), exist in some $\mathbb{C}_M$ and have the form (4.4).

Here $R(t,z) = R(0,t,z) = \big\{R_{ik}(t,z)\big\}_{i,k=1}^2$ is defined by the equalities

$$\frac{d}{dt}R(t,z) = \frac{1}{\mathrm{i}(z+c)}\mathrm{e}^{-\mathrm{i}d(t)j}\begin{bmatrix} \cos(2h_2(t)) & \mathrm{i}\sin(2h_2(t)) \\ -\mathrm{i}\sin(2h_2(t)) & -\cos(2h_2(t)) \end{bmatrix}\mathrm{e}^{\mathrm{i}d(t)j}R(t,z),$$

$$R(0,z) = I_2, \quad d(t) := h_3(0) - \frac{1}{2}h_4 + \int_0^t h_3'(\xi)\big(\sin h_2(\xi)\big)^{-2}d\xi;$$

and $\varphi(0,z)$ is the GW-function of (3.1) where $m_1 = m_2 = 1$ and $v = h_1$.

Corollary 4.6. *There is at most one triple $\{\psi(x,t),\, \chi(x,t),\, \omega(x,t)\}$ of real-valued and twice continuously differentiable functions on Ω_a such that $\sin(2\psi) \neq 0$, that v is bounded, and that complex sine-Gordon (4.8), constraints (4.9) and initial-boundary conditions (4.14) are satisfied.*

5. Reduction of the initial-boundary conditions and unbounded solutions

5.1. Reduction of the initial-boundary conditions in a quarter-plane

1. In this subsection we consider integrable wave equations in the quarter-plane

$$\Omega_\infty = \{(x,\,t) : 0 \le x < \infty,\, 0 \le t < \infty\}. \tag{5.1}$$

First, we consider the defocusing NLS (2.3) with the initial-boundary conditions

$$v(x,0) = h_1(x), \quad v(0,t) = h_2(t), \quad v_x(0,t) = h_3(t). \tag{5.2}$$

Given $h_1(x)$, we recover $\varphi(0,z)$ via (2.29) and given $h_2(t)$ and $h_3(t)$ we recover $R(0,t,z)$ (i.e., $R_{ik}(t,z)$ where $i,k = 1,2$) using (2.22). Hence, given the initial-boundary conditions h_k ($k = 1,2,3$) we recover the right-hand side of (2.31) and obtain the evolution $\varphi(t,z)$ of the Weyl function.

Theorem 5.1. *[58] Let v satisfy the conditions of Theorem 2.6 and boundary conditions given by the second and third equalities in (5.2). Moreover, let the boundary value functions h_2 and h_3 be continuous and bounded, i.e.,*

$$\sup_{0 \leq t < \infty} \|h_2(t)\| < \widehat{M}, \quad \sup_{0 \leq t < \infty} \|h_3(t)\| < \check{M} \tag{5.3}$$

for some $\widehat{M}, \check{M} \in \mathbb{R}_+$. Then, in the domain

$$\mathcal{D} = \{z : \quad \Im(z) \geq 1/2, \quad \Re(z) \leq -\widehat{M}\} \tag{5.4}$$

we have the equality

$$\varphi(0, z) = -\lim_{t \to \infty} R_{22}(t, z)^{-1} R_{21}(t, z). \tag{5.5}$$

Scheme of the proof. The contractiveness of $\varphi(t, z)$ yields

$$\begin{bmatrix} I_{m_1} & \varphi(t, z)^* \end{bmatrix} j \begin{bmatrix} I_{m_1} \\ \varphi(t, z) \end{bmatrix} \geq 0. \tag{5.6}$$

Putting $R(t, z) := R(0, t, z)$ and using formula (2.31) for $\varphi(t, z)$, we derive

$$R(t, z) \begin{bmatrix} I_{m_1} \\ \varphi(0, z) \end{bmatrix} = \begin{bmatrix} I_{m_1} \\ \varphi(t, z) \end{bmatrix} (R_{11}(t, z) + R_{12}(t, z)\varphi(0, z)). \tag{5.7}$$

It is immediate from (5.6) and (5.7) that

$$\begin{bmatrix} I_{m_1} & \varphi(0, z)^* \end{bmatrix} R(t, z)^* j R(t, z) \begin{bmatrix} I_{m_1} \\ \varphi(0, z) \end{bmatrix} \geq 0. \tag{5.8}$$

Moreover, the first inequality in (5.3) implies that

$$\frac{d}{dt}\big(- R(t, z)^* j R(t, z) \big) \geq \widehat{M} R(t, z)^* R(t, z), \quad z \in \mathcal{D}. \tag{5.9}$$

From (5.8) and (5.9) we see that

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(0, z)^* \end{bmatrix} R(s, z)^* R(s, z) \begin{bmatrix} I_{m_1} \\ \varphi(0, z) \end{bmatrix} ds \leq (1/\widehat{M}) I_{m_1}. \tag{5.10}$$

After some considerations, using the boundedness of the left-hand side of (5.10) and the inequalities (5.3), one can show that

$$\lim_{t \to \infty} \left\| R(t, z) \begin{bmatrix} I_{m_1} \\ \varphi(0, z) \end{bmatrix} \right\| = 0. \tag{5.11}$$

On the other hand, it easily follows from (5.9) that

$$R_{22}(t, z)^* R_{22}(t, z) \geq I_{m_2}.$$

Therefore, formula (5.11) implies (5.5). An analogue of (5.5) for a scalar dNLS (and with a somewhat different proof) appeared already in [61].

In view of analyticity of Weyl functions, formula (5.5) means that $\varphi(0, z)$, $\varphi(t, z)$, and thus also $v(x, t)$, may be recovered from the boundary conditions.

2. A similar result is valid for the focusing nonlinear Schrödinger equation:

$$2v_t + \mathrm{i}(v_{xx} + 2vv^*v) = 0, \tag{5.12}$$

where v is an $m_1 \times m_2$ matrix function. Equation (5.12) admits [66,67] representation (2.4) where

$$G = izj + jV, \quad F = i\big(z^2 j - izjV - (V_x + jV^2)/2\big), \tag{5.13}$$

and j and V are defined in (2.2). Evolution of the Weyl function is described in this case by the following theorem.

Theorem 5.2. *[29] Let an $m_1 \times m_2$ matrix function $v(x,t)$ be continuously differentiable on Ω_a and let v_{xx} exist. Assume that v satisfies the focusing nonlinear Schrödinger equation (5.12) as well as the following inequalities :*

$$\sup_{(x,t)\in\Omega_a} \|v(x,t)\| \le M, \qquad \sup_{(x,t)\in\Omega_c} \|v_x(x,t)\| < \infty \quad \text{for each} \quad 0 < c < a.$$

Then, the evolution $\varphi(t,z)$ of the GW-functions of the skew-selfadjoint Dirac systems $y_x = Gy$ is given (for $z \in \mathbb{C}_M$) via the Möbius transformation (2.31), where the coefficients R_{ik} are determined by the formula (2.22) and the equality $\{R_{ik}(t,z)\}_{i,k=1}^2 := R(0,t,z)$. Here G and F have the form (5.13).

Remark 5.3. It is immediate that if the conditions of Theorem 5.2 hold for each $a < \infty$, then (2.4) describes the evolution of $\varphi(t,z)$ for all $0 \le t < \infty$.

In order to derive an analogue of Theorem 5.1 we shall require boundedness of v on Ω_∞ (instead of boundedness on all Ω_a) and boundedness of $v_x(0,t)$ on $\mathbb{R}_+$:

$$\sup_{(x,t)\in\Omega_\infty} \|v(x,t)\| \le M, \qquad \sup_{t\in\mathbb{R}_+} \|v_x(0,t)\| < \infty. \tag{5.14}$$

Then, using (2.22) (see, e.g., [29, formula (4.14)]), we derive inequality (5.9) for $R(t,z) = R(0,t,z)$ in a domain

$$\mathcal{D} = \{z : \quad \Im(z) > M_1, \quad \Re(z) > M_2\}, \tag{5.15}$$

and with some $\widehat{M} > 0$, $M_1 > M$, $M_2 > 0$. After that, the same considerations as in the scheme of the proof of Theorem 5.1 yield our next theorem.

Theorem 5.4. *Let v satisfy the conditions of Theorem 5.2 (for each $a \in \mathbb{R}_+$) and inequalities (5.14). Then, in some domain (5.15) we have the equality*
$\varphi(0,z) = -\lim_{t\to\infty} R_{22}(t,z)^{-1} R_{21}(t,z).$

3. Finally, let us consider SGE (4.1), where $v = -\psi_x$ is bounded in Ω_∞:

$$\sup_{(x,t)\in\Omega_\infty} |\psi_x(x,t)| \le M. \tag{5.16}$$

The next proposition follows (see [60, Corollary 6.25]) from Theorem 4.1.

Proposition 5.5. *Let ψ given in Ω_∞ satisfy inequality (5.16). Assume also that the conditions of Theorem 4.1 and initial-boundary conditions (4.5) hold for each Ω_a ($a \in \mathbb{R}_+$). Then, for values of $z \in \mathbb{C}_M$, such that the inequalities*

$$\big(\cos(2h_2(t)) - \varepsilon(z)\big)\Im(z) \ge |\Re(z)\sin(2h_2(t))| \tag{5.17}$$

are valid for some $\varepsilon(z) > 0$ and for all $t \ge 0$, we have

$$\varphi(0,z) = -\lim_{t\to\infty} R_{21}(t,z)/R_{22}(t,z). \tag{5.18}$$

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5.2. Unbounded solutions of SGE

Theorem 4.2, Corollary 4.4 and Proposition 5.5 allow us to construct wide classes of unbounded (in Ω_∞) solutions of SGE, that is, solutions which do not satisfy (5.16). The simplest case is the case $h_2(t) \equiv 0$ (see [60, Proposition 6.27]). In this case, (5.17) holds for all $z \in \mathbb{C}_M$ ($M > 0$) and, if other conditions of Proposition 5.5 are valid, (5.18) yields $\varphi(0, z) \equiv 0$. Since $\mathfrak{M}(0) \equiv 0$, we derive $h_1 \equiv 0$. In other words, any solution of SGE with $h_2(t) \equiv 0$ and $h_1(x) \not\equiv 0$ does not satisfy conditions of Proposition 5.5. Solutions of initial-boundary value problems for SGE may be constructed using Theorem 4.2 and Corollary 4.4.

Proposition 5.6. *Assume that $h_1(x) = \overline{h_1(x)} \not\equiv 0$ is three times differentiable for $x \geq 0$, that*

$$h_1', \ h_1'', \ h_1''' \in L^1(\mathbb{R}_+), \quad h_1(0) = 0,$$

and that $h_2 \equiv 0$. Then one can use the procedure given in Theorem 4.2 in order to construct a solution ψ of the initial-boundary value problem (4.1), (4.5) for SGE, and the absolute value of the derivative $\frac{\partial}{\partial x}\psi$ (for the constructed ψ) is always unbounded in the quarter-plane Ω_∞.

Indeed, the solution constructed in Proposition 5.6 satisfies the conditions of Proposition 5.5 excluding, possibly, (5.16), and so (5.16) does not hold.

Some classes of unbounded solutions of the KdV equation with the minus sign before the dispersion term are constructed in [54] using low energy asymptotics of the Weyl functions.

5.3. Reduction of the initial-boundary conditions in a semistrip

1. In this paragraph we present some results from [57, Section 4] on the N-wave equation (3.12) in the semistrip Ω_a. By $\varphi(t, z)$ we denote evolution of the GW-function corresponding to this equation or, more precisely, $\varphi(t, z)$ are GW-functions of the systems $y_x = G(x, t, z)y$, where G is expressed via ϱ using (3.15) and (3.16). The matrix function $R(t, z) = R(0, t, z)$ is determined via (2.22), where F (similar to G) is given by (3.15) and (3.16). First, we express normalized (by (3.23)) GW-functions $\varphi(t, z)$ via $\varphi(0, z)$ and $R(t, z)$ (in other words, we derive the evolution of the normalized GW-functions). Next, we consider the subcase when the entries of $\widehat{D}$ are ordered in the same way as the entries of D:

$$\widehat{D} = \text{diag}\{\widehat{d}_1, \ \widehat{d}_2, \ldots, \widehat{d}_m\}, \quad \widehat{d}_1 > \widehat{d}_2 > \ldots > \widehat{d}_m > 0. \tag{5.19}$$

Theorem 5.7. *Let $\varrho = \varrho^*$ satisfy the N-wave equation (3.12), where D and $\widehat{D}$ have the form (3.13) and (3.14), respectively. Assume that $\varrho(x, t)$ is uniformly bounded and continuously differentiable on Ω_a.*
  Then the matrix functions

$$\psi_k(t, z) := \begin{bmatrix} I_k & 0 \end{bmatrix} R(t, z)\varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix}$$

$$\times \left(\begin{bmatrix} 0 & I_{m-k} \end{bmatrix} R(t, z)\varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \right)^{-1}, \tag{5.20}$$

where $\varphi(0, z)$ is the normalized GW-function of the system $y_x = G(x, 0, z)$, are well-defined for $1 \leq k < m$. The normalized GW-functions $\varphi(t, z)$ are given (in $\mathbb{C}_M^-$ for some $M > 0$) by the formula

$$\{\varphi_{i,k+1}(t, z)\}_{i=1}^k = \psi_k(t, z) \begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{bmatrix} \quad (z \in \mathbb{C}_M^-), \tag{5.21}$$

and by the normalization conditions (3.23).

From Theorems 3.7 and 5.7 follows the next result.

Theorem 5.8. *For the case where the entries of the matrix $\widehat{D}$ in (3.12) are positive and ordered as in (5.19), there is no more than one uniformly bounded and continuously differentiable on Ω_a solution $\varrho = \varrho^*$ (of the N-wave equation (3.12)), having the initial values $\varrho(x, 0)$ such that $\varphi(0, z)$ is bounded and (3.19) holds. That is, there is no more than one solution of the corresponding initial value problem.*

Taking into account the results on fundamental solutions from [11], the requirements on $\varphi(0, z)$ given in Theorem 5.8 are reformulated below in terms of the sufficient requirements on the initial condition

$$\varrho(x, 0) = \rho(x) = \rho(x)^* \quad (0 \leq x < \infty). \tag{5.22}$$

Proposition 5.9. *Suppose that the initial condition $\rho(x)$ is absolutely continuous on $[0, \infty)$ and the entries of $\rho(x)$ and $\rho'(x)$ belong $L^1(\mathbb{R}_+)$. Then, the normalized GW-function $\varphi(0, z)$ of the system*

$$y'(x, z) = (\mathrm{i}zD - \zeta(x))y(x, z) \quad (x \geq 0), \quad \zeta = [D, \rho] \tag{5.23}$$

is analytic and bounded in $\mathbb{C}_M^-$ (for some $M > 0$) and (3.19) is valid.

Theorem 5.8 is proved in [57, Section 4] using the fact that

$$\sup_{t \in [0, a),\, \Im(z) < -M} \|R(t, z)\varphi(0, z)\exp\{-\mathrm{i}zt\widehat{D}\}\| < \infty, \tag{5.24}$$

which yields (see [60, pp. 108, 109]) that $\varphi(0, z)$ determines not only the initial but also the boundary values of ϱ. In particular, when $\varphi(0, z)$ satisfies the conditions of Theorem 3.8, then, according to Remark 3.9, we have a procedure to recover the initial and boundary conditions from $\varphi(0, z)$. More precisely, we have

$$\zeta(x, 0) = \mathfrak{M}(D, \varphi(0, z)), \quad \widehat{\zeta}(0, t) = \mathfrak{M}(\widehat{D}, \varphi(0, z)), \tag{5.25}$$

and, using (3.16), we easily obtain $\varrho(x, 0)$ and $\varrho(0, t)$. Indeed, without loss of generality we may assume that all the entries of ϱ on the main diagonal are equal to zero and recover $\varrho = \{\varrho_{ik}\}_{i,k=1}^m$ from ζ or $\widehat{\zeta}$ via formulas

$$\varrho_{ii} \equiv 0, \quad \varrho_{ik} = \zeta_{ik}/(d_i - d_k) = \widehat{\zeta}_{ik}/(\widehat{d}_i - \widehat{d}_k) \text{ for } i \neq k.$$

2. In this and the next paragraph we deal with the cases of dNLS (2.3) with quasi-analytic boundary or initial, respectively, conditions, (i.e., the corresponding boundary or initial value functions belong to quasi-analytic classes). The definition of quasi-analytic classes $C([0, \ell]; \widetilde{M})$ is given in Appendix A.

Further we present some results from [57, Section 3]. Let us consider $m_1 \times m_2$ matrix functions $v(x, t)$, which are continuously differentiable and such that v_{xx} exists on the semi-strip Ω_a. Moreover, we require some smoothness of v in the neighborhood of the point $(0, 0)$. More precisely, we require that for each k there is a value $\varepsilon_k = \varepsilon_k(v) > 0$ such that v is k times differentiable with respect to x in the square

$$\mathcal{D}(\varepsilon_k) = \{(x, t) : \quad 0 \leq x \leq \varepsilon_k, \quad 0 \leq t \leq \varepsilon_k\}, \quad \mathcal{D}(\varepsilon_k) \subset \Omega_a. \tag{5.26}$$

The class of such functions $v(x, t)$ is denoted by $C_\varepsilon(\Omega_a)$.

Proposition 5.10. *Assume that $v(x, t) \in C_\varepsilon(\Omega_a)$ satisfies the dNLS equation (2.3) on Ω_a and that a matrix function $v(0, t)$ or $v_x(0, t)$ is quasi-analytic.*

Then a matrix function $v(0, t)$ or $v_x(0, t)$, respectively, is uniquely determined by the initial condition

$$v(x, 0) = h(x). \tag{5.27}$$

Proposition 5.10 is proved by presenting formulas to recover the derivatives $\left(\frac{\partial^k}{\partial t^k}v\right)(0,0)$ and $\left(\frac{\partial^k}{\partial t^k}v_x\right)(0,0)$ $(k \geq 0)$ from $h(x)$ (see the proof of [57, Proposition 3.2]).

Using Proposition 2.2, Theorems 2.4 and 2.6 and Proposition 5.10 we derive the next theorem.

Theorem 5.11. *Assume that $v \in C_\varepsilon(\Omega_a)$ satisfies the dNLS equation (2.3) on Ω_a, that (2.30) holds and that the functions $v(0,t)$ and $v_x(0,t)$ (boundary values) belong to some quasi-analytic classes $C([0,a]; \widetilde{M})$ and $C([0,a]; \widetilde{M}^+)$, respectively. Then v is uniquely determined by the initial condition (5.27).*

3. Using Proposition 2.2 and Theorems 2.4 and 2.6 one can also derive Theorem 4.3 from [58], which we formulate in this paragraph. The proof is based on the formulas to recover the derivatives $\left(\frac{\partial^k}{\partial x^k}v\right)(0,0)$ $(k \geq 0)$ from the boundary conditions

$$v(0,t) = h_0(t), \quad v_x(0,t) = h_1(t) \quad (0 \leq t < a). \tag{5.28}$$

Theorem 5.12. *Assume that $v \in C_\varepsilon(\Omega_a)$ satisfies the dNLS equation (2.3) on Ω_a, that (2.30) holds and that the initial value function $v(x,0)$ belongs to some quasi-analytic class $C([0,\infty); \widetilde{M})$. Then v is uniquely determined by the boundary conditions (5.28).*

We note that the case of quasi-analytic boundary (or initial) conditions is important also because related suggestions that initial and boundary conditions (or even solutions) belong to the so called Schwartz class of functions are often used for simplicity (see, e.g., [27]). Our result shows that one should be rather careful with such suggestions, so that they agree with the established interrelations between initial and boundary conditions.

6. Dynamical Dirac system and response function

6.1. Introduction

In this section we give several statements from our paper [59], which appeared in arXiv in 2015. Classical Dirac systems (2.1) are also called spectral Dirac systems and various new results on inverse problems for these systems appeared quite recently (see some references in Subsection 2.1). At the same time, a great and growing interest in dynamical systems and control methods is reflected in the active study of the dynamical inverse problems and, in particular, in the study of the inverse problems for dynamical Schrödinger and Dirac systems [7, 8, 13, 14, 34, 44] (see also the references therein).

Dynamical Dirac system (Dirac system in the *time-domain setup*) was studied in the important recent paper [14]. The dynamical Dirac system considered in [14] is an evolution system of hyperbolic type and has the following form:

$$iY_t + JY_x + \mathcal{V}Y = 0 \quad (x > 0, \quad t > 0); \tag{6.1}$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}, \quad Y_t := \frac{\partial Y}{\partial t}, \tag{6.2}$$

where $p = p(x)$ and $q = q(x)$ are real-valued functions of x, and initial-boundary conditions are given by the equalities

$$Y(x,0) = 0, \ x \geq 0; \quad Y_1(0,t) = f(t), \ t \geq 0. \tag{6.3}$$

Here f is a complex-valued function (called boundary control by [14]) and the input-output map (response operator) $R : Y_1(0,\cdot) \to Y_2(0,\cdot)$ is of the convolution form $Rf = if + r * f$. Thus, R and r denote in this section the response operator and response function, respectively. ($\widehat{Y}$ stands in this section for the Fourier transformation of Y, see (6.16).)

The inverse problem consists in recovery of the potential $\mathcal{V}$ from the *response function r*. This inverse problem was considered in [14] using boundary control methods.

We note that recent results on dynamical Schrödinger and Dirac equations are based on several earlier works. In his paper [17] from 1971, A.S. Bla-goveščenskii considered dynamical system

$$Y_{tt} - Y_{xx} + \mathcal{Q}(x)Y_x = 0 \tag{6.4}$$

with boundary control $Y(0,t) = f(t)$, and solved inverse problem to recover $\mathcal{Q}$ from f. A.S. Blagoveščenskii established important connections between his problem and spectral theory of string equations. This work was developed further in [12] (see also references therein), where response operator appears in inverse problem. Finally, the inverse problem to recover the matrix potential $\mathcal{Q}(x)$ of the dynamical Schrödinger equation

$$Y_{tt} - Y_{xx} + \mathcal{Q}(x)Y = 0 \tag{6.5}$$

from the response function was considered in [6] by S. Avdonin, M. Belishev, and S. Ivanov.

Similar to the case of the spectral Dirac and Schrödinger equations, the dynamical Dirac equation is a more general object than the dynamical Schrödinger equation. More precisely, setting in (6.1)

$$p(x) = 0, \quad q(x) = g_x(x)/g(x), \quad \text{where} \quad g_{xx}(x) = \mathcal{Q}(x)g(x), \tag{6.6}$$

and rewriting (6.1), (6.2) in the form

$$(Y_1)_t = \mathrm{i}\big((Y_2)_x + qY_2\big), \quad (Y_2)_t = \mathrm{i}\big(-(Y_1)_x + qY_1\big), \tag{6.7}$$

we obtain a dynamical Schrödinger equation

$$\begin{aligned} (Y_1)_{tt} &= (Y_1)_{xx} - (q_x + q^2)Y_1 = (Y_1)_{xx} - \big(g_{xx}/g\big)Y_1 \\ &= (Y_1)_{xx} - \mathcal{Q}Y_1. \end{aligned} \tag{6.8}$$

For interesting applications of the interconnections between spectral Dirac and Schrödinger equations see, for instance, the papers [18, 23–25, 31, 32] and references therein.

6.2. Preliminaries and estimates

According to [14, Theorem 1], in the case where p, q, f (in (6.2) and in (6.3)) are continuously differentiable and $f(0) = f'(0) = 0$, there is a unique classical solution Y of (6.1), (6.3) and this solution admits representation

$$Y = Y_*^f + w^f; \qquad Y_*^f(x,t) = f(t-x)\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}, \quad f(t) = 0 \quad (t < 0); \tag{6.9}$$

$$w^f(x,t) = \int_x^t f(t-s)\kappa(x,s)ds \quad (t \ge x \ge 0), \tag{6.10}$$

$$w^f(x,t) = 0 \quad (x > t \ge 0), \tag{6.11}$$

where $\kappa(x,s)$ $(x \le s)$ is continuously differentiable. In particular, formulas (6.9) and (6.11) yield:

$$Y(x,t) = 0 \text{ for } 0 \le t < x \quad (\text{finiteness of the domain of influence}). \tag{6.12}$$

Representation (6.9)–(6.11) is proved in [14] using Duhamel formula.

Let us also assume that $\mathcal{V}$, f and f' are bounded:

$$\sup_{x>0} \|\mathcal{V}(x)\| < M_1, \quad \sup_{t>0} \left\|f(t)\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}\right\| < c_0, \quad \sup_{t>0} \left\|f'(t)\begin{bmatrix}1\\ \mathrm{i}\end{bmatrix}\right\| < \widetilde{c}_0. \tag{6.13}$$

The following estimates are proved in [59, Section 2].

Proposition 6.1. *Let* p, q, f *be continuously differentiable and let equalities* $f(t) = f'(t) = 0$ *hold for* $t \leq 0$. *Assume that* (6.13) *is valid. Then the solution* Y *of the dynamical Dirac system* (6.1), *such that* (6.3) *and* (6.12) *are valid, satisfies the following inequalities*

$$\|Y(x,t)\| \leq c_0 \mathrm{e}^{Mt}, \quad \|Y_t(x,t)\| \leq \widetilde{c}_0 \mathrm{e}^{Mt}, \quad \|Y_x(x,t)\| \leq M_2 \mathrm{e}^{Mt}, \tag{6.14}$$

where $x \geq 0$, $t \geq 0$, $M_2 > 0$ *is some constant, and* $M = 2\sqrt{2}M_1$.

The kernel κ *of the integral operator in* (6.10) *satifies the inequality*

$$\|\kappa(x,t)\| \leq M \mathrm{e}^{Mt}. \tag{6.15}$$

In view of (6.14), we can apply to Y the transformation:

$$\widehat{Y}(x,z) = \int_0^\infty \mathrm{e}^{\mathrm{i}zt} Y(x,t) dt, \quad z \in \mathbb{C}_M, \tag{6.16}$$

and $\widehat{Y}$ stands in this section for the Fourier transformation of Y (which is taken, for the sake of convenience, for the fixed values x, z). Moreover, the same transformation can be applied to Y_t and Y_x, and we have

$$\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt} Y_t(x,t) dt = z\widehat{Y}(x,z), \tag{6.17}$$

$$\int_0^\infty \mathrm{e}^{\mathrm{i}zt} Y_x(x,t) dt = \frac{d}{dx}\widehat{Y}(x,z) =: \widehat{Y}'(x,z), \quad z \in \mathbb{C}_M. \tag{6.18}$$

Now, applying the Fourier transformation to the dynamical Dirac system (6.1), we derive

$$z\widehat{Y}(x,z) + J\widehat{Y}'(x,z) + \mathcal{V}(x)\widehat{Y}(x,z) = 0. \tag{6.19}$$

Note that, according to [14], the *response function* r is given by

$$r(t) = \kappa_2(0,t). \tag{6.20}$$

Indeed, for r of the form (6.20), using (6.3), (6.9) and (6.10) we obtain $Y_1(0,t) = f(t)$ and

$$Y_2(0,t) = \mathrm{i}f(t) + \int_0^t r(t-s)f(s)ds = \mathrm{i}Y_1(0,t) + \int_0^t r(t-s)Y_1(0,s)ds. \tag{6.21}$$

6.3. Response and Weyl functions

In this subsection we always assume that the conditions of Proposition 6.1 are fulfilled. However, it seems possible (and would be interesting) to modify representation (6.9)-(6.11) for the case of matrix functions p and q and for weaker smoothness conditions, in which case the solution of the inverse problem in a much more general situation will follow.

The equivalence transformation between spectral Dirac systems in the form (6.19), where J and $\mathcal{V}$ are given in (6.2), and in the form (2.1), (2.2) ($m_1 = m_2 = 1$) is given by the relations

$$y = K\widehat{Y}, \quad v = \mathrm{i}q - p, \quad K = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{bmatrix}. \tag{6.22}$$

Since, according to [59, Section 3], $\widehat{u}(x,z)$ ($z \in \mathbb{C}_M$) is a Weyl solution of (6.19) (i.e., $\widehat{u} \in L_2^2(0,\infty)$), we see that $y = K\widehat{Y}$ is a Weyl solution of (2.1), (2.2), where $m_1 = m_2 = 1$. Hence, in view of (6.21), definition (2.7) yields [59, Proposition 3.1]:

Proposition 6.2. *The response function $r(t)$ of the dynamical Dirac system* (6.1) *is connected with the Weyl function $\varphi(z)$ of the corresponding spectral Dirac system* (2.1) *(where $m_1 = m_2 = 1$) via the equality*

$$\varphi(z) = \widehat{r}(z)/(\widehat{r}(z) + 2\mathrm{i}), \quad z \in \mathbb{C}_M. \tag{6.23}$$

Remark 6.3. Proposition 6.2 jointly with Theorem 2.4 and the equality $v = \mathrm{i}q - p$ in the equivalence transformation (6.22) (which yields $p = -\Re(v)$, $q = \Im(v)$) provide a procedure to solve the inverse problem of the recovery of the potential $\mathcal{V}$ of a dynamical Dirac system from its response function.

Since $m_1 = m_2$, it is possible to introduce a slightly different class of Weyl functions (that is, Weyl functions φ_H) via the inequality:

$$\int_0^{\infty} \big[I_k \; \mathrm{i}\varphi_H(z)^* \big] \, \Theta u(x,z)^* u(x,z) \Theta^* \begin{bmatrix} I_k \\ -\mathrm{i}\varphi_H(z) \end{bmatrix} dx < \infty, \tag{6.24}$$

$$\Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & -I_k \\ I_k & I_k \end{bmatrix}, \quad z \in \mathbb{C}_+, \tag{6.25}$$

where $k = m_1 = m_2$. Comparing (6.24) with (2.7) we see that φ_H is a linear fractional transformation of φ. It is convenient that Weyl functions φ_H belong to Herglotz class (i.e., $\mathrm{i}(\varphi_H(z)^* - \varphi_H(z)) \geq 0$). In our case we have $k = 1$ and the connection between φ_H and $\widehat{r}$ is simpler than (6.23). Namely, we have

$$\varphi_H(z) = \widehat{r}(z) + \mathrm{i}. \tag{6.26}$$

Remark 6.4. Clearly, we can recover v from φ_H after an easy transformation which maps φ_H into φ. However, there is also an independent procedure in [51] (see [59,60] as well) to recover v from φ_H. Instead of the structured operators S_l given by (2.15) and (2.13), convolution operators $\widetilde{S}_l$:

$$\widetilde{S}_l = \frac{d}{dx} \int_0^l \widetilde{s}(x - t) \cdot dt, \qquad \widetilde{s}(x) = -\widetilde{s}(-x)^* \quad (x > 0), \tag{6.27}$$

$$\widetilde{s}(x)^* := \frac{d}{dx} \left(\frac{\mathrm{i}}{4\pi} \mathrm{e}^{\eta x} \mathrm{l.i.m.}_{a \to \infty} \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}\xi x} (\xi + \mathrm{i}\eta)^{-2} \varphi_H(\xi + \mathrm{i}\eta) d\xi \right) \tag{6.28}$$

are used for this purpose. Formulas (6.26) and (6.28) imply that $r(t) = 2\mathrm{i}\overline{\widetilde{s}'(t)}$. Moreover, it is shown in [59] that the response function r coincides with a so called A-amplitude from [33] (more precisely, with the analogue of A-amplitude for the corresponding spectral Dirac system). We note that in M.G. Krein's terminology $\widetilde{s}'$ is called the accelerant.

Interconnections between response functions, Weyl functions and A-amplitudes for Schrödinger equations are discussed in the interesting paper [8].

Some explicit solutions of the inverse problem for dynamical Dirac system are obtained in [59, Section 5]:

Theorem 6.5. *[59] Let $r(t)$ be the response function of a dynamical Dirac system and assume that $r(t)$ admits representation*

$$r(t) = -2\mathrm{i}\vartheta_2^* \mathrm{e}^{-\mathrm{i}t\alpha} \vartheta_1,$$

where the $n \times n$ ($n \in \mathbb{N}$) matrix α and the column vectors $\vartheta_i \in \mathbb{C}^n$ ($i = 1, 2$) satisfy the identity

$$\alpha - \alpha^* = -\mathrm{i}\big(\vartheta_1 + \vartheta_2\big)\big(\vartheta_1 + \vartheta_2\big)^*.$$

Then the potential $\mathcal{V}$ of this dynamical Dirac system is given (in terms of α and ϑ_i) by the equalities

$$\mathcal{V} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}, \quad p = -\Re(v), \quad q = \Im(v);$$

$$v(x) = -2\mathrm{i}\vartheta_1^* \mathrm{e}^{\mathrm{i}xA^*} S(x)^{-1} \mathrm{e}^{\mathrm{i}xA} \vartheta_2, \quad A := \alpha + \mathrm{i}\vartheta_1\big(\vartheta_1 + \vartheta_2\big)^*;$$

$$S(x) = I_n + \int_0^x \Lambda(t)\Lambda(t)^* dt, \quad \Lambda(t) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}tA}\vartheta_1 & \mathrm{e}^{\mathrm{i}tA}\vartheta_2 \end{bmatrix}.$$

A. Quasi-analytic functions and matrix functions

The class $C\big(\{\widetilde{M}_k\}\big)$ consists of all infinitely differentiable on $[0, \ell)$ scalar functions f such that for some $c(f) \geq 0$ and for fixed constants $\widetilde{M}_k > 0$ $(k \geq 0)$ we have

$$\left| \frac{d^k f}{dt^k}(t) \right| \leq (c(f))^{k+1} \widetilde{M}_k \quad \text{for all} \quad t \in [0, \ell), \quad 0 < \ell \leq \infty. \tag{A.1}$$

Recall that $C\big(\{\widetilde{M}_k\}\big)$ is called quasi-analytic if for the functions f from this class and for any $0 \leq t < \ell$ the equalities $\frac{d^k f}{dt^k}(t) = 0$ $(k \geq 0)$ yield $f \equiv 0$. According to the famous Denjoy–Carleman theorem, the equality

$$\sum_{n=1}^{\infty} \frac{1}{L_n} = \infty, \quad L_n := \inf_{k \geq n} \widetilde{M}_k^{1/k} \tag{A.2}$$

implies that the class $C\big(\{\widetilde{M}_k\}\big)$ is quasi-analytic.

In the case of matrix functions $\phi(t)$ (e.g, in the case of $\phi(t) = v(0, t)$) we say that ϕ is quasi-analytic if the entries of ϕ are quasi-analytic and we say that $\phi \in C([0, \ell); \widetilde{M})$, where $\widetilde{M} = \{\widetilde{M}_k(i, j)\}$, if the entries ϕ_{ij} of ϕ belong to quasi-analytic classes $C(\{\widetilde{M}_k(i, j)\})$.

The recovery of a function belonging to some quasi-analytic class $C\big(\{\widetilde{M}_k\}\big)$ from its Taylor coefficients (and from the sequence $\{\widetilde{M}_k\}$) is discussed in important works [9, 38] and [16, Section III.8] but many interesting problems are still open.

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