

Variable Moving Average Transform Stitching Waves

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Abstract. A moving average transform in the plane with a variable size and shape window depending on the position and the 'time' is studied. The main objective is to select the window parameters in such a way that the new transform converges smoothly to the identity transform at the boundary of a prescribed bounded plane region. A new approximation of solitary waves arising from Korteweg-de Vries equation is obtained based on results in the paper. Numerical implementation and examples are included.

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1. Introduction

Complex models often require localization of different processes in space and time. In turn the fragmentation requires exchanging information among the parts of the model. The final outcome often is also expected to be at least C^2 smooth function and to satisfy certain initial conditions. In the current paper we consider a modification of the moving average transform (MAT) and its inverse for particular classes of functions that transition to the identity transform on prescribed open and bounded sets in the plane. MAT has rich history in modeling, Kirchhoff diffraction integral to name one. We discuss a different use of MAT. The general idea is to smoothly blend two surfaces over disjoint domains with common edge by preserving the surfaces outside a strip containing the edge. Since the process remains stitching we refer to it also as Stitching Transform.

In more precise mathematical terms we consider two problems. The first one is to develop a constructive method for generating smooth functions with prescribed bounded supports. The second one can be formulated as follows. Let $\Omega \subset \mathbb{R}^2$ and $\Omega_0 \subset \Omega$ be bounded and simply connected plane domains, i.e. open and bounded sets, with piece-wise C^m boundaries. For $f, g \in C^m(\Omega)$ construct a function $h \in C^m(\Omega)$ such that $h = g$ on $\Omega_0 \setminus \delta$, and $h = f$ on $\Omega \setminus (\Omega_0 \cup \delta)$, where δ is a region containing the boundary of Ω_0 . We propose local, smooth, and computationally effective method for solving the problems. The main tool in the construction is a modification of the moving average transform which we call *Variable Moving Average Transform* (VMAT). The difference from MAT is that the region, over which the average is taken, depends on the position in the plane and an extra parameter that we call 'time'. The moving average is well known and used technique that amounts to a convolution of a plane integrable

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function f and a characteristic function of a domain. Box splines, see [2], with their convolution definition are closely related to the moving average on directional vectors. The method can be considered also as a kernel method, see [6]. The applications range from numerical solutions of PDE's, [8] and [3], to computer graphics, [10]. Although MAT provides smoothness and has low computational cost its applications are limited due to the lack of locality i.e. in general f and its moving average differ on the entire $\mathbb{R}^2$. In the case of single valued functions adaptive moving average method was introduced in finance, [5]. We considered the problem of smoothly connecting two functions in a neighborhood of a point on the real line, see [11]. The method is based on extrapolating the functions by partial Taylor's polynomials and connecting the polynomial pieces. In the case of more variables the Taylor polynomials extrapolation in general fails. In the current paper in order to localize the averaging process in the plane we consider two different approaches. In the general case the averaging region is smoothly contracted to a single point, thus VMAT to the identity transform. In some special cases it is possible to identify the inverse operator and mimic the one dimensional case. The process is governed by a smooth locally supported function α . The smoothness is in the sense that at the point $P = (x, y)$ there exists an algebraic polynomial in Taylor's sense. In the next two sections we introduce and study the properties of the transform. In section 4 we discuss and construct locally supported smooth functions. Section 5 is a general discussion on numerical implementation with examples.

In section 6 we discuss possible application of VMAT in modeling solitons and a water flow around an object. The smoothing properties of VAMT allow the blended surfaces to be continuously differentiable to a high order. On the other hand the nature in which the support of the transform propagates agrees with the Huygens-Fresnel principle for the wave front propagation.

2. Stitching Transform

For a continuous function α with bounded support and $P, P_j \in \mathbb{R}^2, j = 1, \dots, r$ the variation of the averaging region is realized by the transformations $\mu_P^\alpha(P_1) = P + \alpha(P)P_1$, and

$$\nu_P^\alpha(P_1) = \begin{cases} \frac{1}{\alpha(P)}(P_1 - P), & \text{if } \alpha(P) \neq 0; \\ 0, & \text{if } \alpha(P) = 0. \end{cases}$$

Notice that for $\alpha(P) \neq 0$ the identity $\mu_P^\alpha(\nu_P^\alpha(S)) = S$ holds for any $S \in \mathbb{R}^2$. Fig. 1 visualizes the dynamics of the domains of integration.

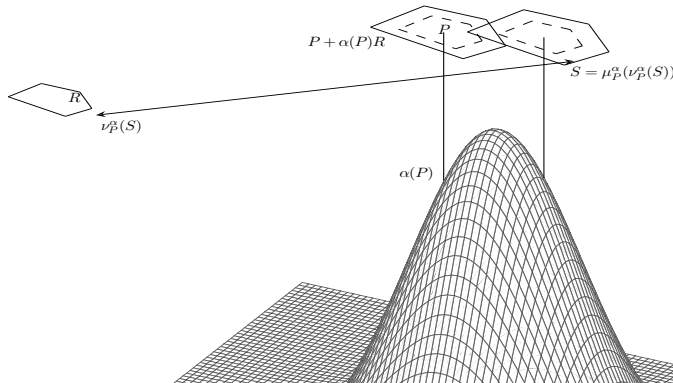


FIGURE 1. The surface is α , R is a region with a centroid at $(0,0)$, the dashed regions represent translations of R (the case of $\alpha = 1$), the solid regions represent $\mu_P^\alpha(R)$.

Iterations of μ are defined as

$$\begin{aligned}
 \mu_{P,P_1}^\alpha(P_2) &= \mu_{\mu_P^\alpha(P_1)}^\alpha(P_2) = \mu_P^\alpha(P_1) + \alpha(\mu_P^\alpha(P_1))P_2, \\
 &= P + \alpha(P)P_1 + \alpha(P + \alpha(P)P_1)P_2, \\
 &\vdots \\
 \mu_{P,P_1,\dots,P_r}^\alpha(P_{r+1}) &= \mu_{\mu_{P,P_1,\dots,P_{r-1}}^\alpha(P_r)}^\alpha(P_{r+1}) \\
 &= \mu_{P,P_1,\dots,P_{r-1}}^\alpha(P_r) + \alpha(\mu_{P,P_1,\dots,P_{r-1}}^\alpha(P_r))P_{r+1}.
 \end{aligned}$$

To localize the effect of the operator we consider the moving average as a kernel integral with a kernel depending on the position. For an integrable function f , a convex plane domain R with a piece-wise C^m boundary and area $A(R)$, and $\alpha \in C^m(\mathbb{R}^2)$ we define VMAT as follows: If $\alpha(P) = 0$ then $\Lambda_{0,R}^1 f(P) = f(P)$, and if $\alpha(P) \neq 0$ then

$$\Lambda_{\alpha,R}^1 f(P) = \frac{1}{\alpha(P)^2 A(R)} \int_{\mu_P^\alpha(R)} f(Q) dQ \quad (2.1)$$

$$= \frac{1}{A(R)} \int_R f(P + \alpha(P)Q) dQ \quad (2.2)$$

$$= \frac{1}{A(R)} \int_{\mathbb{R}^2} f(Q) \chi_R \left(\frac{Q - P}{\alpha(P)} \right) dQ. \quad (2.3)$$

If instead of a function with bounded support, α is constant over the entire plane then the above definition of $\Lambda_{\alpha,R}^1$ is the standard moving average. That case is considered in more detail in section 4. If R is a polygon spanned by two linearly independent vectors v_1 and v_2 , and α is constant then VMAT can be expressed in terms of box splines, for definition and properties of box splines see [2]. Box splines are defined recursively by

$$B_0(v_1, v_2; P) = \frac{1}{A(R)} \chi_R(P),$$

and for more directions, $v_3, \dots, v_K$, inductively as

$$B_s(v_1, \dots, v_{s+2}; P) = \int_0^1 B_{s-1}(P - tv_{s+2}) dt.$$

In this case $\Lambda_{1,R}^1 f(P) = f * B_0(v_1, v_2)(P)$. The iterations could be considered in terms of 'centered' box splines, namely if χ_{v_1, v_2} is the characteristic function of the polygon spanned by v_1, v_2 then

$$\begin{aligned}
 C_0(v_1, v_2; P) &= \frac{1}{A(R)} \chi_{v_1, v_2}(P) \\
 C_{s+1}(v_1, \dots, v_{s+2}; P) &= \frac{1}{A(R)} \int_{-1/2}^{1/2} C_s(P - \alpha_{s+2}(t)v_{s+2}) dt.
 \end{aligned}$$

A repeated Variable Moving Average Transform is defined as

$$\Lambda_{\alpha,R}^{s+1} f(P) = \Lambda_{\alpha,R}^1 (\Lambda_{\alpha,R}^s f)(P).$$

The iterations of Λ can be also expressed in terms of the family of kernel functions

$$K_{\alpha,R}^r(P, Q) = \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \chi_R(\nu_{S_1}^\alpha(Q)) \chi_R(\nu_{S_2}^\alpha(S_1)) \dots \chi_R(\nu_P^\alpha(S_r)) dS_1 \dots dS_r,$$

or

$$K_{\alpha,R}^r(P, Q) =$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \chi_R \left(\frac{Q - S_1}{\alpha(S_1)} \right) \chi_R \left(\frac{S_1 - S_2}{\alpha(S_2)} \right) \cdots \chi_R \left(\frac{S_r - P}{\alpha(P)} \right) dS_1 dS_2 \dots dS_r.$$

The dilation can be done by using different α_j at each iteration. The inductive definition of box splines allows convolution with a single direction vector while for VMAT at least two vectors are needed. On the other hand it is interesting to notice that the domain in VMAT could be any, including non convex, and the $K_\alpha(P, Q)$ are position dependent.

The iterations of the transform can be expressed explicitly

$$\Lambda_{\alpha,R}^s f(P) = \frac{1}{A(R)^s} \int_R \cdots \int_R f(\mu_{P,P_1,\dots,P_{s-1}}(P_s)) dP_1 \dots dP_s \quad (2.4)$$

$$= \frac{1}{A(R)^s} \int_{\mathbb{R}^2} f(Q) K_{\alpha,R}^{s-1}(P, Q) dQ \quad (2.5)$$

$$= \frac{1}{A(R)} \int_{\mathbb{R}^2} \Lambda_{\alpha,R}^{s-1} f(Q) \chi_R(\nu_P^\alpha(Q)) dQ \quad (2.6)$$

In the next section we derive some properties of VMAT. In section 4 explicit constructions of compactly supported C^m functions(used as α) are discussed.

3. Properties of Variable Moving Average Transform

The properties of $\Lambda_{\alpha,R}^s$ regarding smoothness are similar to the properties of the standard moving average but for functions α , with a bounded support, the position dependent transform is also supported on a bounded region. Throughout the section α is supported on Ω , $\partial = \partial\Omega$ is the piece-wise smooth closed boundary of Ω , Ω^c is the complement of Ω , $\Omega^0 = \Omega \setminus \partial$, and $A(R) = 1$. In order VMAT to be a smoothing operator we have to require f to be continuous on ∂ . Indeed, since $\Lambda_{\alpha,R}$ is the identity on $\partial\Omega$ and in case f is discontinuous then so is $\Lambda_{\alpha,R}f$.

Lemma 3.1. *Let f be continuous at $P \in \partial\Omega$ then $\Lambda_{\alpha,R}^1 f$ is also continuous at P and $\lim_{P_1 \rightarrow P} \Lambda_{\alpha,R}^1 f(P_1) = f(P)$. Furthermore, if $f \in C^m(\mathbb{R}^2)$ in a neighborhood of P and $\alpha \in C^m(\mathbb{R}^2)$ then $\Lambda_{\alpha,R}^1 f$ is C^m at P and $\frac{\partial^{s+k}}{\partial x^s \partial y^k} \Lambda_{\alpha,R}^1 f(P) = \frac{\partial^{s+k}}{\partial x^s \partial y^k} f(P)$ for all integers $s, k \geq 0$ such that $s+k = 0, \dots, m$.*

Proof. We first make an observation that is used also in other proofs in the section. Since Λ is the identity on Ω^c and we consider f continuous on that domain, it follows that $|\Lambda_{\alpha,R}^1 f(P_1) - f(P)| = |f(P_1) - f(P)| \rightarrow 0$ when $P_1 \rightarrow P$ and $P_1 \in \Omega^c$ and hence in order to establish continuity for any P_1 we need to show that $|\Lambda_{\alpha,R}^1 f(P_1) - f(P)|$ can be made as small as we wish only for $P_1 \in \Omega^0$. Indeed, let $P_1 \in \Omega^0$, then by using (2.2) we get that

$$\begin{aligned} |\Lambda_{\alpha,R}^1 f(P_1) - f(P)| &= \left| \int_R f(P_1 + \alpha(P_1)Q) dQ - \int_R f(P) dQ \right| \\ &= \left| \int_R (f(P_1 + \alpha(P_1)Q) - f(P_1)) + (f(P_1) - f(P)) dQ \right|. \end{aligned}$$

Since $\alpha(P_1) \rightarrow 0$ as $P_1 \rightarrow P$ and f is continuous the two terms in the integral tend to zero and since R is bounded the integral tends to zero, too.

Next, let f be continuously differentiable at P . By using the Leibniz integral rule and (2.2) we get that

$$\begin{aligned} \frac{\partial}{\partial x} \Lambda_{\alpha,R}^1 f(P) &= \int_R \frac{\partial}{\partial x} f(P + \alpha(P)Q) dQ \\ &= \int_R f_x(P + \alpha(P)Q)(1 + \alpha_x(P)u) + f_y(P + \alpha(P)Q)\alpha_x(P)v dQ, \end{aligned}$$

where $Q = (u, v)$. Since α is identically zero outside Ω and $\alpha \in C^m$ then $\frac{\partial^{s+k}}{\partial x^s \partial y^k} \alpha(P) = 0$ when $P \in \partial\Omega$. It follows that $\frac{\partial}{\partial x} \Lambda_{\alpha,R}^1 f(P) = f_x(P)$ and is continuous at P . In a similar fashion we get that $\frac{\partial}{\partial y} \Lambda_{\alpha,R}^1 f$ is continuous at P and hence $\Lambda_{\alpha,R}^1 f$ is differentiable at P since for $P \in \Omega^c$ $\Lambda_{\alpha,R}^1 f \equiv f$. For the higher order derivatives the statement follows from the recurrence formula (2.3) and induction. $\square$

From the definition of $\Lambda_{\alpha,R}^1$, it follows that it is the identity operator on Ω^c . For each point $P \in \Omega^o$ we have that $\alpha(P) \geq \epsilon > 0$ in a neighborhood of P and this fact allows us to expand the class of functions to piece-wise continuous on Ω^o . We remind that we assume $A(R) = 1$.

Lemma 3.2. *Let R be bounded and convex with a piece-wise C^m boundary. Then for a piece-wise continuous f and $P \in \Omega^o$ we have that $\Lambda_{\alpha,R}^1$ is continuous at P . Furthermore, if $\alpha, f \in C^k(\mathbb{R}^2)$ then $\Lambda_{\alpha,R}^1 f \in C^{k+1}(\Omega_0)$ for $k = 0, 1, \dots, m$.*

Proof. Since R is convex it can be partitioned into vertical strips $R = \cup_{j=1}^M V_j$, where

$$V_j = \{(u_1, v_1) | x_j \leq u_1 \leq x_{j+1}, r_j(u_1) \leq v_1 \leq q_j(u_1)\}, r_j, q_j \in C^m(\mathbb{R}).$$

Let V represents any of the strips, for shorter notation we use x_1, x_2, r and s , and assume $A(V) = 1$. Then for $P = (x, y)$ we have

$$\begin{aligned} \mu_P^\alpha(V) = \{(u, v) | & x + \alpha(P)x_1 \leq u \leq x + \alpha(P)x_2, \\ & y + \alpha(P)r\left(\frac{u-x}{\alpha(P)}\right) \leq v \leq y + \alpha(P)q\left(\frac{u-x}{\alpha(P)}\right)\}. \end{aligned}$$

Let f be piece-wise continuous, then for $P_1 \in V$ from (1) we have that

$$\begin{aligned} \Lambda_{\alpha,V}^1 f(P) - \Lambda_{\alpha,V}^1 f(P_1) &= \frac{1}{\alpha^2(P)} \int_{\mu_P^\alpha(V)} f(Q) dQ - \frac{1}{\alpha^2(P_1)} \int_{\mu_{P_1}^\alpha(V)} f(Q) dQ \\ &= \frac{1}{\alpha^2(P)} \int_{\mu_P^\alpha(V) \cap \mu_{P_1}^\alpha(V)} \left(1 - \frac{\alpha^2(P)}{\alpha^2(P_1)}\right) f(Q) dQ \\ &\quad + \frac{1}{\alpha^2(P)} \int_{\mu_P^\alpha(V) \setminus \mu_{P_1}^\alpha(V)} f(Q) dQ \\ &\quad - \frac{1}{\alpha^2(P_1)} \int_{\mu_{P_1}^\alpha(V) \setminus \mu_P^\alpha(V)} f(Q) dQ \rightarrow 0, \text{ as } P_1 \rightarrow P. \end{aligned}$$

Indeed, $\alpha \geq \epsilon > 0$ and from the continuity of $\alpha(P)$ and $\mu_P^\alpha(V)$ it follows that $\frac{\alpha(P)}{\alpha(P_1)} \rightarrow 1$ and $\mu_{P_1}^\alpha(V) \rightarrow \mu_P^\alpha(V)$ as $P_1 \rightarrow P$.

Let f be continuous, $P = (x, y)$ and $Q = (u, v)$, then from the Fubini's theorem it follows that

$$\begin{aligned} \Lambda_{\alpha,V}^1 f(P) &= \frac{1}{\alpha^2(P)} \int_V f(P + \alpha(P)Q) dQ \\ &= \frac{1}{\alpha^2} \int_{x+\alpha x_1}^{x+\alpha x_2} \int_{y+\alpha r((u-x)/\alpha)}^{y+\alpha q((u-x)/\alpha)} f(u, v) dv du. \end{aligned}$$

To simplify the writings of the formulas let $\alpha = \alpha(P)$. By applying the Leibniz integral rule on the line twice we get that

$$\begin{aligned} \frac{\partial}{\partial x} \Lambda_{\alpha,V}^1 f(x, y) &= -\frac{2\alpha_x}{\alpha^3} \int_{x+\alpha x_1}^{x+\alpha x_2} \int_{y+\alpha r((u-x)/\alpha)}^{y+\alpha q((u-x)/\alpha)} f(u, v) dv du \\ &\quad + \frac{1}{\alpha^2} \int_{y+\alpha r(x_2)}^{y+\alpha q(x_2)} f(x + \alpha x_2, v) dv (1 + \alpha_x x_2) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\alpha^2} \int_{y+\alpha r(x_1)}^{y+\alpha q(x_1)} f(x + \alpha x_1, v) dv (1 + \alpha_x x_1) \\
& + \frac{1}{\alpha^2} \int_{x+\alpha x_1}^{x+\alpha x_2} f(u, y + \alpha q((u-x)/\alpha) \\
& \quad \left(\alpha_x q((u-x)/\alpha) - q'((u-x)/\alpha) \left(\frac{\alpha_x}{\alpha} (u-x) + 1 \right) \right) du \\
& - \frac{1}{\alpha^2} \int_{x+\alpha x_1}^{x+\alpha x_2} f(u, y + \alpha r((u-x)/\alpha) \\
& \quad \left(\alpha_x r((u-x)/\alpha) - r'((u-x)/\alpha) \left(\frac{\alpha_x}{\alpha} (u-x) + 1 \right) \right) du \\
& = -2 \frac{\alpha_x(P)}{\alpha(P)} \Lambda_{\alpha,V}^1 f(P) + \frac{1}{\alpha^2(P)} \int_{\mu_P^\alpha(\partial V)} (n \cdot (\alpha_x \nu_P^\alpha(S) + e_1)) f dS,
\end{aligned}$$

where n is the outward normal to $\mu_P^\alpha(V)$ and $e_1 = (1, 0)$. Since all functions above are continuous and $\alpha \geq \epsilon > 0$, it follows that $\frac{\partial}{\partial x} \Lambda_{\alpha,V}^1$ is also continuous at P . Similarly the derivative with respect to y is also continuous. Since $\Lambda_{\alpha,R}^1 = \sum_{j=1}^K \frac{A(V_j)}{A(R)} \Lambda_{\alpha,V_j}^1$ we obtain the statement with R in lieu of V .

If $f \in C^m$ then we can differentiate m more times f in the above expression for the derivative of $\frac{\partial}{\partial x} \Lambda_{\alpha,R}^1$. $\square$

Some of the properties of $\Lambda_{\alpha,R}^m$ are summarized in the next theorem.

Theorem 3.3. *Let $\alpha \in C^m(\mathbb{R}^2)$ be positive and supported on a bounded and simply connected domain Ω with a piece-wise smooth boundary ∂ . If f is continuous in a neighborhood of ∂ then the following properties hold true.*

- (P1)(Continuity) *If f is piece-wise continuous on Ω^o then $\Lambda_{\alpha,R}^1 f \in C(\Omega)$.*
- (P2)(Continuous Derivatives) *If $f \in C^m(\mathbb{R}^2)$ then $\Lambda_{\alpha,R}^1 f \in C^{m+1}(\Omega^o)$.*
- (P3)(Positiveness) *If $f > 0$ then $\Lambda_{\alpha,R}^1 > 0$.*
- (P4)(Bounded Support) *The function $f - \Lambda_{\alpha,R}^1 f$ has bounded support contained in Ω and in particular $\Lambda_{\alpha,R}^1 f = f$ on $\Omega^c \cup \partial$.*
- (P5)(Gradient) *For any $P \in \mathbb{R}^2$*

$$\nabla \Lambda_{\alpha,R}^1 f(P) = -2 \frac{\nabla \alpha(P)}{\alpha(P)} \Lambda_{\alpha,R}^1 f(P) + \frac{1}{\alpha^2(P)} \int_{\mu_P^\alpha(\partial R)} (n \cdot (\nu_P^\alpha(S) \nabla \alpha(P)) + n) f(S) dS,$$

where n is the outward normal to $\mu_P^\alpha(R)$. If f is C^1 at P , then

$$\nabla \Lambda_{\alpha,R}^1 f(P) = \int_R (\nabla f(\mu_P^\alpha(Q)) + (\nabla f(\mu_P^\alpha(Q)) \cdot Q) \nabla \alpha(P)) dQ.$$

Proof. Properties (P1) and (P2) follow from Lemmas 3.1,2, (P3) and (P4) are inherited from the properties of α . The formulas in (P5) follow from the estimate in the proof of Lemma 3.2 and (2.2). $\square$

Clearly f and $\Lambda_{\alpha,R}^1$ differ only on the support of α . The smoothness of VMAT depends on the smoothness of α . In the next section we discuss constructions, by using MAT, of smooth functions with a bounded support.

4. Construction of smooth functions with bounded plane support

In this section $R = [-d, d] \times [-d, d]$ is a constant. Box splines can be used for the construction of smooth functions α on convex polygonal domains. We consider the convolution definition

$$\Lambda_{\alpha,R}^1 f(P) = \frac{1}{A(R)} f * \chi_R(P) = \frac{1}{A(R)} \int_{\mathbb{R}^2} f(Q) \chi_R(P - Q) dQ,$$

to include larger variety of domains. The construction of α starts with an initial function defined on a subset of Ω and then expanding the support by repeated convolutions with $\frac{1}{A(R)} \chi_R$.

The inverse of $\Lambda_R^s = \Lambda_{1,R}^s$ can be explicitly obtained in some cases. Let $\Lambda_R^{-s} f$ be the inverse of $\Lambda_R^s f$ i.e. $\Lambda_R^{-s}(\Lambda_R^s f) = \Lambda_R^s(\Lambda_R^{-s} f) = f$ and $\text{sinc}(z) = \frac{\sin z}{z}$.

Lemma 4.1. *For any $P = (x, y)$ the following holds true.*

(i) *For the real polynomial $g(x, y) = a_{11}x^2 + a_{12}xy + a_{22}y^2 + b_1x + b_2y + c$ we have that*

$$\Lambda_R^{-s} g(P) = g(P) - \frac{2sd^2}{3}(a_{11} + a_{22}).$$

(ii) *For $\sigma, \rho \in \mathbb{R}$, we have that*

$$\Lambda_R^{-s} e^{\sigma x + \rho y} = (\sinh(\sigma d) \sinh(\rho d))^{-s} e^{\sigma x + \rho y}.$$

(iii) *For $\sigma \in \mathbb{R}$ we have that*

$$\Lambda_R^{-s} e^{i\sigma x} = \text{sinc}(s\sigma d) e^{i\sigma x}.$$

Proof. In (i) we can consider any region R with a centroid at $(0, 0)$. By straight computations we get that

$$\Lambda_R^1 g(P) = g(P) + a_{11} \frac{I_x(R)}{A(R)} + a_{22} \frac{I_y(R)}{A(R)} + a_{12} \frac{I_{xy}(R)}{A(R)},$$

where I_{xx}, I_{xy}, I_{yy} are the moments of R . Since the square has axis of symmetry, it follows that $I_{xy} = 0$. Parts (ii) and (iii) can be checked by straightforward calculations. $\square$

In order to construct α supported on the prescribed domain Ω we recall a set operation.

Definition 4.2. The Minkowski sum of two sets A and B is

$$A \oplus B = \{a + b | a \in A, b \in B\}.$$

Let $mR = [-md, md] \times [-md, md]$ and $\Lambda_d^n = \Lambda_{1,R}^n$, then we have the following lemma.

Lemma 4.3. *Let $\Sigma = \Omega \setminus (\Omega^c \oplus \frac{n}{2}R) \neq \emptyset$ and $\alpha = \Lambda_d^n(\gamma \chi_\Sigma)$ for a piece-wise continuous and positive function γ , then $\text{supp}(\alpha) = \Omega$, $\alpha \in C^{n-1}$, and $\alpha > 0$ on Ω^o .*

Proof. The smoothness and positiveness follow from Theorem 3.3. The function $\Lambda_d^1(\gamma \chi_\Sigma)(P)$ is nonzero if, and only if, the square $P + R = [x - d, x + d] \times [y - d, y + d]$ intersects Σ . If $Q \in (P + R) \cap \Sigma$ then Q belongs to Ω and to a shift of $\frac{1}{2}R$ with vertices: P , one of the vertices of $P + R$, and the midpoints of the two adjacent edges i.e. $Q \in \Omega \oplus \frac{1}{2}R$. Similarly, $\Lambda_d^1(\gamma \chi_\Sigma)(P) \neq 1$ if, and only if $P + R \not\subset \Sigma$. Inductively, the statement of the lemma follows for any n . $\square$

Remark 4.4. The statement of Lemma 4.3 holds true if R is any regular n -gon with a centroid at 0.

There are numerous algorithms for computing the Minkowski sums for certain types of domains, see [9] and [12], but in our numerical examples we use a technique based on Lemma 4.3. It is clear that if $\gamma = \Lambda_d^{-n}g$ then $\alpha(P) = \Lambda_d^n(\gamma\chi_R)(P) = g(P)$, $P \in \Sigma \setminus (\Sigma^c \oplus \frac{n}{2}R)$.

The locally supported smooth function can be constructed by starting with a function defined on Σ and then repeatedly convolved with χ_R . Good candidates for starting functions are the functions considered in Lemma 4.1. A distance function also can be used but in general the function is not smooth and moving average smoothing should be used. Another choice could be powers of functions that are zeros on the smooth pieces of $\partial\Omega$. Three C^3 smooth functions supported on the unit square are presented on Fig.2. The left is the function $(1-x^2)^4(1-y^2)^4$, the middle one is constructed by starting with the characteristic function of the square $[-1+4d, 1-4d]$ and then smoothed four times. The right function is obtained by smoothing four times the distance $\min(|1-4d-x|, |1-4d+x|, |1-4d-y|, |1-4d+y|)$. In the next section we consider more numerical examples.

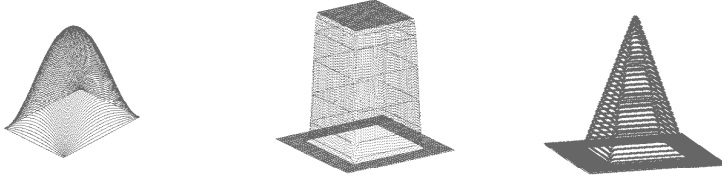


FIGURE 2. C^3 smooth functions supported on the unit square.

5. Numerical Implementation and Examples

The discrete analog of $\Lambda_{\alpha,R}^s$ can be obtained by replacing the integral over R by a discrete sum over a set of points $\{Q_{i,j}\}_{1,j=1}^{N_1M_1}$ contained in R

$$D\Lambda_{\alpha,R}^s f(P) = \sum_{i,j=1}^{N_1M_1} f(Q_{i,j}) K_{\alpha,R}^s(P, Q_{i,j}).$$

The kernel functions $K_{\alpha,R}^s(P, Q_{i,j})$ can be constructed in advance. For a varying α the kernel depends on α but not on f . There is no scaling formula for computations. These kernels are related to the box splines. The continuous VMAT $\Lambda_{\alpha,R}^3 f \in C^2$ but $D\Lambda_{\alpha,R}^3 f \in C^1$.

In the examples we consider a more direct approach by using repeatedly the formula

$$\Lambda_{\alpha,R} f(P) = \frac{1}{A(R_\alpha(P))} \sum_{Q_I \in P + \alpha(P)R} f(Q_I),$$

where I runs through an enumeration of the points $Q_{i,j}$.

The numerical experiments are done by using MatLab on the square grid of integers $[1, 100] \times [1, 100]$.

Example 5.1. The first example is based on Lemma 4.3. For a domain Ω we apply Λ_d^4 to χ_{Ω^c} and select the zero set Σ . Applying Λ_d^4 to χ_Σ we obtain $\alpha \in C^3$ and supported on Ω .

Let

$$\Omega = \{(x, y) | 11 \leq x \leq 90, \frac{(x-11)(x-90)}{80} + 30 \leq y \leq 6 \sin\left(\frac{x}{8} - 2\right) + 80\},$$

with a square hole $[40, 60] \times [40, 60]$, then Ω and the corresponding Σ from Lemma 4.3 are illustrated in the left side of Fig. 3. The region enclosed by the two dashed lines is Σ . The starting function is

defined as $2 - 0.025(x - 50)^2 - 0.025(y - 50)^2 + 0.05$, (the term 0.05 is from part (i) of Lemma 4.1 with $s = 3, d = 1$) on the square with the dashed outline, and 1 on Σ . The function α obtained by a quadruple repetition of the VMAT is plotted on the right side of Fig.3. On the square $[40, 60] \times [40, 60]$ we have that $\alpha(x, y) = 2 - 0.025(x - 50)^2 - 0.025(y - 50)^2$, between the dashed lines $\alpha = 1$, and $\alpha = 0$ on Ω^c . By construction $\alpha \in C^3(\mathbb{R}^2)$.

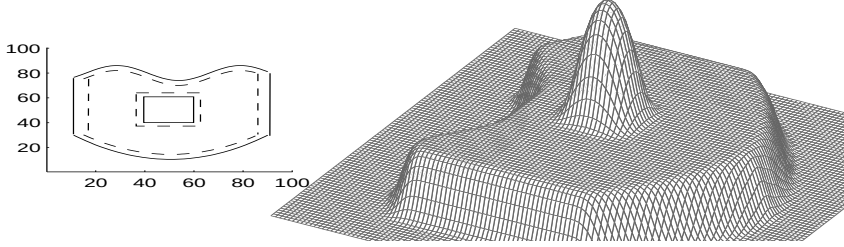


FIGURE 3. Minkowski differences and the resulting C^3 function on $[1, 100] \times [1, 100]$.

Example 5.2. The second example illustrates how VMAT can be used to blend the functions

$$F(x, y) = 4e^{-(|x-80|+|y-20|+|x+y-100|)/100} \quad \text{and} \quad F_1(x, y) = \sin \frac{x}{6} \cos \frac{y}{6}$$

along the triangle with vertices $(28, 29), (68, 64), (84, 19)$. The function α is constructed by quadruple repetition of the VMAT with $d = 1$ to the four times the characteristic function of a triangular strip obtained from subtracting the triangle with vertices $(36, 32), (64, 58), (78, 24)$ and the triangle with vertices $(20, 26), (72, 70), (90, 14)$. The left side of Fig.4 represents the function α and the right side represents the result of a triple repetition of the VMAT with α . It is clear that the sharp edges of F are preserved outside the domain of α i.e. illustration of the local character of VMAT.

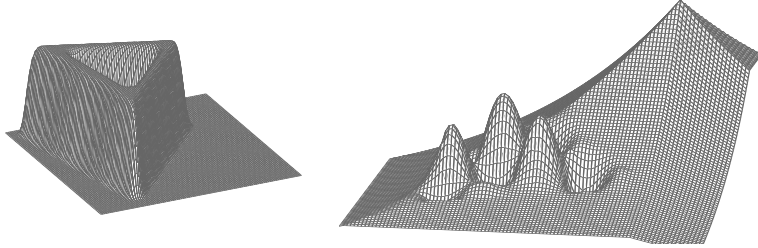


FIGURE 4. The C^3 smooth function α is supported on a triangular band. The resulting function differs from F only inside the 'big' triangle.

6. Applications to Solitary Waves

Solitary waves or solitons arise often in modeling natural phenomena, flow in a narrow channel or tsunami waves for example. In the current section, without aiming at exact models we illustrate and discuss possible applications of the VMAT. A classical result, for details see [1], is that the two dimensional Korteweg-de Vries (KdV, see [7]) equation

$$u_t + 6uu_x + u_{xxx} = 0, u(x, 0) = u_0(x), \quad (6.1)$$

has a solitary wave solution $U(x, t) = \frac{c}{2} \operatorname{sech}^2(\frac{\sqrt{c}}{2}(x - ct))$ for particular initial conditions u_0 . The solution travels along the x -axis and maintains constant shape, single crest, and exponential decay rate of $\sqrt{c}|x|$ as $x \rightarrow \pm\infty$. We are interested in an interaction of a solitary wave traveling along a narrow shallow water channel with an obstacle, in particular a solid box of a width smaller than the channel width. The problem in more general setting is discussed in [4]. We are interested in a short time period when the wave passes the box.

Let t_0 be fixed then the crest is centered at $x_0 = ct_0$ and is of magnitude c . The equation (6.1) is fragmented in three linear IVP corresponding to the crest and decay to $\pm\infty$ of U . Let $c, h > 0$ be fixed and WLOG let $t_0 = 0$ and

(i) For $x \in [-h, h]$ let ϕ_0 be the solution of the IVP

$$-cu_x + 2cu_x + u_{xxx} = 0, \quad u^{(k)}(0) = u_0^{(k)}(x, 0), k = 0, 1, 2.$$

(ii+) For $x > h$ let ϕ_+ be the solution to the BVP

$$-cu_x + u_{xxx} = 0, \quad u(h) = \phi_0(h), \lim_{x \rightarrow \infty} u(x) = 0.$$

(ii-) For $x < -h$ let ϕ_- be the solution to the BVP

$$-cu_x + u_{xxx} = 0, \quad u(-h) = \phi_0(-h), \lim_{x \rightarrow -\infty} u(x) = 0.$$

The composition of the solutions of the above problems provides an approximate solution to (6.1).

Lemma 6.1. *For any real x we have that the function*

$$\phi(x) = \phi_-(x)\chi_{(-\infty, -h)}(x) + \phi_0(x)\chi_{[-h, h]}(x) + \phi_+(x)\chi_{(h, \infty)}(x)$$

is such that $\phi \in C(\mathbb{R})$ and

(a) $|\phi(x) - U(x, 0)| \leq \frac{c^2 h^4}{24}.$

(b) For $x \in (-\infty, -h)$ and $x \in (h, \infty)$ the function ϕ is monotone with rate of decay $e^{-\sqrt{c}|x|}$.

Proof. The proof is by a direct verification. Indeed, the solution of (i) is the function $\phi_0(x) = \frac{c}{2} \cos^2 \frac{\sqrt{c}}{2}x$ and the error estimate in (a) follows from the Taylor's formula about 0 and the fact that $\phi_0^{(3)}(0) = U_0^{(3)}(0)$. For part (b) it is enough to notice that the spectrum for (ii $\pm$) is $0, \pm\sqrt{c}$. The function is continuous due to the choice of the initial conditions. $\square$

The above equations can be interpreted as substituting u in KdV equation by constant approximants on different intervals. On $[-h, h]$ that approximant is $\frac{c}{3}$ which in the case $h = \frac{\arcsin(1/3)}{\sqrt{c}}$ is the best constant approximant. In the 'tail' equations the constant approximant is 0. Similarly, if the initial conditions are modified as follows:

(i) $u(0) = U(0, 0) = c$ and $u^{(k)}(0) = 0, k = 1, 2, 3;$

(ii $\pm$) $u(\pm h) = 0;$

then the corresponding approximate solution is $\phi = c\chi_{[-h, h]}$ with an error of approximation equal to $\frac{c}{2}$. Next we stitch the pieces by using VMAT and the inverse of Λ considered in Lemma 4.1 to obtain smooth functions. From [11], it follows that the resulting smooth approximant will preserve the order of h in the error estimate. In the case when ϕ is the characteristic function we obtain B-spline approximation to the cosine function on $[-h, h]$. Fig. 5 shows the numerical result for $c = 4$, $h = 1$, and Λ_α^4 applied with $\alpha = 1/8$.

The above approximation could be useful in case of more complex waves. The trigonometric functions capture the crests of the waves and the smoothing performed by VMAT(including the inverse) stitches the pieces smoothly. The two-dimensional transform introduced in the paper can be applied in a similar

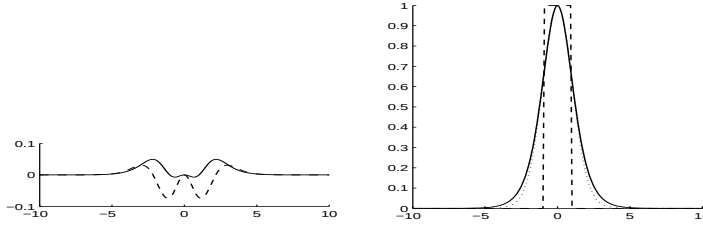


FIGURE 5. On the right side the continuous line is U , the dashed line is $\chi_{[-1,1]}$ and the dotted line is $L = \Lambda_\alpha^4((\Lambda_\alpha^{-4}\phi_-)\chi_{(-\infty,-h)} + (\Lambda_\alpha^{-4}\phi_0)\chi_{[-h,h]} + (\Lambda_\alpha^{-4}\phi_+)\chi_{(h,\infty)})$ for $\alpha = 1/8$. The left graph shows $U - L$, the continuous line and $U - \Lambda_\alpha^4\chi_{[-1,1]}$, the dashed line.

way to stitch smoothly waves supported on disjoint plane regions. To illustrate the latter we include a numerical example related to an interaction of a solitary wave with an obstacle. In [4] propagation of solitons in shallow water was discussed in theoretical and numerical context and similarly to the exposition there we consider the problem to visualize the initial time a soliton reaches the end of a solid stationary block, see Fig. 6. The governing equation of the experiment could be KdV with a modified boundary conditions or the wave equation. For a wave modeled by the characteristic functions of disjoint rectangles the running average transform is used as approximation of the Green's function for the wave equation in a neighborhood of the block's boundary. The result agrees with Hyguen-Fresnel principle and an analytic formula is a simple vector calculus exercise. In case a vortex appears behind the obstacle VMAT can be used to blend it with the flow. The above numerical result is an experimental example and is used as an illustration of application of VMAT not as a model.

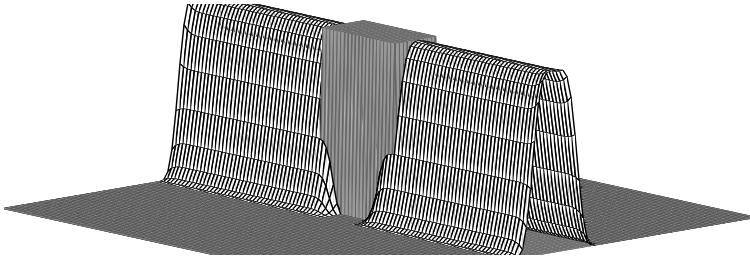


FIGURE 6. The solitary wave is modeled by the characteristics functions of two rectangles disjoint by the support of a solid block of the same height as the wave. The VMAT is performed with a moving square of constant length $1/4$ over the domain with the support of the solid block removed.

7. Conclusion

The VMAT studied in the paper has the smoothing properties of the MAT and although lacks recursive formulas it is not computationally intense due to its bounded support. The smooth functions α with bounded support, considered in section 4 can be constructed by using the fast Fourier transform. The advantage of the method is that for functions of bounded support VMAT tends smoothly to the identity operator toward the boundary of the support of α . The results about the inverse transform can be expanded and applied in more areas, for example Bessel functions, Hermite interpolation, and etc. Interesting future work is to study incorporating observed data into steady state solutions.

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