

Ground States for NLS on Graphs: a Subtle Interplay of Metric and Topology

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Abstract. We review some recent results on the minimization of the energy associated to the nonlinear Schrödinger Equation on non-compact graphs. Starting from seminal results given by the author together with C. Cacciapuoti, D. Finco, and D. Noja for the star graphs, we illustrate the achievements attained for general graphs and the related methods, developed in collaboration with E. Serra and P. Tilli. We emphasize ideas and examples rather than computations or proofs.

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Mathematics Subject Classification: 35Q55, 35P30

1. Introduction

The subject of nonlinear evolution on *graphs* or *networks*, first introduced by F. Ali Mehmeti ([9]) and developed by several authors (e.g. [10, 11, 16, 23, 28]), has rapidly become highly popular in a quite spread scientific community, ranging from experts in pointwise potentials ([12, 13, 20]) up to specialists of the Nonlinear Schrödinger Equation and its standing waves ([3, 15, 19, 22, 25]).

The interest was initially driven by physical applications (for an exhaustive introduction see [24]), that involve propagation in optical fibers and junctions ([14, 18, 30]), and Bose-Einstein condensation ([27, 29, 31]). Subsequently, interesting mathematical issues also arose and have been considered as relevant for themselves.

Here we restrict our scope to the most basic variational problem: proving the existence or the nonexistence of the ground state for the Schrödinger equation endowed with a focusing cubic nonlinearity, on a non-compact graph \mathcal{G} . The dynamics we investigate is then defined by the equation

$$i\partial_t u = -\Delta u - |u|^2 u, \quad (1.1)$$

but, since we concentrate on the problem of the ground state, the central mathematical object in our analysis is the *energy functional* ([1, 6])

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathcal{G})}^2 - \frac{1}{4} \|u\|_{L^4(\mathcal{G})}^4, \quad (1.2)$$

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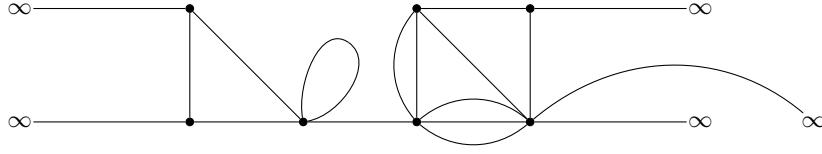


FIGURE 1. **A typical graph.** Vertices are bullets, edges are lines, vertices at infinity are denoted by ∞ . Notice that self-loops and multiple connections are allowed.

whose value, as widely known, is conserved by the dynamics generated by equation (1.1).

Let us take for a moment the exact meaning of the functional spaces involved in formula (1.2) and of the Laplacian appearing in (1.1) apart, and focus preliminarily on the physical interpretation of the problem. In fact, finding the ground state of the energy functional (1.2) means finding the wave function of a Bose-Einstein condensate made of atoms attracting one another, placed in an optical and/or magnetic trap supplied with many branches and ramifications. The non-compactness of the graph is the mathematical translation of the fact that some of the branches of the trap are much longer than others, and also much longer than a characteristic length of the condensate, to be specified later.

We remark that the expression of the energy (1.2) is formally equivalent to the analogous formula for the NLS in \mathbb{R}^N . The non standard elements here are the graph \mathcal{G} , and, therefore, the functions defined on it and the related functional spaces. However, the definition of \mathcal{G} as a *metric graph*, of the functions defined on it, and of the associated L^q -spaces, are the most natural ones. We rapidly review them for the convenience of the reader.

A *graph* \mathcal{G} is defined by two sets: a set V of *vertices*, that are to be thought of as spatial points, and a set E of *edges* (or *bonds*), each of them to be thought of as a line joining a couple of vertices. It is convenient to isolate a subset of V , denoted by V_∞ , that contains the *vertices at infinity*. We establish that two distinct vertices at infinity cannot be connected by an edge. Furthermore, defining the *degree* of a vertex as the number of edges starting or ending on it, we assume that every vertex at infinity has degree one, i.e. it is the endpoint of exactly one halfline. We also notice that there is no loss of generality in supposing that every vertex has degree at least three, and this is what we shall always do.

For the sake of simplicity, we require the resulting structure to be *connected*, i.e. starting from an arbitrary vertex it is possible to reach any other vertex by following a sequence of adjacent edges.

By all this, the *topology* of the graph is fixed. Let us turn to the *metric*.

In order to endow \mathcal{G} with a metric structure, we identify each edge e with a real interval $I_e := [0, \ell_e]$, with $\ell_e > 0$, or, if the edge ends up into a vertex at infinity, with the halfline $I_e := [0, +\infty)$. Thus, \mathcal{G} is a metric space formed by the union of the intervals representing the edges, and the distance between two points is given by the shortest path joining the two points through a connected sequence of edges. In this way we obtain a *metric graph* (Fig.1).

As a next step, the notion of function (or, thinking of quantum mechanical applications, *wave function*) on the metric space \mathcal{G} is natural: the function $u : \mathcal{G} \rightarrow \mathbb{C}$ is defined as the set of functions u_e , $e \in E$, i.e. u_e is the restriction of u to the edge e . For a function u to be continuous, in addition to the continuity of every restriction u_e , one has to require *continuity at vertices*: for instance, if the same vertex v is the initial point of two edges e and e' , then $u_e(0) = u_{e'}(0)$, and an analogous equality holds if the same vertex acts as initial point for a vertex e and as endpoint for e' . In the same way we define differentiability, and denote by ∇u the complex-valued function on \mathcal{G} whose restriction to the edge e is given by u'_e .

Once we have introduced functions on \mathcal{G} , it is immediate to define functional spaces. We need the spaces $L^q(\mathcal{G})$, defined by

$$L^q(\mathcal{G}) = \bigoplus_{e \in E} L^q(I_e), \quad \|u\|_{L^q(\mathcal{G})}^q = \sum_{e \in E} \|u_e\|_{L^q(I_e)}^q,$$

and the space $H^1(\mathcal{G})$, defined by

$$\begin{aligned} H^1(\mathcal{G}) &= \{u \in \bigoplus_{e \in E} H^1(I_e), \text{s.t. } u \text{ is continuous at every vertex}\} \\ \|u\|_{H^1(\mathcal{G})}^2 &= \sum_{e \in E} \|u_e\|_{H^1(I_e)}^2. \end{aligned} \quad (1.3)$$

Notice that the energy space is not defined as the direct sum of the energy spaces of the edges (that would guarantee continuity at internal points of the vertices only), but there is the additional requirement of the continuity at vertices too. Notice that this is not the only possible option: choosing, for instance, the space $\bigoplus_{e \in E} H^1(I_e)$ to be the domain of the energy functional, one would find that minimizers satisfy homogeneous Neumann conditions at every vertex instead of continuity: obviously, the ground state would change, and in general both existence and shape of the ground states, as well as of the stationary waves, strongly depend on conditions at the vertices, and therefore on the choice of the energy domain.

In more plain terms, the energy functional (1.2) can be rewritten as

$$E(u, \mathcal{G}) = \frac{1}{2} \sum_{e \in E} \int_0^{\ell_e} |u'_e|^2 dx - \frac{1}{4} \sum_{e \in E} \int_0^{\ell_e} |u_e|^4 dx,$$

where, with a slight abuse of notation, we allowed $\ell_e = +\infty$,

It is immediately seen that there is no absolute minimum of $E(u, \mathcal{G})$: indeed, for any non-vanishing u , one has $E(\lambda u, \mathcal{G}) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$, i.e. for large data the energy is unbounded from below. However, since the flow associated to the NLS preserves the L^2 -norm, or *mass*, it is physically meaningful to seek minimizers of the energy functional (1.2) under the constraint of *constant mass*. Indeed, in current experiments on Bose-Einstein condensates, what we are calling the mass denotes the number of atoms (in different contexts related to quantum physics, it denotes the overall probability of finding a particle on a certain region and so on), therefore the attainable configuration of minimal energy is always conditioned by the choice of the initial mass and this identifies a natural constraint.

Then, the problem we are interested in is the following.

Problem P. Given $\mu > 0$, defined the space

$$H_\mu^1(\mathcal{G}) = \{u \in H^1(\mathcal{G}), \|u\|_{L^2(\mathcal{G})}^2 = \mu\},$$

and introduced the notation

$$\mathcal{E}_\mathcal{G}(\mu) := \inf_{u \in H_\mu^1(\mathcal{G})} E(u, \mathcal{G}),$$

find a function $u \in H_\mu^1(\mathcal{G})$ such that

$$E(u, \mathcal{G}) = \mathcal{E}_\mathcal{G}(\mu).$$

Notice that this is a standard variational problem (on a non-standard environment), and involves two main elements: first, the *boundedness from below* of the *constrained* energy functional; second, the existence of a function u whose energy equals the infimum of the constrained functional. Such a function is also called a *minimizer*.

As a preliminary fact we stress that, despite the non-boundedness of the energy functional (1.2) on the domain $H^1(\mathcal{G})$, the constrained energy is bounded from below: indeed, Gagliardo-Nirenberg's type estimates hold for graphs too [8, 26], in particular

$$\|u\|_{L^p(\mathcal{G})}^p \leq C \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|\nabla u\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1}, \quad (1.4)$$

that, applied to the energy functional (1.2), gives

$$E(u, \mathcal{G}) \geq \frac{1}{2} \|\nabla u\|_{L^2(\mathcal{G})}^2 \left(1 - C \frac{\mu^{\frac{3}{2}}}{\|\nabla u\|_{L^2(\mathcal{G})}} \right) \quad (1.5)$$

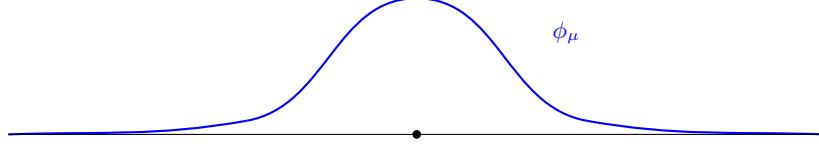


FIGURE 2. **The soliton** ϕ_μ . It is the unique ground state on the line, up to translations and multiplications by a phase factor.

which, for fixed μ , is a lower bounded quantity, so that

$$\inf_{u \in H^1_\mu} E(u, \mathcal{G}) > -\infty. \quad (1.6)$$

It remains then to establish whether the infimum is attained or not. In order to suitably review the result on that, it is convenient to recall some preliminary notions.

1.1. Preliminary results

Since $E(|u|, \mathcal{G}) \leq E(u, \mathcal{G})$, from now on we shall denote by the symbol u a real non-negative functions.

First, a ground state u gives a *standing wave* for the dynamical problem (1.1), in the sense that the function

$$\psi(t, x) = e^{i\omega t} u(x)$$

is a solution to (1.1). Of course, plugging the expression of ψ into equation (1.1) gives

$$\Delta u + u^3 = \omega u, \quad (1.7)$$

where Δu is the function whose restriction on the edge e is u''_e , and the requirement that u is a stationary point for the functional $E(\cdot, \mathcal{G})$ provides the so-called Kirchhoff condition at vertices, namely

$$\sum_{e \succ v} \frac{du_e}{dx_e}(v) = 0 \quad (1.8)$$

where $e \succ v$ denotes that the edge e is incident at the vertex v . Of course, there can be stationary solutions to (1.1) that are not ground states, and they are usually called *excited states*.

It is well-known that the version on the line of Problem P, i.e. when $\mathcal{G} = \mathbb{R}$, is solved by the *soliton* (see Fig.2)

$$\phi_\mu(x) = \frac{\mu}{2\sqrt{2}} \operatorname{sech}\left(\frac{\mu}{4}x\right).$$

and by the functions obtained by translating it. Notice that the existence and the uniqueness of the soliton at mass μ fixes a natural lengthscale on the graph as μ^{-1} . There is then a correspondence between *large mass* and *long edges*, rooted in the fact that a soliton with a large mass is squeezed, and thus it sees edges as long. The energy level of ϕ_μ reads

$$\mathcal{E}_{\mathbb{R}}(\mu) = -\frac{\mu^3}{96}$$

and plays a crucial role in Problem P. Indeed, being the graph \mathcal{G} non-compact, it contains at least one halfline, where it is always possible to approximate arbitrarily well a soliton. So that,

$$\mathcal{E}_{\mathcal{G}}(\mu) \leq -\frac{\mu^3}{96}. \quad (1.9)$$

On the other hand, it is clear that for some graphs it is possible to go below the level $\mathcal{E}_{\mathbb{R}}(\mu)$. For instance, if $\mathcal{G} = \mathbb{R}^+$, then the ground states at mass μ are obviously given by half solitons $\chi_+ \phi_{2\mu}$ centred at the origin of the halfline, and therefore

$$\mathcal{E}_{\mathbb{R}^+}(\mu) = -\frac{\mu^3}{24}.$$

The level $\mathcal{E}_{\mathbb{R}^+}(\mu)$ of the halfline plays an important role too, as it represents the minimal level among all non-compact graphs. In order to understand this point, one must introduce the following result:

Proposition 1.1 (Monotone rearrangement). *Given $u \in H_\mu^1(\mathcal{G})$, there exists a function $u^* \in H_\mu^1(\mathbb{R}^+)$ s.t.*

$$\|u^*\|_{L^4(\mathbb{R}^+)} = \|u\|_{L^4(\mathcal{G})}, \quad \|(u^*)'\|_{L^2(\mathbb{R}^+)} = \|\nabla u\|_{L^2(\mathcal{G})}$$

In practice, the function u^* can be constructed as the *monotone rearrangement* of u on the halfline (see [7, 17]): first, define the *distribution function* as

$$\rho(t) = |\{x \in \mathcal{G}, \text{s.t. } u(x) > t\}|$$

where $|A|$ denotes the Lebesgue measure of the set A . Of course, ρ is a monotonically decreasing function supported in $[0, \max u]$. The monotone rearrangement u^* is then defined as

$$u^*(x) := \inf\{t \geq 0, \rho(t) \leq x\}, \quad x \in \mathbb{R}^+,$$

and it transpires that, if ρ is injective, then u^* is the inverse of ρ .

The main feature of u^* lies in the fact that it is *equimeasurable* w.r.t. u , namely, it has the same distribution function as u . As a consequence, the so-called layer-cake representation (see e.g. Theorem I.13 in [21]) yields

$$\|w\|_{L^p(\mathcal{X})}^p = p \int_0^{+\infty} t^{p-1} \rho(t) dt \tag{1.10}$$

for any w defined on a generic graph \mathcal{X} , belonging to the space $H^1(\mathcal{X})$, endowed with the distribution function ρ and for every $p > 1$. Formula (1.10) implies that every L^p -norm of a function is determined by the distribution function, so that two functions, even if defined on different spaces, share the same L^p -norms if their distribution functions are the same. In particular, if $p = 4$, then

$$\|u\|_{L^4(\mathcal{G})} = \|u^*\|_{L^4(\mathbb{R}^+)}. \tag{1.11}$$

Concerning the kinetic energy, the well-known *Pólya-Szegő* inequality holds, stating that

$$\|(u^*)'\|_{L^2(\mathbb{R}^+)} \leq \|\nabla u\|_{L^2(\mathcal{G})}. \tag{1.12}$$

The proof of Pólya-Szegő inequality is complicated yet classical ([21]), but it is quite intuitive that the monotone rearrangement is a way of reorganizing the mass of the function u in such a way that the quadratic average slope is reduced, or at least not increased. This observation can be brought to a quantitative level by recognizing that, in general, oscillations make the average slope increase and provide *proliferation of preimages*: a single oscillation produces indeed an interval in the range of the function, in which every point *has at least two preimages*. Monotonically rearranging yields to a function such that every point in its range has exactly one preimage (and therefore no oscillations take place). This observation is included in the formula

$$\|u'\|_{L^2(\mathcal{G})}^2 \geq \int_0^{\|u\|_{L^\infty(\mathcal{G})}} dt N(t)^2 \left(\sum_{x_i \text{ s.t. } u(x_i)=t} |u'(x_i)|^{-1} \right)^{-1} = \int_0^{\|u\|_{L^\infty(\mathcal{G})}} dt \frac{N(t)^2}{|\rho'(t)|},$$

where $N(t)$ is the number of the preimages of t for the function u and equality holds if and only if $|u'(x_i)| = |u'(x_j)|$ whenever $u(x_i) = u(x_j)$. It is then clear that in order to minimize the kinetic energy,

once the distribution function is assigned, one has to reduce the number of preimages and fulfil some symmetry requirement.

Finally, from the definition of the functional (1.2) and formulas (1.11) and (1.12), one immediately has

$$E(u^*, \mathbb{R}^+) \leq E(u, \mathcal{G}). \quad (1.13)$$

It is possible to extend the rearrangement theory to the case of compact graphs, i.e., made of finite edges only. Let us call L the *total length* of the graph, i.e., the sum of the length of all edges. Then, as shown in [17] the construction illustrated for non-compact graphs, provides for every w a monotone rearrangement w^* defined on $[0, L]$. In other words, the monotone rearrangement preserves the size of the domain.

The results in (1.9) and (1.13) can be put together to state the following proposition:

Proposition 1.2 (Pinching). *For every non-compact graph \mathcal{G} , the following estimates hold:*

$$-\frac{\mu^3}{24} \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq -\frac{\mu^3}{96}$$

In Theorem 2.2 we shall exhibit quite a huge class of graphs whose topology prevents Problem P from having a solution: according to Proposition 1.2, for all those cases $\mathcal{E}_{\mathcal{G}}(\mu) = -\mu^3/96$, but in none of them the infimum is attained.

It proves useful to have criteria in order to establish that the infimum is attained, even without being able to compute it. The key result to this aim is the following

Proposition 1.3. *If $\mathcal{E}_{\mathcal{G}}(\mu) < -\mu^3/96$, then such an infimum is attained.*

A more flexible version of the previous proposition is given by the following

Corollary 1.4 (Comparison). *The infimum of $E(\cdot, \mathcal{G})$ in $H_{\mu}^1(\mathcal{G})$ is attained if and only if there exists $u \in H_{\mu}^1(\mathcal{G})$ such that*

$$E(u, \mathcal{G}) \leq -\frac{\mu^3}{96}.$$

Such a corollary is relevant under both a practical and a conceptual point of view. At a practical level, it shows that in order to prove existence of the ground state, it is sufficient to exhibit a function whose energy level is lower than the level of the soliton. Conceptually, first, as already explained, there are functions whose energy is arbitrarily close to that of the soliton, but, except for some particular graphs, a soliton cannot be exactly constructed, so that the energy level of a soliton may in general not be attained, because sequences that approximate a soliton run away at infinity, weakly converging to zero.

Now, Corollary 1.4 establishes that if there is a function whose energy is lower than the energy of the soliton with the same mass, then there are minimizing sequences that are compact, i.e., they do not run away in order to reconstruct a far away soliton, but converge to a minimizer.

In other words, there is a competition between the soliton and the function that can be hosted on a graph. If the latter wins the competition, then the infimum is attained.

In what follows we shall give examples and results that illustrate how topology (Section 2) or metric (Section 3) may influence the existence of a ground state.

2. Topology

As mentioned in the previous section, the halfline can be seen as a graph with one vertex at the origin and the other at infinity. On the other hand, the line can be understood as a graph made of two halflines and three vertices: one joining the two halflines, the two others at infinity. These two cases can also be described as *infinite star graphs*, i.e. graphs made of a certain number of halflines, meeting at the unique vertex. Immediately beyond the case of the halfline and of the line, lies the case of the star graph S_3 made of three halflines connected at a single vertex (Fig.3):

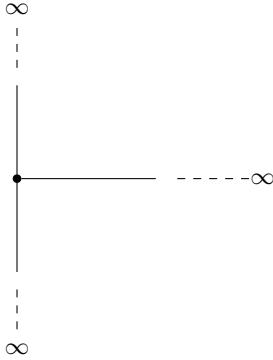


FIGURE 3. **The infinite three-star graph S_3 .** Three halflines meeting at a single vertex. For this graph, whatever the value of the mass μ , there is no ground state, as minimizing sequences run away towards infinity mimicking the shape of the soliton ϕ_μ on a single halfline.

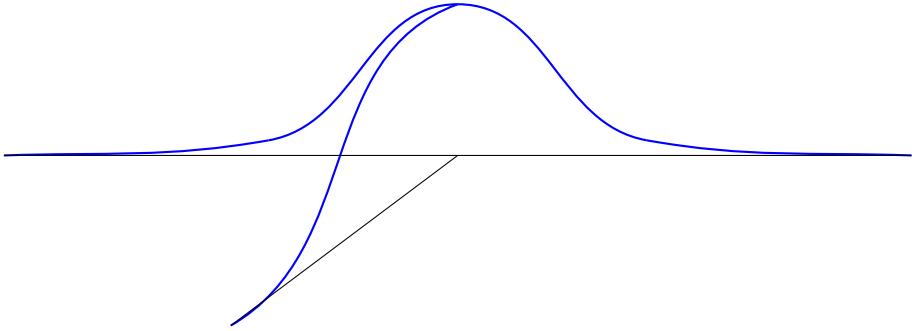


FIGURE 4. **Three half-solitons.** This is the only stationary state for the infinite star-graph made of three half-line. However, it is not a ground state.

It has been shown in [2] (see also [4,5]) that in this case

$$\mathcal{E}_G(\mu) = -\frac{\mu^3}{96}$$

but such an infimum is not attained. The situation can be displayed as follows: the only stationary state u_s is made of three half-solitons of mass $\mu/3$ with their maximum at the vertex (Fig.4).

The energy level of u_s can be easily computed as

$$E(u_s, G) = -\frac{\mu^3}{216} > -\frac{\mu^3}{96}$$

so that u_s cannot be a ground state, in spite of the fact that it is the only stationary state, i.e. the only solution to equation (1.7). The same phenomenon occurs for all star-graphs made of $N \geq 3$ halflines, as outlined in [2]. The deep reason for that has been later investigated in [7], and turns out to be rooted in the rearrangement theory. In order to illustrate this point, one has to introduce the symmetric rearrangement on the line.

2.1. Rearrangements: counting preimages

The monotone rearrangement u^* , introduced in Proposition 1.1, is not the unique rearrangement that can be made in order to pass from functions on the graph to functions on the line. Another kind of

rearrangement is the *symmetric* one, that associates to any $u \in H_\mu^1(\mathcal{G})$ a function $\hat{u} \in H_\mu^1(\mathbb{R})$. The following result holds:

Proposition 2.1 (Symmetric rearrangement). Given $u \in H_\mu^1(\mathcal{G})$, there exists a function $\hat{u} \in H_\mu^1(\mathbb{R})$ s.t.

$$\|\hat{u}\|_{L^4(\mathbb{R})} = \|u\|_{L^4(\mathcal{G})}.$$

Furthermore, if almost every element of $\text{Ran } u$ has at least two preimages in the domain of u , then

$$\|(\hat{u})'\|_{L^2(\mathbb{R})} \leq \|\nabla u\|_{L^2(\mathcal{G})}.$$

For practical purposes, \hat{u} can be constructed as the unique function on \mathbb{R} , that shares with u the same distribution function and is symmetric and monotonically decreasing in $[0, +\infty)$. The relationship with the monotone rearrangement writes simply as

$$\hat{u}(x) := u^*(2|x|).$$

Roughly speaking, Proposition 2.1 shows that the rearranged function makes the kinetic energy decrease as it lowers the measure of the set of the preimages of almost all elements of $\text{Ran } u$: being the set of preimages a zero-dimensional manifold, the natural measure is given by counting points. In this respect, the monotonic rearrangement u^* introduced in the previous section is the best one could find, since for any element in $\text{Ran } u$ it provides *one* preimage only. On the other hand, the symmetric rearrangement \hat{u} provides *two* preimages for every element of $\text{Ran } u$ (except for the maximum of u), so that symmetrically rearranging lowers the energy under the additional hypothesis that for almost every point in $\text{Ran } u$ there are at least two preimages.

We are now in position to understand the reason of the negative result for the infinite star graph made of three halflines: consider a function $u \in H_\mu^1(\mathcal{S}_3)$, and imagine to follow the function starting from its maximum point: you can move rightwards or leftwards (unless the maximum is not at the vertex, where one has possibly still more options), but whatever direction you choose, in order to belong to $H^1(\mathcal{S}_3)$ the value of the function you are following must end up at zero. In other words, for every point in $\text{Ran } u$ (except at most for $\max u$), there are at least two preimages, one for each direction that can be taken starting from the maximum of u , so that

$$E(u, \mathcal{S}_3) \geq E(\hat{u}, \mathbb{R}) \geq E(\phi_\mu, \mathbb{R}) = -\frac{\mu^3}{96}.$$

Furthermore, in order to attain the level $-\frac{\mu^3}{96}$, the rearrangement \hat{u} must coincide with ϕ_μ , then, owing to the construction of the rearrangement \hat{u} (see [7]), the support of u should be infinite, so that u can be considered as a function on the line. As a consequence, in order to avoid having more than two preimages in an interval of elements of its range, u should be entirely supported in a halfine and attain zero at the origin, so u would have a *strictly greater energy* than the soliton, contradicting the assumption that the level of the soliton is attained. This is why there is no ground state if \mathcal{G} is an infinite star-graph with more than two halflines, regardless of the value of the mass.

2.2. Assumption (H): graphs as bunches of lines

It is worth remarking that, for an infinite star graph with more than two halflines, the impossibility of having less than two preimages is dictated by the topological structure of the graph: metrics has no role when one considers all functions in $H_\mu^1(\mathcal{G})$, however shrunk or stretched. More specifically, in our argument the crucial fact was that, starting from the maximum point of \mathcal{G} , a point at infinity was available in both directions, so that, in order to belong to $H^1(\mathcal{G})$, the function u was forced to go to zero in both directions, running through its values at least twice. The same necessity is not present in the case of the halflines, where, if one places the maximum at the origin, then it is possible to attain each value in the range of u exactly once.

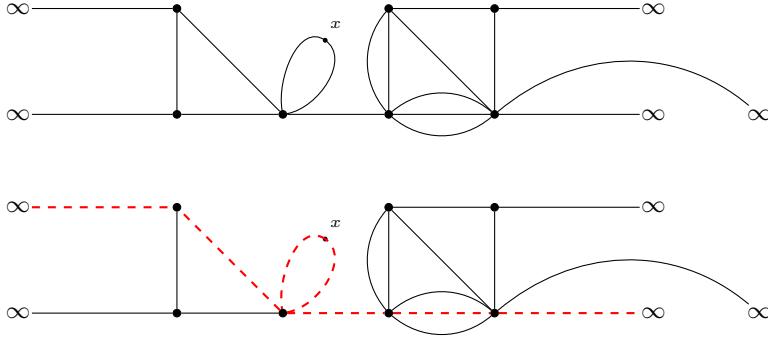


FIGURE 5. **Assumption (H):** every point of the graph lies on a trail connecting two distinct vertices at infinity. The point x belongs to several such trails: in the picture we put in evidence one of them, by representing it through dashed lines.

In the effort of understanding the relationship between topology of a graph and existence of ground states, a non-trivial task then was involved in the search for topological conditions that ensure that

Given an arbitrary function $u \in H^1(\mathcal{G})$, almost every point in $\text{Ran } u$ is endowed with at least two preimages.

The key point is to find a simple condition capable to ensure that, when following a function starting from its maximum, a point at infinity is available in both directions. Besides, since the maximum may be located at any point of the graph, one has to guarantee that, starting from *every* point on the graph, an infinity point is available when moving in either direction.

We finally got the following condition (Fig.5):

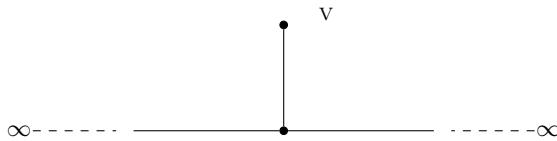
Assumption (H). First version. *Every point of \mathcal{G} lies in a trail that connects two different vertices at infinity.*

A trail is a connected sequence of non-repeated edges. Assumption (H) states exactly that from every point one can get to infinity through two disjoint paths, so the argument used for the infinite star graphs still holds. In other words, every edge can be considered as part of a line, so it is not possible to get an energy level lower than the energy of the soliton.

An equivalent version of Assumption (H), that proves more manageable for proofs, is the following:

Assumption (H). Second version. *Removing an arbitrary edge, every resulting connected component contains a vertex at infinity.*

Owing to this version, it is easier to visualize how Assumption (H) can fail. Clearly, it fails when there is one halfline only (for instance, if $\mathcal{G} = \mathbb{R}^+$), and it fails when there is a terminal edge, or “pendant” too:



Notice that, in the latter case, violation of Assumption (H) is due to the presence of the pendant, whose removal yields a compact connected component (Fig.6).

Assumption (H) leads to a negative result, that is the main goal of [7].

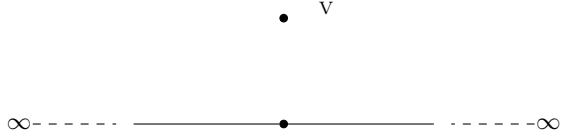


FIGURE 6. **Failure of Assumption (H) for the line with a pendant.** Removing the pendant, two connected components are produced: one of them is a line, the second one is made by a vertex at a finite point of the space. So the line with a pendant, although endowed with two halflines, violates Assumption (H).

Theorem 2.2 (Nonexistence). *Assume that \mathcal{G} satisfies assumption (H). Then*

$$\inf_{u \in H_\mu^1(\mathcal{G})} E(u, \mathcal{G}) = E(\phi_\mu, \mathbb{R})$$

and it is never attained, except if \mathcal{G} is a “bubble tower”.

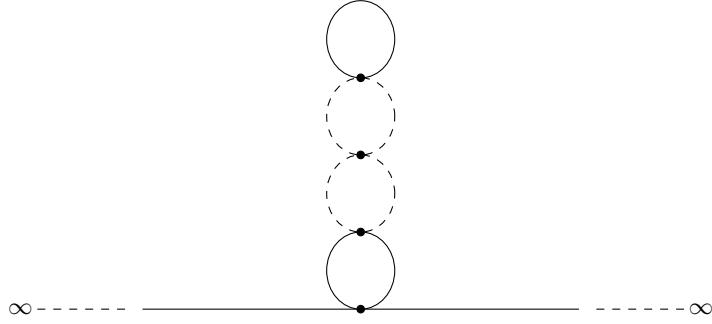


FIGURE 7. **Bubble tower.** Bubble towers are the only graphs that satisfy Assumption (H) but admit ground states (again, regardless of the choice of μ). Notice that every bubble is cut in two arcs of equal length by the vertices lying on it.

The only graphs satisfying assumption (H) and admitting a minimum are the “bubble towers”, since one can cut a soliton on the line and paste it on the top of the tower (see Fig.7). As the energy is not affected by the placement of the function on the graph (Fig.8), Corollary 1.4 guarantees the existence of a ground state.

2.3. Violating (H): a line with a pendant

As already stated, one of the simplest graphs that does not satisfy assumption (H) is the real line with a pendant. We show that, for every $\mu > 0$, there exists a ground state. The procedure will also highlight the shape of such a ground state. In particular, we prove that, denoted by \mathcal{P} the graph made of a line and a pendant,

$$\inf_{u \in H_\mu^1(\mathcal{P})} E(u, \mathcal{P}) < -\frac{\mu^3}{96},$$

so that, owing to Corollary 1.4, the infimum is attained. To this aim, We use a *graph surgery* together with rearrangements.

As a first step, we cut the soliton ϕ_μ centred at a width ℓ (see Fig.9).

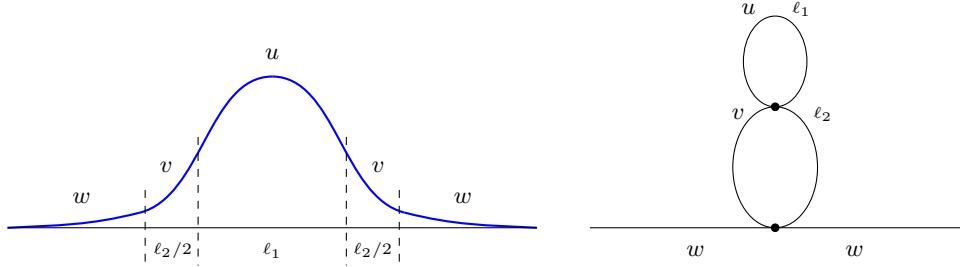


FIGURE 8. **A soliton on the line with two bubbles.** A soliton can be cut and pasted on a bubble tower. Its energy is not affected by the procedure, so that the infimum is attained.

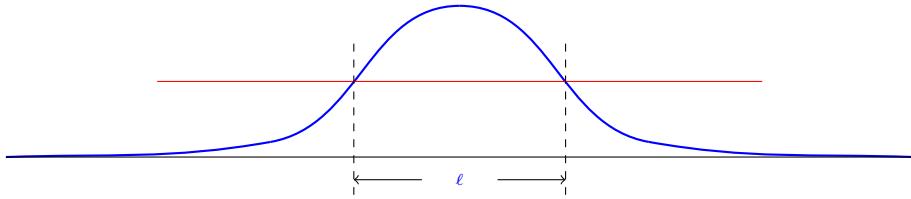


FIGURE 9. **Graph surgery, step 1:** A soliton is cut at the height such that the corresponding width equals the length of the pendant in the graph \mathcal{P} .

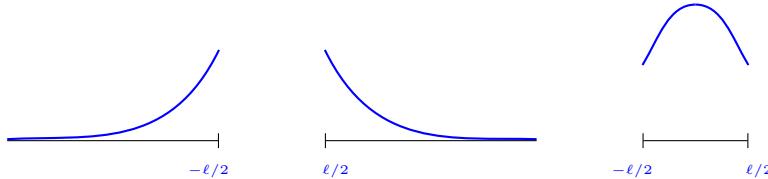


FIGURE 10. **Pieces of soliton.** After the previous cut, one is left with three pieces: two tails and one head.

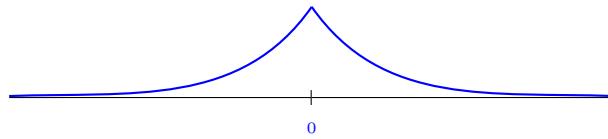


FIGURE 11. **Graph surgery, step 2.** The two tails produced by the first surgery step and shown in Fig. 10 are pasted together in such a way that the maximum is located at the vertex. Notice also that the maximum is a corner point.

Now we join the two resulting soliton tails (see Fig.10) together at their maximum, and place them on the line of the graph \mathcal{P} , with the maximum at the vertex, as illustrated in Fig.11.

Then, we rearrange the head of the soliton monotonically on the pendant, namely on the interval $[0, \ell]$ (see Fig.12). Notice that for every point on the range, except the maximum, the number of preimages passes from two to one. This makes the energy strictly decrease.

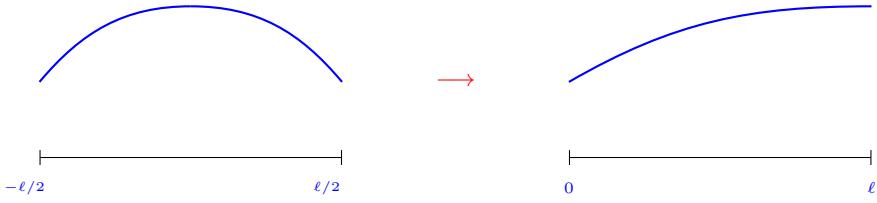


FIGURE 12. **Monotone rearrangement on the pendant.** The monotone rearrangement of the soliton head lowers the energy.

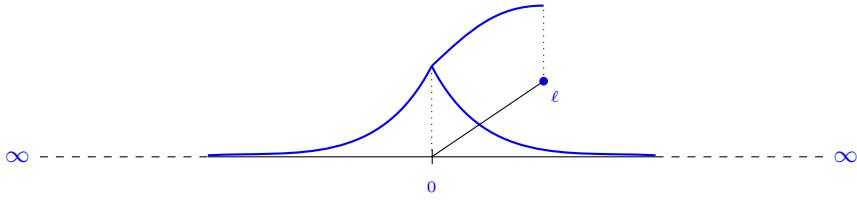


FIGURE 13. **The competitor \tilde{u} .** Through the procedure we illustrated a new function \tilde{u} is constructed, that turns out to be a good competitor for the soliton.

Finally, we put all pieces together and construct a function \tilde{u} on \mathcal{P} (Fig.13), such that

$$E(\tilde{u}, \mathcal{P}) < -\frac{\mu^3}{96}.$$

Then, by Corollary 1.4, the infimum is attained.

It is worth stressing that in this example the construction does not provide a ground state, but rather a good competitor only. This is apparent since, as illustrated in Figure 16, the function \tilde{u} cannot satisfy the Kirchhoff condition (1.8), as the slope on the pendant at the vertex denoted by 0 is too low. Anyway, the important fact is to produce by surgery a function whose energy is lower than the energy of the soliton and then apply Corollary 1.4.

In this example, metric turns out to have no role in the existence of the ground state. So, this result relies on topology only.

In the next section we shall examine a case in which topology is not sufficient to establish the existence of a ground state.

3. Metric

In the preceding section we stressed the role of topology in the existence of ground states. But for graphs where assumption (H) fails, existence and nonexistence can be conditioned by the metric. Let us stress that, according to the correspondence between long edges and large mass, the example we are giving can be understood as a case in which, for a fixed graph, the existence of a ground state depends on the choice of the mass. We will adopt the first point of view, in which the existence depends on the metric, because it is more intuitive.

Let \mathcal{G}_ℓ be a graph made of three halflines and a finite pendant of length ℓ (see Fig.14). We will try to follow the reasoning applied in the case of the line with the pendant, and show why it fails. It immediately appears that, fixed μ , if the pendant is long enough, then it is possible to place almost exactly half a soliton on it, producing then a function whose energy is less than the energy of the soliton, so, by Corollary 1.4, the infimum is attained and then a ground state exists.

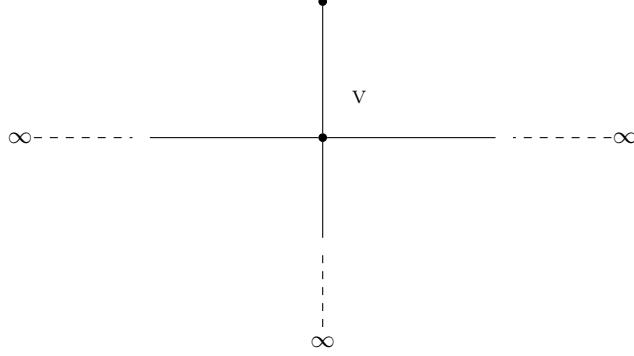


FIGURE 14. The graph \mathcal{G}_ℓ : three halflines and a finite pendant of length ℓ that meet together at a unique vertex v .

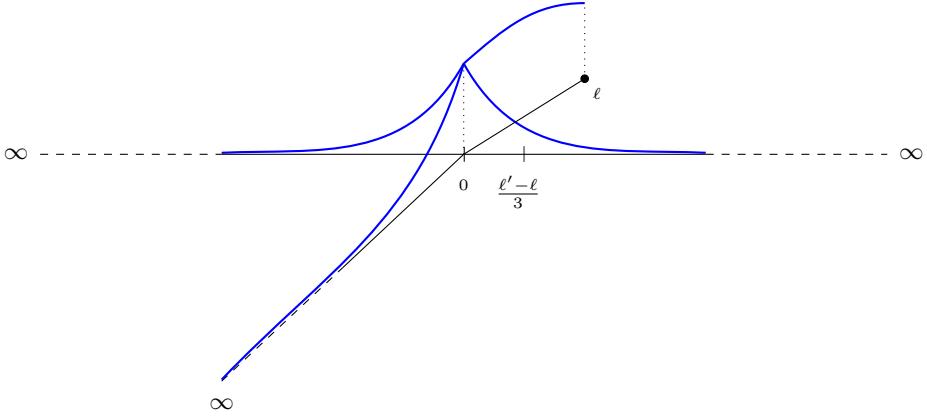


FIGURE 15. The ground state ψ_ℓ . Assuming that the pendant is long enough to trap a ground state, one can prove that the shape of such a ground state must be the one displayed here, with the maximum at the tip of the pendant.

Conversely, if ℓ is small enough, then one can prove that a ground state does not exist: indeed, owing to Theorem 4.3 in [8], if there is a ground state for every $\ell > 0$, then there is a ground state for $\ell = 0$ too. But for $\ell = 0$ the graph \mathcal{G} reduces to the star graph made of three halflines, that falls into the scope of Theorem 2.2. It follows that, as the pendant grows, there is at least one transition from nonexistence to existence. Let us show that there is *one* transition only, i.e. that there exist a unique *critical length* $\ell^* > 0$ such that a ground state exists if and only if $\ell \geq \ell^*$.

To this aim, let ℓ be such that a ground state ψ_ℓ correspondingly exists (Fig.15). Then, in analogy with the case of the line with a pendant, it is possible to show, just arguing on the number of counterimages, that the ground state looks like as shown in Fig.15. Consider now the graph $\mathcal{G}_{\ell'}$ with $\ell' > \ell$. Then, cut on every halfline the interval $[0, (\ell' - \ell)/3]$ and rearrange the three pieces monotonically together on the interval $[0, \ell' - \ell]$ (see Fig.16).

Again, the loss of preimages yielded by the rearrangement lowers the energy level, thus one can mount the three pieces, obtaining a function whose energy is lower than the energy of ψ_ℓ .

In this way we obtained a function $\tilde{\psi} \in H_\mu^1(\mathcal{G}_{\ell'})$ (Fig.17) such that

$$E(\tilde{\psi}, \mathcal{G}_{\ell'}) < E(\psi_\ell, \mathcal{G}_\ell) \leq -\frac{\mu^3}{96},$$

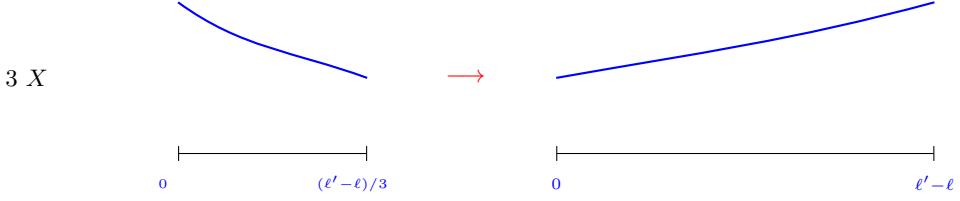


FIGURE 16. **Rearranging the top of the tails of ψ_ℓ .** On the exceeding length of the pendant of $\mathcal{G}_{\ell'}$ one puts the collective monotone rearrangement of the tips of the three soliton tails of ψ_ℓ , as portrayed in Fig.16. This step lowers the energy level.

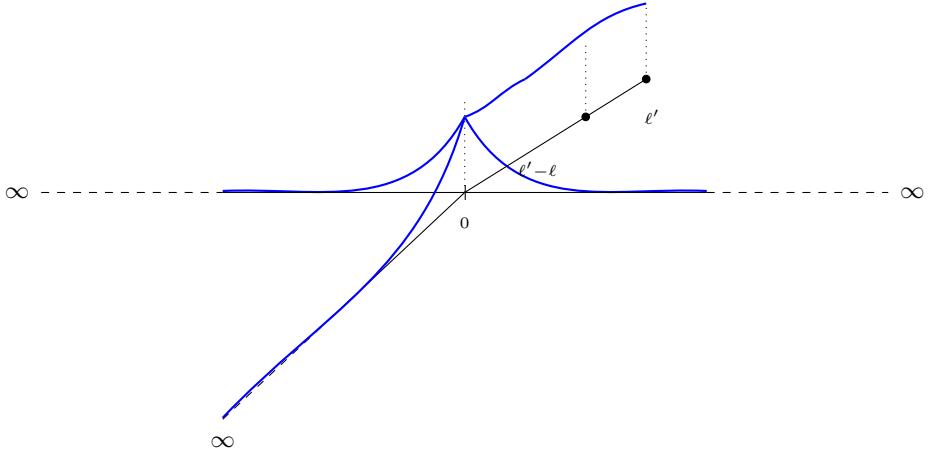


FIGURE 17. **A better competitor.** The function $\tilde{\psi}$, resulting from the surgery and rearrangement procedure illustrated for ψ_ℓ , has an energy content that is lower than the one of ψ_ℓ . Again, the maximum of $\tilde{\psi}$ is located at the tip of the pendant.

and therefore a ground state at mass μ exists for the graph $\mathcal{G}_{\ell'}$.

Besides, if $\ell' > \ell$, then

$$\mathcal{E}_{\mathcal{G}_{\ell'}}(\mu) < \mathcal{E}_{\mathcal{G}_\ell}(\mu),$$

so that $\mathcal{E}_{\mathcal{G}_\ell}(\mu)$ is a monotonically decreasing function of ℓ' . Thus, once $\mathcal{E}_{\mathcal{G}_\ell}(\mu) \leq -\mu^3/96$, as ℓ increases a ground state still exists. Moreover, according to our analysis in the case of large ℓ , one has

$$\mathcal{E}_{\mathcal{G}_\ell}(\mu) \rightarrow -\frac{\mu^3}{24}, \quad \ell \rightarrow \infty.$$

At first sight, this relationship may look misleading, as the star graph made of four halflines does not admit ground states, so that every function defined on it has energy greater than $-\mu^3/96$. To avoid such a difficulty, one has to remember that such a star graph is *by no way* the limit of a graph made of three halflines and a pendant: indeed, the tip of the pendant has no zero-condition (in particular, minimizers satisfy a homogenous Neumann condition on it, and not a homogeneous Dirichlet condition), whereas at every vertex at infinity all functions in H^1 must fulfil a zero condition.

4. Perspectives

The problem of the existence of ground states on graphs is physically relevant, for instance in the context of Bose-Einstein condensates. Our results show that there is a competition between the tendency of the

condensate to stay on the compact core of the trap and the tendency to run away along long leads. The topology of the graph can solve the alternative: in particular, the condensate can choose the first option only if hypothesis (H) is not satisfied by the graph, that means that something like a bottleneck is present in the structure of the trap.

The analysis on the existence of ground states for the energy of a system living on a graph, can be extended in many different directions: first of all, we remark once again that the same results illustrated here for the cubic case hold for all subcritical nonlinearity powers $\|u\|_{L^p(\mathcal{G})}^p$ with $2 < p < 6$. However, for non-cubic nonlinearities it is not possible to perform exact computations, and then to write down the value of the energy level in such a precise form. On the other hand, a different behaviour is expected for the so-called *critical case* $p = 6$, that coincides with the Schrödinger equation with a quintic nonlinearity. This case is somewhat trivial in \mathbb{R}^N due to the presence of an additional symmetry, but we conjecture that the topology of the graph could somewhat inhibit, at least partially, the arising symmetry, leading to new and unexpected effects: this will be the subject of a forthcoming paper.

Richer structures can be constructed by joining together pieces of different dimensions. On the other hand, higher dimensions could be obtained by a limit procedure on the structure of graphs becoming more and more "dense". It is then natural to ask in which component should a ground state locate.

Finally, let us stress that the research on nonlinear effects on graphs is still at its beginning. Many developments are expected in the next future, both in the mathematics and physics: for instance, in modelling periodic structures and/or cells, with the possible presence of a magnetic field.

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