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Comb Model with Slow and Ultraslow Diffusion

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Abstract. We consider a generalized diffusion equation in two dimensions for modeling diffusion on a comb-like structures. We analyze the probability distribution functions and we derive the mean squared displacement in x and y directions. Different forms of the memory kernels (Dirac delta, power-law, and distributed order) are considered. It is shown that anomalous diffusion may occur along both x and y directions. Ultraslow diffusion and some more general diffusive processes are observed as well. We give the corresponding continuous time random walk model for the considered two dimensional diffusion-like equation on a comb, and we derive the probability distribution functions which subordinate the process governed by this equation to the Wiener process.

Keywords and phrases: comb-like model, anomalous diffusion, mean squared displacement, probability density function

Mathematics Subject Classification: 87.19.L-, 05.40.Fb, 82.40.-g

1. Introduction

Anomalous diffusion is typically characterized by the power-law time dependence

$$\langle x^2(t)\rangle \simeq K_\alpha t^\alpha \tag{1.1}$$

of the mean squared displacement (MSD) [1–3]. Here K_α denotes the generalized diffusion coefficient. Based on the value of the anomalous diffusion exponent α we distinguish subdiffusion ($0 < \alpha < 1$) and superdiffusion ($\alpha > 1$). The special value $\alpha = 1$ describes normal diffusion, while ballistic, completely directed motion, corresponds to $\alpha = 2$. Subdiffusion, for instance, is routinely measured for the charge carrier motion in amorphous semiconductors [4], in the cytoplasm of living biological cells [5], [6], or

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in artificially crowded liquids [7], [8]. Superdiffusion is also observed in live cells due to active motion [9], [10]. Large scale simulations in three [11], [12] and two dimensional systems [13], [14] also exhibit (transient) anomalous diffusion. In what follows we will also consider an ultraslow, logarithmic growth

$$\langle x^2(t)\rangle \simeq \log^\gamma t \tag{1.2}$$

of the MSD, on which we comment further below.

Anomalous diffusive transport has been investigated in low dimensional percolation clusters and comb-like structures by introducing comb models [15–19]. The corresponding dynamic equation for the description of the continuous comb was introduced in Ref. [19] and extensively studied [19, 20, 22–25]. In this approach the effective comb model equation reads

$$\frac{\partial}{\partial t}P(x,y,t) = \mathcal{D}_x\delta(y)\frac{\partial^2}{\partial x^2}P(x,y,t) + \mathcal{D}_y\frac{\partial^2}{\partial y^2}P(x,y,t), \tag{1.3}$$

where $-\infty < x, y < \infty$, and the initial condition is

$$P(x,y,0) = \delta(x)\delta(y), \tag{1.4}$$

and the boundary conditions for $P(x,y,t)$ and $\frac{\partial}{\partial q}P(x,y,t)$, $q = \{x,y\}$ are set to zero at infinity, $x = \pm\infty$, $y = \pm\infty$. $P(x,y,t)$ is the probability density function (PDF) to find the test particle at position (x,y) at time t. Here, x measures the direction along the backbone of the comb, while y is the distance along the teeth away from the backbone. This analytical form of the model is suggested by heuristic arguments on inhomogeneous two-dimensional diffusion, where the diagonal components of a diffusion tensor, $\mathcal{D}_x\delta(y)$ and $\mathcal{D}_y$ are the diffusion coefficients in the x and y directions, correspondingly. In this effective formulation of the comb model the individual teeth of the comb are considered smeared out and averaged along the x direction.

The comb model can be described in the framework of the continuous time random walk (CTRW) theory [2], where the particle moving along the backbone can be trapped by diffusion along the teeth during a waiting time. The waiting time PDF for this process in the Laplace space is given by $\psi(s) = \frac{1}{1+s^{1/2}}$, where $\psi(s) = \mathcal{L}[\psi(t)]$ is the Laplace transform of $\psi(t)$, meaning that the waiting time PDF in the long time limit is of power-law form $\psi(t) \simeq t^{-3/2}$. The distribution of jump lengths $\lambda(x)$ is of Gaussian form with variance σ^2, $\lambda(k) \simeq 1 - \frac{1}{2}\sigma^2k^2$, where $\lambda(k) = \mathcal{F}[\lambda(x)]$ is the Fourier transform of $\lambda(x)$ (see Section 2). Thus, the MSD along the backbone is of the form $\left\langle x^2(t)\right\rangle \simeq t^{1/2}$. This can be concluded following the procedure in Section 2 (see also [22]). From this approach one also finds that the PDF $p_1(x,t) = \int_{-\infty}^{\infty} dy P(x,y,t)$ satisfies the following time fractional diffusion equation

$$\frac{\partial^{1/2}}{\partial t^{1/2}}p_1(x,t) = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}\frac{\partial^2}{\partial x^2}p_1(x,t), \tag{1.5}$$

where

$$\frac{\partial^\alpha}{\partial t^\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t dt'\,\frac{\frac{d}{dt'}f(t')}{(t-t')^\alpha} \tag{1.6}$$

is the Caputo fractional derivative [26] and $0 < \alpha < 1$. The diffusion along the teeth is normal which can be directly concluded by integration of Eq. (1.3) over x, i.e., $p_2(y,t) = \int_{-\infty}^{\infty} dx P(x,y,t)$, and taking the boundary conditions $P(\pm\infty,y,t) = 0$ and $\frac{\partial}{\partial x}P(\pm\infty,y,t) = 0$. Thus, one obtains the diffusion equation for Brownian motion, i.e., $\frac{\partial}{\partial t}p_2(y,t) = \mathcal{D}_y\frac{\partial^2}{\partial y^2}p_2(y,t)$ and $\left\langle y^2\right\rangle \simeq t$.

There exist different generalizations of the comb model (1.3). For example, Mendez and Iomin [27], [28] recently considered the following fractional comb model

$$\frac{\partial^\alpha}{\partial t^\alpha}P(x,y,t) = \mathcal{D}_x\delta(y)\,{}_tI_{0+}^{1-\alpha}\frac{\partial^2}{\partial x^2}P(x,y,t) + \mathcal{D}_y\frac{\partial^2}{\partial y^2}P(x,y,t), \tag{1.7}$$

where

$$ {}_{RL}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t dt' (t-t')^{\alpha-1} f(t'), \tag{1.8} $$

is the Riemann-Liouville (R-L) fractional integral. A comb-like model of the form (1.7) was used to describe anomalous diffusion in spiny dendrites, where the MSD along the x direction has the power-law dependence $\langle x^2(t) \rangle \simeq t^{1-\frac{\alpha}{2}}$ on time [27], [28]. Similar comb-like models were used in different contexts for describing subdiffusion on a fractal comb [29], the mechanism of superdiffusion of ultra-cold atoms in a one dimensional polarization optical lattice [30], to describe diffusion processes on a backbone structure in presence of a drift term [31], to model anomalous transport in low-dimensional composites [32] and electron transport in disordered nanostructured semiconductors [33], or developing an effective comb-shaped configuration of antennas [34]. A comb model approach to study and simulate flows in cardiovascular and ventilatory systems, especially for multiscale biomathematical models related to virtual physiology is given in [35]. Furthermore, different random walk models were used to describe diffusion on a comb [16, 17, 20, 36–38], to mention but a few. Here we note that, as shown in [39], a generalization of the comb model with correlations in the y direction affects the diffusive behavior in the x direction in a non-trivial fashion, resulting in a quite rich diffusive scenario of anomalous transport in the x direction. Recently, it was shown [40] that if we consider a discrete model with a finite number of backbones, which means that the diffusion along the x direction may occur on many backbones, located at $y = l_j$, $j = 1, 2, \dots, N$, $0 < l_1 < l_2 < \cdots < l_N$, the transport exponent does not change, i.e., it is equal to 1/2. Contrary to this, in case of an infinite number of backbones, which are at positions y which belong to the fractal set S_ν with fractal dimension $0 < \nu < 1$, the transport exponent depends on the fractal dimension ν. We stop to mention that the diffusion on a comb in the x direction effectively is of the continuous time random walk class and thus weakly non-ergodic and ageing [41], [42], while the random motion on a fractal such as a critical percolation cluster is ergodic [43], [44].

This paper is organized as follows. In Section 2 we introduce our generalized comb-like model (2.1) that has two different memory kernels. The PDF and MSD are derived and we find the corresponding CTRW models. In Section 3 we analyze the subordination of a process governed by Eq. (2.1) to the Wiener process. The conditions that should be satisfied by the memory kernels in order that the PDFs are non-negative functions are discussed. Analytical results for different forms of the memory kernels are given in Section 4. It is shown that anomalous and ultraslow diffusion may occur in both directions, x and y. A summary is provided in Section 5.

2. Generalized comb model

Here we introduce and analyze the following generalized comb-like model

$$ \int_0^t dt'\, \gamma(t-t') \frac{\partial}{\partial t'} P(x,y,t') = \mathcal{D}_x \delta(y) \int_0^t dt'\, \eta(t-t') \frac{\partial^2}{\partial x^2} P(x,y,t') + \mathcal{D}_y \frac{\partial^2}{\partial y^2} P(x,y,t), \tag{2.1} $$

with the same initial and boundary conditions as in Eq. (1.3). In what follows we use dimensionless variables without loss of generality. Thus, we set $\mathcal{D}_x = \mathcal{D}_y = 1$. Here $\gamma(t)$ and $\eta(t)$ are integrable non-negative memory kernels which approach zero in the long time limit (see next section for more details). The memory kernel $\gamma(t)$ represents the memory effects of the system, which means that the particles moving along the y direction, i.e., along the teeth, may also be trapped, thus the diffusion along the y direction may also be anomalous [45]. This can be directly concluded by integration of Eq. (2.1) over x, and taking the boundary conditions which are of the form $P(\pm\infty, y, t) = 0$ and $\frac{\partial}{\partial x} P(\pm\infty, y, t) = 0$, from where we obtain the generalized diffusion equation recently considered in [45]

$$ \int_0^t dt'\, \gamma(t-t') \frac{\partial}{\partial t'} p_2(y,t') = \frac{\partial^2}{\partial y^2} p_2(y,t), \tag{2.2} $$

where $p_2(y,t) = \int_{-\infty}^{\infty} dx P(x,y,t)$. It is shown that this model represents a useful tool to describe anomalous diffusion, in particular, ultraslow logarithmic diffusion, and that it corresponds to a CTRW model with a Gaussian type jump length PDF $\lambda(y)$, and waiting time PDF given in the Laplace space by $\psi(s) = \frac{1}{1+s\gamma(s)}$. Going back to Eq. (2.1), the case $\gamma(t) = \eta(t) = \delta(t)$ yields the diffusion equation (1.3). The memory kernel $\eta(t)$ was introduced in [27] to compensate the anomalous transport in the backbone. Thus, here, we call the memory kernel $\eta(t)$ as a generalized compensation kernel.

2.1. Solution in Fourier-Laplace space

In order to solve Eq. (2.1) we use the Laplace

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty dt f(t) e^{-st} = F(s), \tag{2.3}$$

and Fourier transform,

$$\mathcal{F}\left[f(x)\right] = \int_{-\infty}^\infty dx f(x) e^{\imath kx} = F(k), \tag{2.4}$$

and consequently the inverse Laplace $\mathcal{L}^{-1}\left[F(s)\right] = f(t)$, and Fourier transform $\mathcal{F}^{-1}\left[F(k)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk F(k) e^{-\imath kx} = f(x)$. In what follows the transformed quantities are denoted by explicit dependence on the respective variables. By Laplace transform we find

$$\gamma(s)\left[sP(x,y,s) - P(x,y,t=0)\right] = \delta(y)\eta(s)\frac{\partial^2}{\partial x^2}P(x,y,s) + \frac{\partial^2}{\partial y^2}P(x,y,s), \tag{2.5}$$

where $P(k,y,t=0)$ is the initial condition to be specified below. By Fourier transform with respect to both space variables x and y it follows that

$$\gamma(s)\left[sP(k_x,k_y,s) - P(k_x,k_y,t=0)\right] = -k_x^2\eta(s)P(k_x,y=0,s) - k_y^2 P(k_x,k_y,s). \tag{2.6}$$

Therefore, PDF in the Fourier-Laplace space is given by

$$P(k_x,k_y,s) = \frac{\gamma(s)P(k_x,k_y,t=0) - k_x^2\eta(s)P(k_x,y=0,s)}{s\gamma(s)+k_y^2}. \tag{2.7}$$

The initial condition is given by (1.4), thus by inverse Fourier transform in respect to k_y we find

$$P(k_x,y,s) = \frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}e^{-\sqrt{s\gamma(s)}|y|} - \frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}\frac{\eta(s)k_x^2}{\gamma(s)}P(k_x,y=0,s)e^{-\sqrt{s\gamma(s)}|y|}. \tag{2.8}$$

If we substitute $y = 0$ we obtain

$$P(k_x,y=0,s) = \frac{\frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}}{1+\frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}\frac{\eta(s)}{\gamma(s)}k_x^2}, \tag{2.9}$$

from where it follows that

$$P(k_x,y,s) = \frac{\frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}}{1+\frac{1}{s}\sqrt{\frac{s\gamma(s)}{4}}\frac{\eta(s)}{\gamma(s)}k_x^2}e^{-\sqrt{s\gamma(s)}|y|}, \tag{2.10}$$

and

$$P(k_x,k_y,s) = \frac{s\gamma(s)\xi(s)}{\left(s\gamma(s)+k_y^2\right)\left(s\xi(s)+\frac{1}{2}k_x^2\right)}, \tag{2.11}$$

where

$$\xi(s) = \frac{1}{\eta(s)}\sqrt{\frac{\gamma(s)}{s}}. \tag{2.12}$$

Relation (2.11) for $k_y = 0$ yields

$$P(k_x, k_y = 0, s) = \frac{\xi(s)}{s\xi(s) + \frac{1}{2}k_x^2}, \tag{2.13}$$

and for $k_x = 0$

$$P(k_x = 0, k_y, s) = \frac{\gamma(s)}{s\gamma(s) + k_y^2}, \tag{2.14}$$

corresponding to the spatial averages in y and x directions, respectively.

2.2. PDF and MSD in the x direction

Let us now analyze the PDF $p_1(x,t) = \int_{-\infty}^{\infty} dy P(x,y,t)$. Its Fourier-Laplace transform reads $p_1(k_x, s) = P(k_x, k_y = 0, s)$, i.e. following Eq. (2.13),

$$p_1(k_x, s) = \frac{\xi(s)}{s\xi(s) + \frac{1}{2}k_x^2}. \tag{2.15}$$

From the results given in [45] we conclude that the PDF $p_1(k_x, s)$ corresponds to the one obtained from the CTRW theory for a process with Gaussian jump length PDF and waiting time PDF whose Laplace transform is $\psi_1(s) = \frac{1}{1+s\xi(s)}$. The normalization condition requires $\lim_{s\to 0} s\xi(s) = \lim_{t\to\infty} \xi(t) = 0$. Thus, the memory kernel (2.12) is of the form that should satisfy $\lim_{s\to 0} \frac{\sqrt{s\gamma(s)}}{\eta(s)} = 0$. From Eq. (2.15) it follows that

$$\xi(s)\left[sp_1(k_x, s) - 1\right] = -\frac{1}{2}k_x^2 p_1(k_x, s), \tag{2.16}$$

from where one finds [45] that the PDF $p_1(x,t)$ satisfies the following generalized diffusion equation

$$\int_0^t dt'\, \xi(t - t')\frac{\partial}{\partial t'}p_1(x, t') = \frac{1}{2}\frac{\partial^2}{\partial x^2}p_1(x, t), \tag{2.17}$$

with initial condition $p_1(x, 0) = \delta(x)$.

The MSD therefore yields in the form [45]

$$\left\langle x^2(t)\right\rangle = \mathcal{L}^{-1}\left[-\frac{\partial^2}{\partial k_x^2}p_1(k_x, s)\right]\Big|_{k_x=0} = \mathcal{L}^{-1}\left[\frac{1}{s^2\xi(s)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\frac{\eta(s)}{\sqrt{s\gamma(s)}}\right]. \tag{2.18}$$

From this relation we directly obtain the MSD for the simple comb-like model (1.3).

2.3. PDF and MSD in the y direction

Next we analyze the PDF $p_2(y,t) = \int_{-\infty}^{\infty} dx P(x,y,t)$, for which we find that $p_2(k_y, s) = P(k_x = 0, k_y, s)$, i.e, following Eq. (2.14)

$$p_2(k_y, s) = \frac{\gamma(s)}{s\gamma(s) + k_y^2}. \tag{2.19}$$

According to the results given in [45], the PDF $p_2(y,t)$ corresponds to the one obtained from the CTRW theory for a process with Gaussian type jump length PDF and waiting time PDF given by its Laplace trasnform as $\psi_2(s) = \frac{1}{1+s\gamma(s)}$. The normalization condition requires $\lim_{\to 0} s\gamma(s) = \lim_{t\to\infty}\gamma(t) = 0$ as well. We rewrite (2.19) as

$$\gamma(s)\left[sp_2(k_y,s)-1\right] = -k_y^2 p_2(k_y,s), \tag{2.20}$$

from where by inverse Fourier-Laplace transform it is obtained that the PDF $p_2(x,t)$ satisfies the following generalized diffusion equation

$$\int_0^t dt'\,\gamma(t-t')\frac{\partial}{\partial t'}p_2(y,t') = \frac{\partial^2}{\partial y^2}p_2(y,t), \tag{2.21}$$

with initial condition $p_2(y,0) = \delta(y)$.

Thus, for the MSD along y direction one finds [45]

$$\left\langle y^2(t)\right\rangle = \mathcal{L}^{-1}\left[-\frac{\partial^2}{\partial k_y^2}p_2(k_y,s)\right]\Big|_{k_y=0} = 2\mathcal{L}^{-1}\left[\frac{1}{s^2\gamma(s)}\right], \tag{2.22}$$

i.e., the MSD along the y direction depends only on the memory kernel $\gamma(t)$.

3. Subordination to the Wiener process

Let us now find the corresponding PDFs which subordinate the diffusion processes, governed by equations (2.15) and (2.19), from time scale t (physical time) to the Wiener processes on a time scale u (operational time). In such a scheme the PDF $P(x,t)$ of a given random process $x(t)$ can be represented as [46–48]

$$P(x,t) = \int_0^\infty P_0(x,u)h(u,t)du \tag{3.1}$$

where

$$P_0(x,u) = \frac{1}{\sqrt{4\pi u}}\exp\left(-\frac{x^2}{4u}\right), \tag{3.2}$$

is the famed Gaussian PDF, i.e., the PDF of the Wiener process, and $h(u,t)$ is a PDF *subordinating* the random process $x(t)$ to the Wiener process. Note that here we use that diffusion coefficient is equal to one ($\mathcal{D} = 1$).

Relations (2.15) and (2.19) can be rewritten in the form

$$p_1(k_x,s) = \int_0^\infty du\, e^{-u\frac{1}{2}k_x^2}h_1(u,s), \tag{3.3}$$

where

$$h_1(u,s) = \xi(s)e^{-us\xi(s)} = -\frac{\partial}{\partial u}\frac{1}{s}L_1(s,u), \tag{3.4}$$

$$L_1(s,u) = e^{-us\xi(s)}, \tag{3.5}$$

and

$$p_2(k_y,s) = \int_0^\infty du\, e^{-uk_y^2}h_2(u,s), \tag{3.6}$$

moreover,

$$h_2(u,s) = \gamma(s)e^{-us\gamma(s)} = -\frac{\partial}{\partial u}\frac{1}{s}L_2(s,u), \tag{3.7}$$

and

$$L_2(s,u) = e^{-us\gamma(s)}, \tag{3.8}$$

respectively. By inverse Fourier-Laplace transform of (3.3) and (3.6) one finds

$$p_1(x,t) = \int_0^\infty du\, \frac{1}{\sqrt{2\pi u}} e^{-\frac{x^2}{2u}} h_1(u,t), \tag{3.9}$$

and

$$p_2(y,t) = \int_0^\infty du\, \frac{1}{\sqrt{4\pi u}} e^{-\frac{y^2}{4u}} h_2(u,t), \tag{3.10}$$

which means that the PDFs $h_1(u,t)$ and $h_2(u,t)$ provide subordination of the random processes governed by equations (2.15) and (2.19), respectively, to the Wiener process by using the operational time u. Here we note that the functions $h_1(u,t)$ and $h_2(u,t)$ are normalized in respect to u. This can be shown as follows. From Eq. (3.4) we find

$$\int_0^\infty du\, h_1(u,t) = \mathcal{L}_s^{-1}\left[\int_0^\infty du\, \xi(s)e^{-us\xi(s)}\right] = \mathcal{L}_s^{-1}\left[\frac{1}{s}\right] = 1. \tag{3.11}$$

The same procedure can be applied to $h_2(u,t)$.

By using the subordination approach one can see that the PDFs $p_1(x,t)$ and $p_2(y,t)$ are non-negative if respectively, $h_1(u,t)$ and $h_2(u,t)$ are non-negative, therefore, it is sufficient to show that their Laplace transforms $h_1(u,s)$ and $h_2(u,s)$ are completely monotone functions in respect to s (see Appendix A for definitions). In order $h_1(u,s)$, Eq. (3.4), to be a completely monotone function, both functions $\xi(s) = \frac{1}{\eta(s)}\sqrt{\frac{\gamma(s)}{s}}$ and $e^{-us\xi(s)}$ should be completely monotone. The function $e^{-us\xi(s)}$ is completely monotone if $s\xi(s) = \frac{1}{\eta(s)}\sqrt{s\gamma(s)}$ is a Bernstein function [45]. Similarly, in order the function $h_2(u,s)$, Eq. (3.7), to be a completely monotone function, the function $\gamma(s)$ should be completely monotone and $s\gamma(s)$ should be a Bernstein function. These conditions give restrictions on the possible choice of the kernels $\gamma(t)$ and $\eta(t)$ in Eq. (2.1).

4. Different diffusion regimes

Here we consider different forms of the memory kernels $\gamma(t)$ and $\eta(t)$, namely, Dirac δ memory kernel, power-law memory kernel, two power-law memory kernels, distributed order memory kernels, as well as their combinations.

4.1. Cases with $\eta(t) = \delta(t)$

Firstly we consider cases with $\eta(t) = \delta(t)$. For the classical comb model (1.3), $\eta(t) = \gamma(t) = \delta(t)$, i.e., $\eta(s) = \gamma(s) = 1$, and $\xi(s) = s^{-1/2}$. Since $\xi(s) = s^{-1/2}$ is completely monotone, and $s\xi(s) = s^{1/2}$ is a Bernstein function, then $p_1(x,t)$ is a non-negative function. Conversely, since $\gamma(s) = 1$ is completely monotone, and $s\gamma(s) = s$ is a Bernstein function, it follows that $p_2(y,t)$ is non-negative function. The solution of Eq. (1.3) reads

$$p_1(x,t) = \mathcal{F}^{-1}\left[E_{1/2}\left(-\frac{1}{2}k_x^2 t^{1/2}\right)\right] = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\left(t^{1/2}/2\right)^{1/2}}\left|\begin{matrix}(1,1/4)\\(1,1)\end{matrix}\right.\right], \tag{4.1}$$

and the PDF along the y direction has Gaussian form. For these memory kernels it follows from Eqs. (2.22) and (2.18) that the diffusion along the y direction is normal $\left\langle y^2(t)\right\rangle = 2t$, and anomalous subdiffusion is in the x direction $\left\langle x^2(t)\right\rangle = 2\frac{t^{1/2}}{\Gamma(1/2)}$.

Next we use power-law memory kernel $\gamma(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, i.e., $\gamma(s) = s^{\alpha-1}$, where $0 < \alpha < 1$. Since $\xi(s) = s^{\alpha/2-1}$ is completely monotone for $0 < \alpha/2 < 1$, and $s\xi(s) = s^{\alpha/2}$ is a Bernstein function for $0 < \alpha/2 < 1$, then it follows that $p_1(x,t)$ is a non-negative function for $0 < \alpha < 1$. Thus, the PDF along the x direction, by using relations (B.1), (B.2) and (B.6), becomes

$$p_1(x,t) = \mathcal{F}^{-1}\left[E_{\alpha/2}\left(-\frac{1}{2}k_x^2 t^{\alpha/2}\right)\right] = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\left(\frac{1}{2}t^{\alpha/2}\right)^{1/2}}\left|\begin{matrix}(1,\alpha/4)\\(1,1)\end{matrix}\right.\right], \tag{4.2}$$

and the MSD, by employing the relation (B.7) grows subdiffusively,

$$\left\langle x^2(t)\right\rangle = \frac{t^{\alpha/2}}{\Gamma\left(1+\alpha/2\right)}. \tag{4.3}$$

For the y direction, in the same way, we find

$$p_2(y,t) = \mathcal{F}^{-1}\left[E_{\alpha}\left(-k_y^2 t^{\alpha}\right)\right] = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|y|}{t^{\alpha/2}}\left|\begin{matrix}(1,\alpha/2)\\(1,1)\end{matrix}\right.\right], \tag{4.4}$$

and

$$\left\langle y^2(t)\right\rangle = 2\frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}, \tag{4.5}$$

i.e. subdiffusion along the y direction is faster than the subdiffusion along the x direction. Note that in the same way as in the case of one fractional exponent, one can show that $p_1(x,t)$ is non-negative in case of the memory kernel with two fractional exponents, which we consider below.

Thus, for a memory kernel of the form $\gamma(t) = C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$, $C_1, C_2 > 0$, $C_1 + C_2 = 1$, $0 < \alpha_1 < \alpha_2 < 1$, for the MSD in the x direction we find

$$\left\langle x^2(t)\right\rangle = \frac{1}{\sqrt{C_2}}\mathcal{L}^{-1}\left[\frac{s^{-\alpha_1/2-1}}{\left(s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}\right)^{1/2}}\right] = \frac{1}{\sqrt{C_2}}t^{\alpha_2/2}E_{\alpha_2-\alpha_1,\alpha_2/2+1}^{1/2}\left(-\frac{C_1}{C_2}t^{\alpha_2-\alpha_1}\right). \tag{4.6}$$

From here we conclude that decelerating subdiffusion appears along the x direction since for the short time limit the MSD behaves as $\left\langle x^2(t)\right\rangle \simeq \frac{1}{\sqrt{C_2}}\frac{t^{\alpha_2/2}}{\Gamma(1+\alpha_2/2)}$, where we use the definition (B.1), and in the long time limit as $\left\langle x^2(t)\right\rangle \simeq \frac{1}{\sqrt{C_1}}\frac{t^{\alpha_1/2}}{\Gamma(1+\alpha_1/2)}$, where we apply the asymptotic expansion formula (B.3). Along the y direction it follows that

$$\left\langle y^2(t)\right\rangle = \frac{2}{C_2}\mathcal{L}^{1}\left[\frac{s^{-\alpha_1-1}}{s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}}\right] = \frac{2}{C_2}t^{\alpha_2}E_{\alpha_2-\alpha_1,\alpha_2+1}\left(-\frac{C_1}{C_2}t^{\alpha_2-\alpha_1}\right), \tag{4.7}$$

i.e., decelerating subdiffusion as well, since the MSD turns from $\left\langle x^2(t)\right\rangle \simeq \frac{2}{C_2}\frac{t^{\alpha_2}}{\Gamma(1+\alpha_2)}$ in the short time limit to $\left\langle x^2(t)\right\rangle \simeq \frac{2}{C_2}\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}$ in the long time limit.

Next we introduce kernels of distributed order. Thus, firstly we consider the following kernel [49–53]

$$\gamma(t) = \int_0^1 d\alpha\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \tag{4.8}$$

i.e., $\gamma(s) = \frac{s-1}{s\log(s)}$, and $\eta(t) = \delta(t)$. Thus, $\xi(s) = \frac{1}{s}\sqrt{\frac{s-1}{\log(s)}}$. Such a memory kernel was used in [49] in the theory of evolution equations with distributed order derivative, and in [54] as a friction kernel in the generalized Langevin equation. The distributed order diffusion equations are introduced by Chechkin et al. [50], [51] and they show that the corresponding MSD shows accelerating, decelerating and ultraslow diffusive behaviors. If we substitute (4.8) in the relations for MSDs (2.18) and (2.22), we obtain the following generalized ultraslow diffusive behaviors in the long time limit,

$$\left\langle x^2(t)\right\rangle \simeq \log^{1/2} t, \tag{4.9}$$

and

$$\left\langle y^2(t)\right\rangle = 2\left[\gamma + \log t + e^t \mathrm{E}_1(t)\right] \simeq 2\log t, \tag{4.10}$$

where $\gamma = 0.577216$ is the Euler-Mascheroni (or Euler's) constant, and $\mathrm{E}_1(t)$ is the exponential integral function [55],

$$\mathrm{E}_1(t) = -\mathrm{Ei}(-t) = \int_t^\infty dx\, \frac{e^{-x}}{x}. \tag{4.11}$$

Here we note that for calculation of MSD (4.9) along the x direction we apply the Tauberian theorem for slowly varying functions (see Appendix C). Thus, ultraslow diffusion occurs along both x and y directions.

A more general distributed order memory kernel is [51]

$$\gamma(t) = \int_0^1 d\alpha\, \nu\alpha^{\nu-1}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \tag{4.12}$$

where $\nu > 0$, and we use $\eta(t) = \delta(t)$. In the long time limit $\gamma(s) \simeq \frac{\Gamma(1+\nu)}{s\log^{\nu}\frac{1}{s}}$, and for the MSDs we obtain

$$\left\langle x^2(t)\right\rangle \simeq \frac{\log^{\nu/2} t}{\sqrt{\Gamma(1+\nu)}}, \tag{4.13}$$

and

$$\left\langle y^2(t)\right\rangle \simeq 2\frac{\log^{\nu} t}{\Gamma(1+\nu)}, \tag{4.14}$$

where we use the Tauberian theorem for slowly varying functions (see Appendix C). This is an ultraslow diffusion again. A prominent example of an ultraslow diffusion is the Sinai diffusion, where the MSD behaves as $\log^4 t$ [56], [57]. Logarithmically slow diffusion also occurs in granular gases with constant restitution coefficient [58], for single file diffusion in disordered environments [59], or in dynamic maps [60]. Stochastic models leading to logarithmically growing MSDs include continuous time random walks with superheavy waiting time densities [57], strongly localized heterogeneous diffusion coefficients [61], ageing continuous time random walks [62], and ultraslow scaled Brownian motion [63].

4.2. Cases with $\eta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$

For power-law memory kernels $\eta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ ($\eta(s) = s^{\alpha-1}$) and $\gamma(t) = \delta(t)$ it follows that $\xi(s) = s^{-\alpha+1/2}$. Since $\eta(t)$ is a non-negative integrable function for $0 < \alpha < 1$, while $\xi(s)$ is completely monotone for $-\alpha + 1/2 < 0$, and $s\xi(s) = s^{-\alpha+3/2}$ is a Bernstein function for $0 < -\alpha + 3/2 < 1$, it follows that $1/2 < \alpha < 1$ should be satisfied in order $p_1(x,t)$ to be non-negative function. By using relation (2.15), Laplace transform formula (B.2) yields

$$p_1(k_x,t) = E_{3/2-\alpha}\left(-\frac{1}{2}k_x^2 t^{3/2-\alpha}\right), \tag{4.15}$$

from where by the Fourier transform formula (B.6), we find that the PDF $p_1(x,t)$ in terms of the Fox H-function (see definition (B.4)) reads

$$p_1(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{\left(\frac{1}{2}t^{3/2-\alpha}\right)^{1/2}}\left|\begin{array}{c}(1,3/4-\alpha/2)\\(1,1)\end{array}\right.\right]. \tag{4.16}$$

For the MSD, relation (B.7) yields

$$\left\langle x^2(t)\right\rangle = \frac{t^{3/2-\alpha}}{\Gamma(5/2-\alpha)}, \tag{4.17}$$

i.e. subdiffusive behavior for $1/2 < \alpha < 1$. The case $\alpha = 1/2$ yields normal diffusion. From the asymptotic behavior of the Fox H-function [64], we find that the PDF $p_1(x,t)$ has a stretched Gaussian form

$$p_1(x,t) \simeq \frac{1}{\sqrt{2\pi(1+2\alpha)}} \frac{|x|^{\frac{1-2\alpha}{1+2\alpha}}}{\left(\frac{1}{2}t^{3/2-\alpha}\right)^{\frac{1}{1+2\alpha}}} \exp\left(-\frac{1+2\alpha}{4}\left(\frac{3-2\alpha}{4}\right)^{\frac{3-2\alpha}{1+2\alpha}} \frac{|x|^{\frac{4}{1+2\alpha}}}{\left(\frac{1}{2}t^{3/2-\alpha}\right)^{\frac{2}{1+2\alpha}}}\right). \tag{4.18}$$

Note that the case $\alpha = 1/2$ yields the Gaussian form of PDF. For the y direction, since $\gamma(t) = \delta(t)$, the diffusion is normal, i.e., $\left\langle y^2(t)\right\rangle = 2t$.

In the case where $\gamma(t) = \eta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$ (note that $\gamma(t)$ and $\eta(t)$ are non-negative integrable functions), we recover Eq. (1.7), for which the PDF along the x direction is given by

$$p_1(x,t) = \mathcal{F}^{-1}\left[E_{1-\alpha/2}\left(-\frac{1}{2}k_x^2 t^{1-\alpha/2}\right)\right] = \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{\left(\frac{1}{2}t^{1-\alpha/2}\right)^{1/2}}\left|\begin{array}{c}(1,1/2-\alpha/4)\\(1,1)\end{array}\right.\right], \tag{4.19}$$

and the MSD becomes

$$\left\langle x^2(t)\right\rangle = \frac{t^{1-\alpha/2}}{\Gamma(2-\alpha/2)}. \tag{4.20}$$

For the y direction the results correspond to (4.4) and (4.5). The non-negativity of $p_1(x,t)$ follows from the fact that $\xi(s) = s^{-\alpha/2}$ is completely monotone, and $s\xi(s) = s^{-\alpha/2+1}$ is a Bernstein function for $0 < \alpha < 1$.

For a memory function with two power-laws $\gamma(t) = C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$, $0 < \alpha_1 < \alpha_2 < 1$, and power-law memory kernel $\eta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$, the MSD along the x direction becomes

$$\left\langle x^2(t)\right\rangle = \frac{1}{\sqrt{C_2}} t^{1+\alpha_2/2-\alpha} E^{1/2}_{\alpha_2-\alpha_1, 2+\alpha_2/2-\alpha}\left(-\frac{C_1}{C_2}t^{\alpha_2-\alpha_1}\right), \tag{4.21}$$

which in the short time limit behaves as $\left\langle x^2(t)\right\rangle \simeq \frac{1}{\sqrt{C_2}}\frac{t^{1+\alpha_2/2-\alpha}}{\Gamma(2+\alpha_2/2-\alpha)}$, and in the long time limit as $\left\langle x^2(t)\right\rangle \simeq \frac{1}{\sqrt{C_1}}\frac{t^{1+\alpha_1/2-\alpha}}{\Gamma(2+\alpha_1/2-\alpha)}$, i.e., decelerating anomalous subdiffusion. Here we note that in order PDF $p_1(x,t)$ to be non-negative, the restriction $0 < \alpha_1/2 < \alpha_2/2 < \alpha < 1$ should be satisfied. This can be shown from the fact that $\xi(s) = \left(C_1 s^{\alpha_1-2\alpha} + C_2 s^{\alpha_2-2\alpha}\right)^{1/2}$ is a completely monotone function if $C_1 s^{\alpha_1-2\alpha}$ and $C_2 s^{\alpha_2-2\alpha}$ are completely monotone functions. This is satisfied if $\alpha_{1,2} - 2\alpha < 0$. Along the y direction the MSD is given by (4.7) since the memory kernel $\gamma(t)$ is the same in both cases.

For the power-law memory kernel $\eta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$, and uniformly distributed order $\gamma(t)$ (4.8) the MSD in the long time limit reads

$$\left\langle x^2(t)\right\rangle \simeq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \log^{-1/2} t. \tag{4.22}$$

This result is obtained with the use of Tauberian theorem (see Appendix C). The MSD along y direction is the same as in Eq. (4.10) since the same memory kernel $\gamma(t)$ is used.

4.3. Cases with $\eta(t) = C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$

Furthermore, we consider kernel with two fractional exponents $\alpha_1 < \alpha_2$, i.e., $\eta(t) = C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$, $C_1, C_2 > 0$, $C_1 + C_2 = 1$, where $1/2 \leq \alpha_1 < \alpha_2 < 1$. This condition should be satisfied in order the corresponding PDF $p_1(x,t)$ to be non-negative and $\eta(t)$ to be a non-negative integrable function. Let us show this. Since $\eta(t)$ should be a non-negative integrable function then $0 < \alpha_{1,2} < 1$. From the kernels $\gamma(t)$ and $\eta(t)$ it follows that $\xi(s) = \frac{1}{C_1 s^{\alpha_1-1/2}+C_2 s^{\alpha_2-1/2}}$. From Appendix A, property 4, $\xi(s)$ is completely monotone if $C_1 s^{\alpha_1-1/2} + C_2 s^{\alpha_2-1/2}$ is a complete Bernstein function. Thus, $0 < \alpha_{1,2} - 1/2 < 1$, i.e., $1/2 < \alpha_{1,2} < 3/2$. From here we obtain the restrictions on the parameters $\alpha_{1,2}$, i.e., $1/2 \leq \alpha_1 < \alpha_2 < 1$. The PDF $p_1(x,t)$ is then obtained in terms of infinite series in Fox H-functions,

$$p_1(x,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{C_1}{C_2}\right)^n \times \frac{1}{2|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\left(\frac{1}{2}C_2 t^{3/2-\alpha_2}\right)^{1/2}} \middle| \begin{array}{c} (1-n,1/2),((\alpha_2-\alpha_1)n+1, 3-\alpha_2/2),(1,1/2) \\ (1,1),(1,1/2),(1,1/2) \end{array}\right], \tag{4.23}$$

from where we find the MSD

$$\left\langle x^2(t)\right\rangle = C_1\frac{t^{3/2-\alpha_1}}{\Gamma(5/2-\alpha_1)} + C_2\frac{t^{3/2-\alpha_2}}{\Gamma(5/2-\alpha_2)}. \tag{4.24}$$

Therefore, in the short time limit we obtain $\left\langle x^2(t)\right\rangle \simeq C_2\frac{t^{3/2-\alpha_2}}{\Gamma(5/2-\alpha_2)}$, whereas for long times $\left\langle x^2(t)\right\rangle \simeq C_1\frac{t^{3/2-\alpha_1}}{\Gamma(5/2-\alpha_1)}$, i.e. accelerating anomalous diffusion. Generalization to the case of a kernel with many fractional exponents α_i, $i = 1, 2, \ldots, N$, is obvious. The diffusion along the y direction is normal since $\gamma(t) = \delta(t)$.

In case of a power-law memory kernel $\gamma(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, such that $0 < \alpha/2 < \alpha_1 < \alpha_2 < 1$ (these restrictions follow from the conditions (i) $\xi(s) = \frac{1}{C_2}\frac{s^{\alpha/2-\alpha_1}}{s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}}$ to be completely monotone, and (ii) $s\xi(s)$ to be a Bernstein function), for the MSD we find

$$\left\langle x^2(t)\right\rangle = C_1\frac{t^{1+\alpha/2-\alpha_1}}{\Gamma(2+\alpha/2-\alpha_1)} + C_2\frac{t^{1+\alpha/2-\alpha_2}}{\Gamma(2+\alpha/2-\alpha_2)}, \tag{4.25}$$

from where for the short time limit MSD behaves as $\left\langle x^2(t)\right\rangle = C_2\frac{t^{1+\alpha/2-\alpha_2}}{\Gamma(2+\alpha/2-\alpha_2)}$, and in the long time limit as $\left\langle x^2(t)\right\rangle = C_1\frac{t^{1+\alpha/2-\alpha_1}}{\Gamma(2+\alpha/2-\alpha_1)}$. In the y direction $\left\langle y^2(t)\right\rangle = 2\frac{t^{\alpha}}{\Gamma(1+\alpha)}$.

In the case where both memory kernels are combinations of two power-law functions $\gamma(t) = \eta(t) = C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$, $0 < \alpha_1 < \alpha_2 < 1$, it follows $\xi(s) = \frac{\sqrt{\frac{1}{C_2}}s^{-\alpha_1/2}}{\left(s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}\right)^{1/2}}$. Thus, for the MSD along the x direction we find

$$\left\langle x^2(t)\right\rangle = C_2^{1/2}\mathcal{L}^{-1}\left[\frac{s^{\alpha_1-2}}{\left(s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}\right)^{-1/2}}\right] = C_2^{1/2}t^{1-\alpha_2/2}E_{\alpha_2-\alpha_1,2-\alpha_2/2}^{-1/2}\left(-\frac{C_1}{C_2}t^{\alpha_2-\alpha_1}\right), \tag{4.26}$$

from where the short time limit yields $\left\langle x^2(t)\right\rangle \simeq C_2^{1/2}\frac{t^{1-\alpha_2/2}}{\Gamma(2-\alpha_2/2)}$, and in the long time limit $\left\langle x^2(t)\right\rangle \simeq C_1^{1/2}\frac{t^{1-\alpha_1/2}}{\Gamma(2-\alpha_1/2)}$. The MSD in y direction is given by Eq. (4.7).

A more complicated combination follows if $\gamma(t)$ is uniformly distributed memory kernel (4.8), for which $\xi(s) = \frac{1}{C_2}\frac{s^{-\alpha_1}\sqrt{\frac{s-1}{\log s}}}{s^{\alpha_2-\alpha_1}+\frac{C_1}{C_2}}$. For the MSD in x direction, by using Tauberian theorem (see Appendix C), we derive

$$\left\langle x^2(t)\right\rangle \simeq C_1\frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\log^{-1/2}t+C_2\frac{t^{1-\alpha_2}}{\Gamma(2-\alpha_2)}\log^{-1/2}t. \tag{4.27}$$

From here the previously obtained result (4.22) follows directly. In y direction the MSD is given by Eq. (4.10).

4.4. Cases with distributed order kernel $\eta(t)$

The case where $\gamma(t)=\delta(t)$ and $\eta(t)$ is a uniformly distributed order memory kernel, i.e., its Laplace transform is given by $\eta(s)=\frac{s-1}{s\log s}$, cannot be considered within our framework, since $\xi(s)\sim\frac{s^{1/2}\log s}{s-1}$ is not a completely monotone function.

A similar situation appears for a power-law memory kernel $\gamma(t)$ (and two-power law memory kernels $\gamma(t)$) and uniformly distributed memory kernel $\eta(t)$, for which $\xi(s)\sim\frac{s^{\alpha/2}\log s}{s-1}$ is not a completely monotone function.

For the case where both kernels are of distributed order, i.e., $\gamma(t)=\eta(t)=\int_0^1 d\alpha\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, we find in the long time limit with the use of Tauberian theorem (see Appendix C),

$$\left\langle x^2(t)\right\rangle \simeq t\log^{-1/2}t. \tag{4.28}$$

The MSD along y direction is the same as in Eq. (4.10).

5. Summary

In this paper we studied anomalous diffusion in a generalized two-dimensional comb-like model governed by a diffusion-like equation with two memory kernels. We solved this equation in the Fourier-Laplace space and gave general expressions for the PDFs and MSDs in x and y directions. The CTRW model, which corresponds to the considered diffusion-like equation on a comb-like structure was presented. We showed that the waiting time PDF which is responsible for the appearance of different diffusive behavior along the backbone (x direction) depends on both memory kernels $\gamma(t)$ and $\eta(t)$, whereas the waiting time PDF along y direction depends only on the memory kernel $\gamma(t)$. We found the PDFs which subordinate the random diffusion processes on the comb-like structure with respect to the Wiener process. We investigated the role of different forms of $\gamma(t)$ and $\eta(t)$, such as Dirac delta, power-law, and distributed order. The results obtained for the PDFs and MSDs are represented by using the Mittag-Leffler and Fox H-functions, from where different diffusive regimes can be observed. It is shown that the considered model may be used to describe anomalous diffusive processes, including decelerating and accelerating anomalous subdiffusion, and ultraslow diffusion as well. The results for the MSDs in x and y directions are summarized in Tables 1 and 2.

TABLE 1. MSD $\left\langle x^2(t)\right\rangle$ along the x direction. It depends on both memory kernels $\gamma(t)$ and $\eta(t)$. The MSDs in case of distributed order memory kernels are calculated in the long time limit by using Tauberian theorem.

$\eta(t)$ \ $\gamma(t)$	$\delta(t)$	$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$	$C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)}+C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$	$\int_0^1 d\alpha\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$
$\delta(t)$	$\sim t^{1/2}$	$\sim t^{\alpha/2}$	Eq. (4.6)	$\sim\log^{1/2}t$
$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$	$\sim t^{3/2-\alpha}$	$\sim t^{1-\alpha/2}$	Eq. (4.21)	$\sim t^{1-\alpha}\log^{-1/2}t$
$C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)}+C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$	Eq. (4.24)	Eq. (4.25)	Eq. (4.26)	Eq. (4.27)
$\int_0^1 d\alpha\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$	$-$	$-$	$-$	$\sim t\log^{-1/2}t$

TABLE 2. MSD $\langle y^2(t)\rangle$ along y direction. It depends only on the memory kernel $\gamma(t)$.

$\gamma(t)$	$\delta(t)$	$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$	$C_1\frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)}+C_2\frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)}$	$\int_0^1 d\alpha\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$
$\langle y^2(t)\rangle$	$\sim t$	$\sim t^{\alpha}$	$\sim t^{\alpha_2}E_{\alpha_2-\alpha_1,\alpha_2+1}\left(-\frac{C_1}{C_2}t^{\alpha_2-\alpha_1}\right)$	$\sim \gamma+\log t+e^{t}\mathrm{E}_1(t)$

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A. Completely monotone and Bernstein functions

Here we give definitions and some properties of completely monotone and Bernstein functions [65].

1. The function $g(s)$ is a completely monotone if $(-1)^n g^{(n)}(s) \geq 0$ for all $n \geq 0$ and $s > 0$. Product of completely monotone functions is completely monotone function too. An example of completely monotone function is s^{α}, where $\alpha < 0$.
2. The Bernstein characterization theorem states that if the Laplace transform $g(s) = \mathcal{L}[G(t)]$ of given function $G(t)$ is completely monotone function, then the function $G(t)$ is non-negative.
3. The function $e^{-us\xi(s)}$ is a completely monotone if $s\xi(s)$ is a complete Bernstein function. A given function $f(s)$ is a Bernstein function if $(-1)^{n-1}f^{(n)}(s) \geq 0$ for all $n \in N$ and $s > 0$. An example of Bernstein function is s^{α}, where $0 < \alpha < 1$.
4. It can be shown that if $f(s)$ is a complete Bernstein function, then $g(s) = 1/f(s)$ is a completely monotonic function [66].
5. Another important property of complete Bernstein function is the one which states that $f(s)$ is a complete Bernstein function if and only if the function $s/f(s)$ is a complete Bernstein function.

In order $h_1(u,s)$ to be completely monotone function, both functions $\xi(s) = \frac{1}{\eta(s)}\sqrt{\frac{\gamma(s)}{s}}$ and $e^{-us\xi(s)}$ should be completely monotone [65]. The function $e^{-us\xi(s)}$ is a completely monotone if $s\xi(s) = \frac{1}{\eta(s)}\sqrt{s\gamma(s)}$ is a complete Bernstein function. Similarly, in order the PDF $h_2(u,s)$ to be completely monotone function, the function $\gamma(s)$ should be completely monotone and $s\gamma(s)$ should be a Bernstein function.

B. Mittag-Leffler and Fox *H*-functions

To calculate the PDFs we use the three parameter Mittag-Leffler (M-L) function [67]

$$E_{\alpha,\beta}^{\delta}(z) = \sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \tag{B.1}$$

where $(\delta)_k = \Gamma(\delta+k)/\Gamma(\delta)$ is the Pochhammer symbol, where the Laplace transform is given by [67]

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}^{\delta}(\pm a t^{\alpha})\right](s) = \frac{s^{\alpha\delta-\beta}}{(s^{\alpha}\mp a)^{\delta}}, \quad \Re(s) > |a|^{1/\alpha}. \tag{B.2}$$

The case $\delta = 1$ yields the two parameter M-L function $E_{\alpha,\beta}(z)$, and the case $\beta = \delta = 1$ - one parameter M-L function $E_{\alpha}(z)$. For the three parameter M-L function the following formula holds [54], [68], [69] (see also [70], [71] for two parameter M-L function)

$$E_{\alpha,\beta}^{\delta}(-z) = \frac{z^{-\delta}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)}{\Gamma(\beta-\alpha(\delta+n))}\frac{(-z)^{-n}}{n!}, \quad |z| > 1, \tag{B.3}$$

in order to analyze the asymptotic behaviors. Thus, the asymptotic expansion formula for three parameter M-L function is given by $E_{\alpha,\beta}^{\delta}(-z) \simeq \frac{z^{-\delta}}{\Gamma(\beta-\alpha\delta)}$ for $z \to \infty$.

The Fox H-function (or simply H-function) is defined by the following Mellin-Barnes integral [64]

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right] = \frac{1}{2\pi\imath}\int_{\Omega}\theta(s)z^s ds, \tag{B.4}$$

where $\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}$, $0 \le n \le p$, $1 \le m \le q$, $a_i, b_j \in C$, $A_i, B_j \in R^+$, $i = 1,...,p$, $j = 1,...,q$. The contour Ω starting at $c - i\infty$ and ending at $c + i\infty$ separates the poles of the function $\Gamma(b_j - B_js)$, $j = 1,...,m$ from those of the function $\Gamma(1 - a_i + A_is)$, $i = 1,...,n$.

The connection between three parameter M-L function and the Fox H-function is given by [64]

$$E_{\alpha,\beta}^{\delta}(-z) = \frac{1}{\Gamma(\delta)}H_{1,2}^{1,1}\left[z\left|\begin{matrix}(1-\delta,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right]. \tag{B.5}$$

The Fourier transform formula for the Fox H-function is [64]

$$\int_0^{\infty} dk\, k^{\rho-1}\cos(kx)H_{p,q}^{m,n}\left[ak^{\delta}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right] = \frac{\pi}{x^{\rho}}H_{q+1,p+2}^{n+1,m}\left[\frac{x^{\delta}}{a}\left|\begin{matrix}(1-b_q,B_q),(\frac{1+\rho}{2},\frac{\delta}{2})\\(\rho,\delta),(1-a_p,A_p),(\frac{1+\rho}{2},\frac{\delta}{2})\end{matrix}\right.\right], \tag{B.6}$$

where $\Re\left(\rho + \delta\min_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right) > 1$, $x^{\delta} > 0$, $\Re\left(\rho + \delta\max_{1\le j\le n}\left(\frac{a_j-1}{A_j}\right)\right) < \frac{3}{2}$, $|\arg(a)| < \pi\theta/2$, $\theta > 0$, $\theta = \sum_{j=1}^{n}A_j - \sum_{j=n+1}^{p}A_j + \sum_{j=1}^{m}B_j - \sum_{j=m+1}^{q}B_j$.

The Mellin transform of the H-function is given by [64]

$$\int_0^{\infty} x^{\xi-1}H_{p,q}^{m,n}\left[ax\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]dx = a^{-\xi}\theta(-\xi), \tag{B.7}$$

where $\theta(-\xi)$ is defined in Eq. (B.4).

C. Tauberian theorem for slowly varying functions [72]

If some function $r(t)$, $t \ge 0$, has the Laplace transform $r(s)$ whose asymptotics behaves as

$$r(s) \simeq s^{-\rho}L\left(\frac{1}{s}\right), \quad s \to 0, \quad \rho \ge 0, \tag{C.1}$$

then

$$r(t) = \mathcal{L}^{-1}\left[r(s)\right] \simeq \frac{1}{\Gamma(\rho)}t^{\rho-1}L(t), \quad t \to \infty. \tag{C.2}$$

Here $L(t)$ is a slowly varying function at infinity, i.e., $\lim_{t\to\infty}\frac{L(at)}{L(t)} = 1$, for any $a > 0$.

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