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The Impact of Hybrid Quarantine Strategies and Delay factor on Viral Prevalence in Computer Networks

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Abstract. Recently, the quarantine approach, which has been applied to infectious disease control, is widely regarded as an effective measure to suppress viral spread in computer networks. Hence, in order to prevent the spread of computer virus in network, and consider the latent period of a latent computer, a new delayed epidemic model of computer virus with hybrid quarantine strategies is presented. By regarding the delay as bifurcating parameter and analyzing the associated characteristic equation, the dynamical behaviors, including local asymptotical stability in which the virus spreading can be controlled, and furthermore, local Hopf bifurcation occurs in the system, which implies computer virus is out of control, are investigated. By applying the normal form and center manifold theorem, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are also determined. Some numerical simulations are provided to support our theoretical results which also imply that hybrid quarantine strategies can inhibit viral spread effectively, and make the model be asymptotically stable.

Keywords and phrases: computer virus, delay, intrusion detection systems, hybrid quarantine strategies, Hopf bifurcation

Mathematics Subject Classification: 35Q53, 34B20, 35G31

1. Introduction

With the rapid development of computer and communication technology, computer network has permeated almost every aspect of our daily life. Meanwhile, computer viruses, of which the number and class rise up sharply, ranging from joke viruses to network worms and spreading over the network quickly, have given rise to enormous economic loss to society. Therefore, for the purpose of effectively suppressing the spread of viruses, it is very important to establish mathematical models which may describe the propagation of viruses, to explore the way of malware propagation and control it. In the past few decades, various multifarious computer virus propagation models have been proposed, such as SIS models [1, 2], SIR models [3], $SIRS$ models [4], $SEIRS$ models [5], $SICS$ models [7], $SLBS$ models [8], SIA models [9], and so on.

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However, the epidemic models cannot be directly used to describe the mechanism of computer virus propagation and the way of how to reduce computer virus. Furthermore, the impact of defensive strategies should be considered and introduced into novel models. Motivated by the methods applied in infectious disease control, quarantine is widely regarded as an effective measure to prevent the spread of computer viruses [10]. Usually, quarantine strategy, which is dependent on the intrusion detection systems ($IDSs$), is studied in [11]. From the technical point of view, there are two kinds of $IDSs$, i.e., signature-based and anomaly-based $IDSs$ [18]. By establishing a database incorporating the features of the known attacks, the signature-based IDS addresses to detect the invaders whose behaviors are in line with one of the collected data. For the known attacks, although this system can report the types of the attacks detailedly and accurately, the effect is limited for the unknown attacks. By constructing a database containing the normal behaviors of operating systems, another system can distinguish the abnormal activities as long as their behaviors differ from the database, and then mark these activities as attacks. Unfortunately, normal activities may be mistaken as attacks sometimes by this system. Overall, both of the $IDSs$ have merits and demerits. So, hybrid $IDSs$, by combining both signature-based and anomaly-based $IDSs$, are presented to make up for the existing defects in the above systems [12–14].

Traditional quarantine strategies, relying largely on the signature-based $IDSs$, are often exploited and applied to establish different computer virus propagation models, especially worm propagation models, in which only infected computers are quarantined [6, 15, 16]. However, there exists a drawback that traditional quarantine strategies can not detect the infectious computer infected by a new virus. To avoid this defect, therefore, hybrid quarantine strategies based on hybrid $IDSs$ are rarely considered in worm propagation models; Wang et al. [17] presented an $SEIQV$ worm model relying on the hybrid quarantine strategies and then analyzed its stability, however, their results required further modification and improvement; Yao et al. [18, 19] have examined the effect of the pulse quarantine which is one of the hybrid quarantine strategies on worm propagation. Hybrid quarantine strategies are also applicable to inhibit the spread of general computer virus. According to what we're informed, however, none of the previous papers investigate the dynamical behaviors of general computer virus models with hybrid quarantine strategies.

In view of the fact that latent characteristic is the common feature of computer virus [20], which means that, when a computer is infected by a computer virus, there is a latent period before it breaks out. Hence, the delay factor should be incorporated into a realistic model [21, 22]. In addition, it is well known that time delays play a key role in investigating the behaviors of dynamical systems, which can cause a stable equilibrium to be unstable, and make a system produce periodic solutions bifurcating from the equilibrium. Therefore, dynamical systems with delay have been studied by many researchers [23–27].

Very recently, combining the fact that latent computers possess infectivity, and the recovered computers can gain temporary immunity, a susceptible-latent-breaking-out-recovered-susceptible ($SLBRS$) model is presented in [28], and its extension in [29]. In this paper, considering the effect of hybrid quarantine strategies and the delay factor, a new computer virus propagation model named delayed susceptible-latent-breaking-out-quarantined-recovered-susceptible ($SLBQRS$) model (see Fig. 1) is established by introducing a new quarantined compartment (Q). In this model, both susceptible and infected computers are quarantined by disconnecting the network as long as they exhibit suspicious behaviors. By analyzing the dynamical properties of this model, the behaviors of virus propagation under the impact of hybrid quarantine strategies and delay factor are studied. Sufficient conditions for the local stability and existence of Hopf bifurcation are obtained by regarding the delay which is due to the latent period of a latent computer as a bifurcating parameter and analyzing the distribution of the roots of the associated characteristic equation. The direction of Hopf bifurcation and the stability of bifurcating periodic solutions are obtained by applying the normal form and center manifold theorem. Numerical simulations illustrate the main results of this paper and imply that hybrid quarantine strategies can inhibit the diffusion of computer virus effectively and make the system be asymptotically stable.

The rest of this paper is organized as follows: Section 2 formulates the new model. Section 3 examines the stability and the existence of local Hopf bifurcation. Section 4 studies the properties of the Hopf bifurcation. Some numerical simulations are illustrated in Section 5. Section 6 summarizes this work.

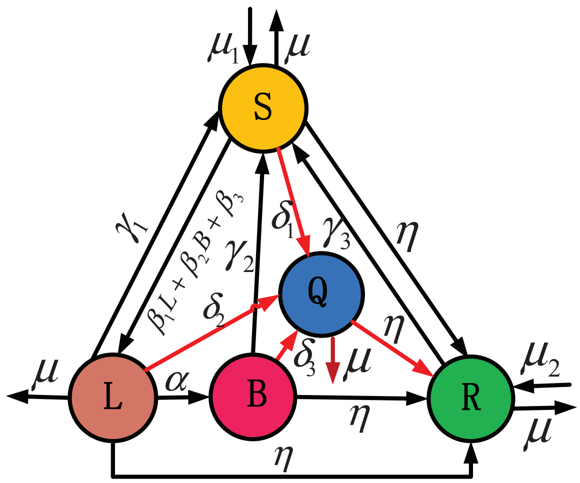

FIGURE 1. State transition diagram for the model.

2. Model description

As usual, at time t, a computer connected to the network or quarantined is regarded as internal, or it is regarded as external. By connecting to the network, an external computer becomes an internal computer. As well, if an internal computer dies out, it becomes an external computer. All the internal computers are classified into five compartments: susceptible, latent, breaking-out, quarantined and recovered computers (see Fig. 1). Let $S(t)$, $L(t)$, $B(t)$, $Q(t)$ and $R(t)$ denote the percentages of susceptible, latent, breaking-out, quarantined and recovered computers at time t, respectively. In order to obtain the propagation model, the following assumptions are needed.

(A1) All external computers newly connected to the internet are susceptible or recovered. What's more, external susceptible and recovered computers connect to the internet at constant rate $\mu_1 > 0$ and $\mu_2 > 0$, respectively.
(A2) Due to the operating system collapse or close down, every computer dies out at constant rate $\mu = \mu_1 + \mu_2$.
(A3) Every internal susceptible computer is infected by internal latent computers with probability $\beta_1 L(t)$, and it is infected by internal breaking-out computers with probability $\beta_2 B(t)$, where $\beta_1, \beta_2 > 0$.
(A4) Every internal susceptible computer is infected by infected removable storage media with probability $\beta_3 > 0$.
(A5) Every internal latent computer breaks out with probability $\alpha > 0$.
(A6) Every internal recovered computer loses immunity with probability $\gamma_3 > 0$.
(A7) Due to repairing or reinstalling the system, every internal latent or breaking-out computer becomes susceptible with probability $\gamma_1 > 0$, $\gamma_2 > 0$, respectively.
(A8) Due to the effect of antivirus programs, every computer is recovered with probability $\eta > 0$.

What's more, the following additional assumptions are made.

(A9) Due to the hybrid quarantine strategies, every susceptible, latent and breaking-out computer is quarantined with probability $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$, respectively. Obviously, $\delta_2 > \delta_1$ and $\delta_3 > \delta_1$, because the quarantined rate of infected computers is higher than that of susceptible computers.

(A10) Due to the latent period of a latent computer, a latent computer breaks out in a period of τ.

Based on the above assumptions, the propagation model can be designed as

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu_1 + \gamma_1 L(t) + \gamma_2 B(t) + \gamma_3 R(t) - (\beta_1 L(t) + \beta_2 B(t) + \beta_3)S(t) - (\delta_1 + \eta + \mu)S(t), \\ \dfrac{dL(t)}{dt} = (\beta_1 L(t) + \beta_2 B(t) + \beta_3)S(t) - (\gamma_1 + \mu + \delta_2 + \eta)L(t) - \alpha L(t-\tau), \\ \dfrac{dB(t)}{dt} = \alpha L(t-\tau) - (\gamma_2 + \mu + \delta_3 + \eta)B(t), \\ \dfrac{dQ(t)}{dt} = \delta_1 S(t) + \delta_2 L(t) + \delta_3 B(t) - (\eta + \mu)Q(t), \\ \dfrac{dR(t)}{dt} = \mu_2 + \eta(S(t) + L(t) + B(t) + Q(t)) - (\mu + \gamma_3)R(t). \end{cases} \tag{2.1}$$

Let $S(t) + L(t) + B(t) + Q(t) + R(t) = 1$, the system (2.1) can be simplified as follows

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu_1 + \gamma_1 L(t) + \gamma_2 B(t) + \gamma_3(1 - S(t) - L(t) - B(t) - Q(t)) - (\beta_1 L(t) + \beta_2 B(t))S(t) \\ \qquad - \beta_3 S(t) - (\delta_1 + \eta + \mu)S(t), \\ \dfrac{dL(t)}{dt} = (\beta_1 L(t) + \beta_2 B(t) + \beta_3)S(t) - (\gamma_1 + \mu + \delta_2 + \eta)L(t) - \alpha L(t-\tau), \\ \dfrac{dB(t)}{dt} = \alpha L(t-\tau) - (\gamma_2 + \mu + \delta_3 + \eta)B(t), \\ \dfrac{dQ(t)}{dt} = \delta_1 S(t) + \delta_2 L(t) + \delta_3 B(t) - (\eta + \mu)Q(t), \end{cases} \tag{2.2}$$

with the initial condition $(S(0), L(0), B(0), Q(0)) \in \Omega$, where

$$\Omega = \{(S, L, B, Q) \in R_+^4 : S + L + B + Q \le 1\},$$

where R_+^4 is a positive region.

3. Stability and Existence of Local Hopf Bifurcation

Theorem 3.1. *System* (2.1) *has no virus-free equilibrium.*

Proof. See Appendix A. □

According to Theorem 3.1, we can see that system (2.2) has no virus-free equilibrium. Then, it is clearly that there is no infection-free state of system (2.2).

Theorem 3.2. *System* (2.1) *has a unique viral equilibrium* $\overline{E}^*(S^*, L^*, B^*, Q^*, R^*)$, *where*

$$\begin{aligned} &B^* = \frac{n + \sqrt{n^2 + 4m\beta_3 b}}{2m}, \quad L^* = aB^*, \quad Q^* = \frac{\delta_1(1 - R^*)}{\eta + \mu + \delta_1} + dB^*, \\ &S^* = 1 - L^* - B^* - Q^* - R^*, \quad R^* = \frac{\mu_2 + \eta}{\mu + \gamma_3 + \eta}, \end{aligned} \tag{3.1}$$

where

$$a = \frac{\gamma_2 + \mu + \delta_3 + \eta}{\alpha}, \qquad b = \frac{(\eta + \mu)(1 - R^*)}{\eta + \mu + \delta_1},$$
$$d = \frac{(\delta_2 - \delta_1)a + (\delta_3 - \delta_1)}{\eta + \mu + \delta_1}, \qquad m = (\beta_1 a + \beta_2)(1 + a + d),$$
$$n = (\beta_1 a + \beta_2)b - \beta_3(1 + a + d) - (\gamma_1 + \mu + \delta_2 + \eta + \alpha)a.$$

Proof. See Appendix B. □

As a direct consequence of Theorem 3.2, we get that system (2.2) has a unique viral equilibrium $E^*(S^*, L^*, B^*, Q^*)$.

The linearized system of system (2.2) at the equilibrium is as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = a_1 S(t) + a_2 L(t) + a_3 B(t) + a_4 Q(t), \\ \dfrac{dL(t)}{dt} = a_5 S(t) + a_6 L(t) + a_7 B(t) + b_1 L(t - \tau), \\ \dfrac{dB(t)}{dt} = b_2 L(t - \tau) + a_8 B(t), \\ \dfrac{dQ(t)}{dt} = a_9 S(t) + a_{10} L(t) + a_{11} B(t) + a_{12} Q(t), \end{cases} \tag{3.2}$$

where

$$a_1 = -(\beta_1 L^* + \beta_2 B^* + \beta_3 + \mu + \delta_1 + \eta + \gamma_3), \quad a_2 = \gamma_1 - \beta_1 S^* - \gamma_3,$$
$$a_3 = \gamma_2 - \beta_2 S^* - \gamma_3, \quad a_4 = -\gamma_3, \quad a_5 = \beta_1 L^* + \beta_2 B^* + \beta_3,$$
$$a_6 = \beta_1 S^* - (\gamma_1 + \mu + \delta_2 + \eta), \quad a_7 = \beta_2 S^*, \quad a_8 = -(\gamma_2 + \mu + \delta_3 + \eta),$$
$$a_9 = \delta_1, \quad a_{10} = \delta_2, \quad a_{11} = \delta_3, \quad a_{12} = -(\eta + \mu), \quad b_1 = -\alpha, \quad b_2 = \alpha.$$

Then, the characteristic equation of system (3.2) at the equilibrium E^* is

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{3.3}$$

where

$$\begin{cases} m_3 = -(a_1 + a_6 + a_8 + a_{12}), \\ m_2 = (a_1 + a_6 + a_{12})a_8 - a_4 a_9 + a_1 a_{12} + a_6 a_{12} + a_1 a_6 - a_2 a_5, \\ m_1 = a_8(a_4 a_9 - a_1 a_{12} - a_6 a_{12} - a_1 a_6 + a_2 a_5) - a_4 a_5 a_{10} + a_4 a_6 a_9 - a_1 a_6 a_{12} + a_2 a_5 a_{12}, \\ m_0 = a_8(a_4 a_5 a_{10} - a_4 a_6 a_9 + a_1 a_6 a_{12} - a_2 a_5 a_{12}), \end{cases} \tag{3.4}$$

$$\begin{cases} n_3 = -b_1, \\ n_2 = a_8 b_1 + a_1 b_1 + a_7 b_1 + a_{12} b_1, \\ n_1 = (a_4 a_9 - a_8 a_{12} - a_1 a_8 - a_1 a_{12} + a_3 a_5 - a_7 a_{12} - a_1 a_7)b_1, \\ n_0 = (a_1 a_8 a_{12} - a_4 a_8 a_9 + a_4 a_5 a_{11} - a_4 a_7 a_9 + a_1 a_7 a_{12} - a_3 a_5 a_{12})b_1. \end{cases} \tag{3.5}$$

For $\tau > 0$, let $\lambda = i\omega(\omega > 0)$ be the root of Eq. (3.3). Then, we have

$$\omega^4 - im_3\omega^3 - m_2\omega^2 + im_1\omega + m_0 + (-in_3\omega^3 - n_2\omega^2 + in_1\omega + n_0)e^{-\omega i\tau} = 0. \tag{3.6}$$

Separating the real and imaginary parts for Eq. (3.6), we get

$$\begin{aligned} &(n_1\omega - n_3\omega^3)\sin\tau\omega + (n_0\omega - n_2\omega^3)\cos\tau\omega = m_2\omega^2 - \omega^4 - m_0, \\ &(n_1\omega - n_3\omega^3)\cos\tau\omega - (n_0\omega - n_2\omega^3)\sin\tau\omega = m_3\omega^3 - m_0\omega. \end{aligned} \tag{3.7}$$

Then, we can gain

$$\omega^8 + c_3\omega^6 + c_2\omega^4 + c_1\omega^2 + c_0 = 0, \tag{3.8}$$

where

$$\begin{aligned} c_3 &= m_3^2 - n_3^2 - 2m_2, \quad c_2 = m_2^2 + 2m_0 - n_2^2 - 2m_1m_3 + 2n_1n_3, \\ c_1 &= m_1^2 - 2m_0m_2 + 2n_0n_2 - n_1^2, \quad c_0 = m_0^2 - n_0^2. \end{aligned} \tag{3.9}$$

Let $z = \omega^2$, then, Eq. (3.8) becomes

$$z^4 + c_3z^3 + c_2z^2 + c_1z + c_0 = 0. \tag{3.10}$$

Define $f(z) = z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$. Then, we have $f'(z) = 4z^3 + 3c_3z^2 + 2c_2z + c_1$. Let

$$4z^3 + 3c_3z^2 + 2c_2z + c_1 = 0. \tag{3.11}$$

Set $y = z + \frac{3}{4}p$, then, Eq. (3.11) becomes

$$y^3 + p_1y + p_2 = 0, \tag{3.12}$$

where $p_1 = \frac{1}{2}c_2 - \frac{3}{16}c_3^2$, $p_2 = \frac{1}{32}c_3^3 - \frac{1}{8}c_2c_3 + c_1$.

Denote

$$\begin{aligned} E &= (\tfrac{p_2}{2})^2 + (\tfrac{p_1}{3})^3, \qquad y_1 = \sqrt[3]{-\tfrac{p_2}{2} + \sqrt{E}} + \sqrt[3]{-\tfrac{p_2}{2} - \sqrt{E}}, \\ y_2 &= \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{-\tfrac{p_2}{2} + \sqrt{E}} + (\frac{-1 + \sqrt{3}i}{2})^2\sqrt[3]{-\tfrac{p_2}{2} - \sqrt{E}}, \\ y_3 &= (\frac{-1 + \sqrt{3}i}{2})^2\sqrt[3]{-\tfrac{p_2}{2} + \sqrt{E}} + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{-\tfrac{p_2}{2} - \sqrt{E}}, \\ z_i &= y_i - \frac{3}{4}c_3, \quad i = 1, 2, 3. \end{aligned} \tag{3.13}$$

Then, we can get the following results (see [30] for details) about the distributions of the positive roots of Eq. (3.10).

Lemma 3.3 (see [30]).

(i) If $c_0 < 0$, then Eq. (3.10) *has at least one positive root.*

(ii) If $c_0 \geqslant 0$ and $E \geqslant 0$, then Eq. (3.10) *has positive roots if and only if $z_1 > 0$ and $f(z_1) \leqslant 0$.*

(iii) If $c_0 \geqslant 0$ and $E < 0$, then Eq. (3.10) *has positive roots if and only if there exists at least one $z^* \in \{z_1, z_2, z_3\}$, such that $z^* > 0$ and $f(z^*) \leqslant 0$.*

Suppose that Eq. (3.10) has positive roots. Then, we can get that Eq. (3.8) has a positive root ω_0 such that $\pm i\omega_0$ is a pair of purely imaginary roots of Eq. (3.3). From (3.7), we have

$$\cos(\omega_0\tau) = \frac{g(\omega_0)}{h(\omega_0)}, \tag{3.14}$$

where

$$\begin{aligned} g(\omega_0) &= (n_2 - m_3n_3)\omega_0^6 + (m_1n_3 - m_2n_2 + m_3n_1 - n_0)\omega_0^4 + (m_0n_2 - m_1n_1 + m_2n_0)\omega_0^2 - m_0n_0, \\ h(\omega_0) &= n_3^2\omega_0^6 + (n_2^2 - 2n_1n_3)\omega_0^4 + (n_1^2 - 2n_0n_2)\omega_0^2 + n_0^2. \end{aligned} \tag{3.15}$$

Thus, for ω_0, the corresponding critical value of time delay is

$$\tau_0 = \frac{1}{\omega_0}\arccos\frac{g(\omega_0)}{h(\omega_0)}. \tag{3.16}$$

For $\tau = 0$, Eq. (3.3) becomes

$$\lambda^4 + m_{33}\lambda^3 + m_{22}\lambda^2 + m_{11}\lambda + m_{00} = 0, \tag{3.17}$$

where

$$m_{33} = m_3 + n_3, \quad m_{22} = m_2 + n_2,$$

$$m_{11} = m_1 + n_1, \quad m_{00} = m_0 + n_0.$$

By means of the Routh-Hurwitz criterion, obviously, if the following condition ($H1$): (3.18) holds, then all the roots of the Eq. (3.17) have negative real parts, and furthermore, E^* is locally asymptotically stable without delay. One has

$$\begin{aligned} D_1 &= m_{33} > 0, \\ D_2 &= \begin{vmatrix} m_{33} & 1 \\ m_{11} & m_{22} \end{vmatrix} > 0, \\ D_3 &= \begin{vmatrix} m_{33} & 1 & 0 \\ m_{11} & m_{22} & m_{33} \\ 0 & m_{00} & m_{11} \end{vmatrix} > 0, \\ D_4 &= \begin{vmatrix} m_{33} & 1 & 0 & 0 \\ m_{11} & m_{22} & m_{33} & 1 \\ 0 & m_{00} & m_{11} & m_{22} \\ 0 & 0 & 0 & m_{00} \end{vmatrix} > 0. \end{aligned} \tag{3.18}$$

Lemma 3.4 (see [30])**.** *Suppose that (H1) holds.*

(i) *If one of the conditions (a) $c_0 < 0$; (b) $c_0 \geqslant 0$, $E \geqslant 0$, $z_1 > 0$ and $f(z_1) \leqslant 0$; (c) $c_0 \geqslant 0$, $E < 0$, and there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $f(z^*) \leqslant 0$ is satisfied, then all roots of Eq.* (3.3) *have negative real parts when $\tau \in [0, \tau_0)$.*

(ii) *If the conditions (a)-(c) of (i) are not satisfied, then all roots of Eq.* (3.3) *have negative real parts for all $\tau \geqslant 0$.*

Taking the derivative of λ with respect to τ on both sides of Eq. (3.3), we obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{4\lambda^4 + 3m_3\lambda^3 + 2m_2\lambda^2 + m_1}{\lambda(\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)} + \frac{3n_3\lambda^3 + 2n_2\lambda^2 + n_1}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)} - \frac{\tau}{\lambda}. \tag{3.19}$$

Then, we have

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{f'(z_*)}{n_3^2\omega_0^6 + (n_2^2 - 2n_1n_3)\omega_0^4 + (n_1^2 - 2n_0n_2)\omega_0^2 + n_0^2}, \tag{3.20}$$

where $z_* = \omega_0^2$. Thus, if condition (H2) i.e., $f'(z) \neq 0$ holds, then $\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} \neq 0$.

From Lemma 3.3 and 3.4 and by the Hopf bifurcation theorem in [31], we have the following results.

Theorem 3.5. *Suppose that (H1) and (H2) hold.*

(i) *If the conditions (a) $c_0 < 0$; (b) $c_0 \geqslant 0$, $E \geqslant 0$, $z_1 > 0$ and $f(z_1) \leqslant 0$; (c) $c_0 \geqslant 0$, $E < 0$, and there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $f(z^*) \leqslant 0$ are not satisfied, then the equilibrium $E^*(S^*, L^*, B^*, Q^*)$ of system* (2.2) *is asymptotically stable for all $\tau \geqslant 0$.*

(ii) *If one of the conditions (a)-(c) of (i) is satisfied, then the equilibrium $E^*(S^*, L^*, B^*, Q^*)$ of system* (2.2) *is asymptotically stable for $\tau \in [0, \tau_0)$ and system* (2.2) *undergoes a Hopf bifurcation at the equilibrium $E^*(S^*, L^*, B^*, Q^*)$ when $\tau = \tau_0$.*

4. Properties of the Hopf Bifurcation

In this section, by applying the normal form and center manifold theorem (see Appendix C), we investigate the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of system (2.2).

Based on Appendix C, we can obtain the coefficients determining the properties of the Hopf bifurcation by using similar computation process given in [23] and applying the algorithms introduced in [31] as follows:

$$\begin{aligned}
g_{20} &= 2\tau_0\overline{D}(\overline{q}_2^* - 1)(\beta_1 q_2 + \beta_2 q_3),\\
g_{11} &= \tau_0\overline{D}(\overline{q}_2^* - 1)(\beta_1(\overline{q}_2 + q_2) + \beta_2(\overline{q}_3 + q_3)),\\
g_{02} &= 2\tau_0\overline{D}(\overline{q}_2^* - 1)(\beta_1\overline{q}_2 + \beta_2\overline{q}_3),\\
g_{21} &= 2\tau_0\overline{D}(\overline{q}_2^* - 1)(\beta_1(\frac{1}{2}W_{20}^{(1)}(0)\overline{q}_2 + W_{11}^{(1)}(0)q_2 + \frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(2)}(0))\\
&\quad + \beta_2(\frac{1}{2}W_{20}^{(1)}(0)\overline{q}_3 + W_{11}^{(1)}(0)q_3 + \frac{1}{2}W_{20}^{(3)}(0) + W_{11}^{(3)}(0))),
\end{aligned} \tag{4.1}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1e^{2i\tau_0\omega_0\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2,
\end{aligned} \tag{4.2}$$

where E_1 and E_2 can be determined by the following equations, respectively,

$$\begin{aligned}
E_1 &= 2\begin{pmatrix} 2i\omega_0 - a_1 & -a_2 & -a_3 & -a_4\\ -a_5 & 2i\omega_0 - a_6 - b_1e^{-2i\omega_0\tau_0} & -a_7 & 0\\ 0 & -b_2e^{-2i\omega_0\tau_0} & 2i\omega_0 - a_8 & 0\\ -a_9 & -a_{10} & -a_{11} & 2i\omega_0 - a_{12}\end{pmatrix}^{-1}\begin{pmatrix} E_1^{(1)}\\ E_1^{(2)}\\ 0\\ 0\end{pmatrix},\\
E_2 &= -\begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ a_5 & a_6 + b_1 & a_7 & 0\\ 0 & b_2 & a_8 & 0\\ a_9 & a_{10} & a_{11} & a_{12}\end{pmatrix}^{-1}\begin{pmatrix} E_2^{(1)}\\ E_2^{(2)}\\ 0\\ 0\end{pmatrix},
\end{aligned} \tag{4.3}$$

with

$$\begin{aligned}
E_1^{(1)} &= -\beta_1q_2 - \beta_2q_3, \quad E_1^{(2)} = \beta_1q_2 + \beta_2q_3,\\
E_2^{(1)} &= -\beta_1(q_2 + \overline{q}_2) - \beta_2(q_3 + \overline{q}_3), \quad E_2^{(2)} = \beta_1(q_2 + \overline{q}_2) + \beta_2(q_3 + \overline{q}_3).
\end{aligned} \tag{4.4}$$

Then, we can get the following formulas:

$$\begin{aligned}
C_1(0) &= \frac{i}{2\tau_0\omega_0}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},\\
\beta_2 &= 2\mathrm{Re}\{C_1(0)\},\\
T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0},
\end{aligned} \tag{4.5}$$

where the sign μ_2 determines the direction of the Hopf bifurcation, the sign β_2 determines the stability of the bifurcating periodic solutions, and the sign of T_2 determines the period of the bifurcating periodic solutions.

In conclusion, we have the following results.

Theorem 4.1. *For system* (2.2)*, if* $\mu_2 > 0(\mu_2 < 0)$*, then the Hopf bifurcation is supercritical (subcritical). If* $\beta_2 < 0(\beta_2 > 0)$*, then the bifurcating periodic solutions are stable (unstable). If* $T_2 > 0(T_2 < 0)$*, then the period of the bifurcating periodic solutions increases (decreases).*

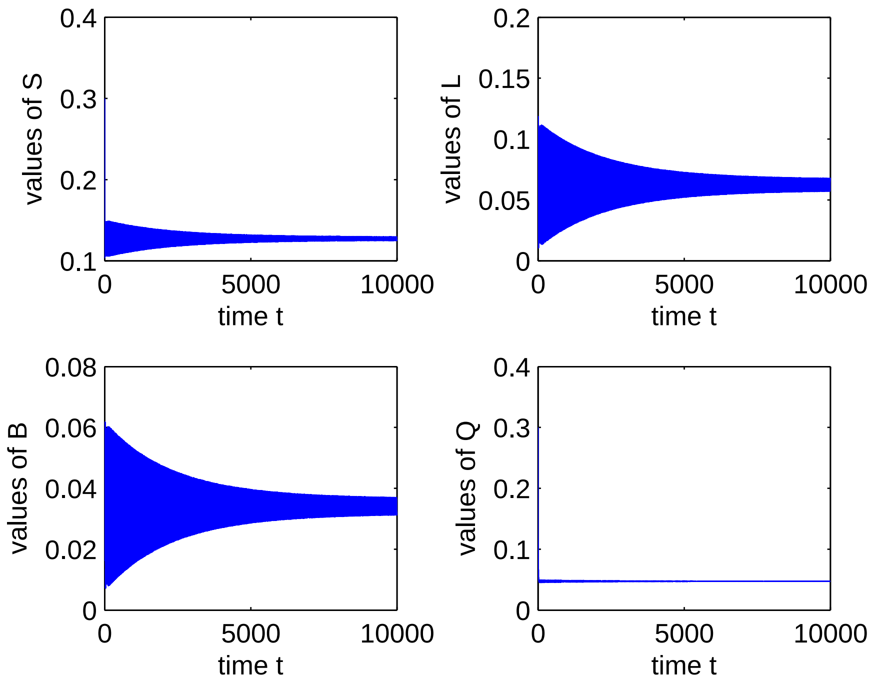

FIGURE 2. Evolutions of S, L, B and Q for $\tau = 13.7 < \tau_0 = 13.7684$ versus time t.

5. Numerical examples

In this section, we illustrate the main results by giving several numerical examples. We choose a set of parameter values in system (2.2): $\mu_1 = 0.001, \mu_2 = 0.004, \mu = 0.005, \beta_1 = 0.3, \beta_2 = 0.62, \beta_3 = 0.35, \gamma_1 = 0.08, \delta_1 = 0.01, \delta_2 = 0.05, \delta_3 = 0.1, \gamma_2 = 0.65, \gamma_3 = 0.06, \alpha = 0.5, \eta = 0.16$. By computer simulation, the equilibrium $E^* = (0.1273, 0.0624, 0.0341, 0.0473)$ of system (2.2) can be obtained. Further, we get that Eq. (3.10) has one positive root $z = 0.0407$. Then, we get $\omega_0 = 0.2018$ and $\tau_0 = 13.7684$.

First, we choose $\tau = 13.7 < \tau_0$. Figure 2 depicts the time plots of S, L, B and Q and Figure 3 and 4 show the corresponding phase plots. From Figures 2-4, it is clear that system (2.2) is asymptotically stable.

Then, we choose $\tau = 13.8 > \tau_0$. Figure 5 displays the time plots of S, L, B and Q and Figure 6 and 7 show the corresponding phase plots. From Figures 5-7, it can be seen that system (2.2) undergoes a Hopf bifurcation, and a family of periodic solutions occurs.

A diagram is constructed to show the model properties depending on the key model parameter. Figure 8 show the values of $L+B$ depending on τ. From Figure 8, we can see that system (2.2) is asymptotically stable for $\tau < \tau_0$, and system (2.2) undergoes a Hopf bifurcation for $\tau > \tau_0$.

Furthermore, we choose $\tau = 5 < \tau_0$. Figure 9 demonstrates the time plots of $L + B$ with and without hybrid quarantine strategies. From this figure, it is easy to see that the value of $L + B$ with hybrid quarantine strategies is significantly less than that without hybrid quarantine strategies, implying that hybrid quarantine strategies can effectively suppress the diffusion of computer viruses and make the system be stable.

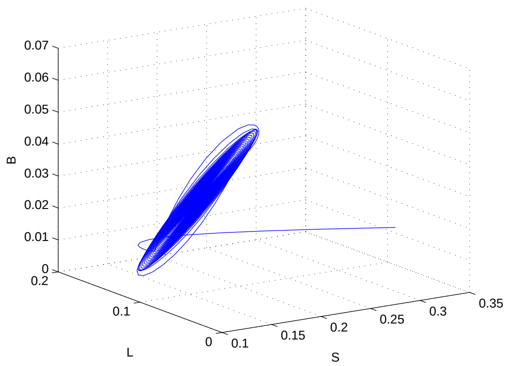

FIGURE 3. The phase plot of the states S, L and B for $\tau = 13.7 < \tau_0 = 13.7684$.

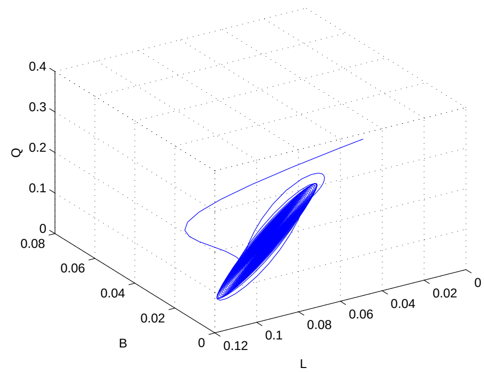

FIGURE 4. The phase plot of the states L, B and Q for $\tau = 13.7 < \tau_0 = 13.7684$.

6. Conclusion

As the quarantine is an effective measure to suppress computer virus propagation, in this paper, a new epidemic model of computer virus named delayed $SLBQRS$ model with hybrid quarantine strategies has been presented. The behaviors of virus propagation under the impact of hybrid quarantine strategies and delay factor have been investigated by analyzing the dynamical properties of this model. The main results have been proposed in terms of local stability and Hopf bifurcation analysis. By regarding the delay which is due to the latent period of a latent computer as a bifurcating parameter and analyzing the distribution of the roots of the associated characteristic equation, sufficient conditions for the local stability and existence of Hopf bifurcation have been obtained. And the critical value τ_0 of the Hopf bifurcation also has been derived. It has been shown that when the delay $\tau < \tau_0$, the model is locally asymptotically stable in which condition of the virus spreading can be controlled. However, when the delay passes through the critical value, the model undergoes a Hopf bifurcation which is not welcome in the networks because the propagation of the computer virus is out of control. Furthermore, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions have been determined by applying the normal form and center manifold theorem. Numerical simulations have been given to verify the main results of this paper, which also have shown that hybrid quarantine strategies can control the viral spread effectively and make the system become stable.

This work is conducive to the understanding of the behaviors of virus propagation in the presence of hybrid quarantine strategies, which are very important and desirable for understanding of the virus spread patterns, as well as for management and control of the spread. In reality, installing and timely updating of hybrid intrusion detection systems on computers, the viral spread can be controlled effectively. This

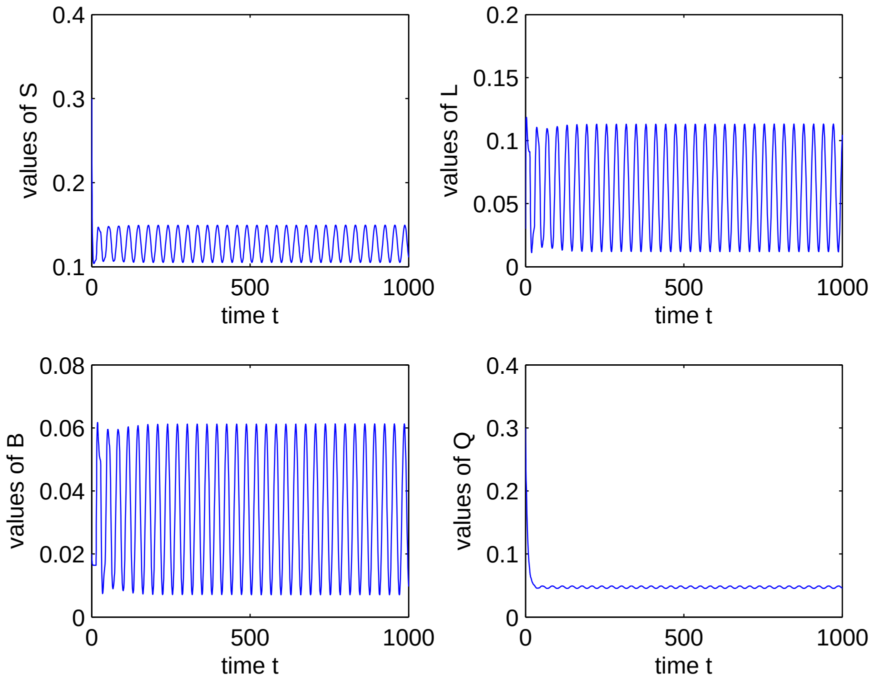

FIGURE 5. Evolutions of S, L, B and Q for $\tau = 13.8 > \tau_0 = 13.7684$ versus time t.

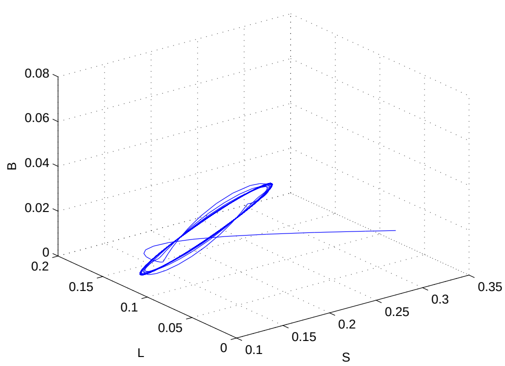

FIGURE 6. The phase plot of the states S, L and B for $\tau = 13.8 > \tau_0 = 13.7684$.

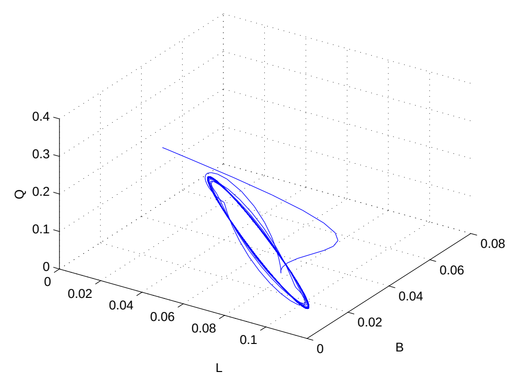

FIGURE 7. The phase plot of the states L, B and Q for $\tau = 13.8 > \tau_0 = 13.7684$.

work is only a starting point for the study of computer virus propagation models with the impact of hybrid quarantine strategies and delay factor. Towards this direction, let us enumerate a few topics of research that are worthy of further study. (1) It is noteworthy that the positivity of solutions of system (2.2) is an open question. (2) To effectively avoid and delay the adverse consequences of Hopf bifurcation, we should do some research on the control of the Hopf bifurcation. (3) Due to the effect of antivirus programs, in the real world, a computer in different compartment is recovered with different

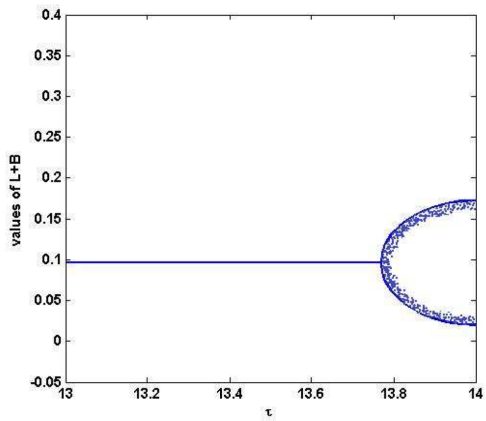

FIGURE 8. The values of $L+B$ for $\tau \in (13, 14)$.

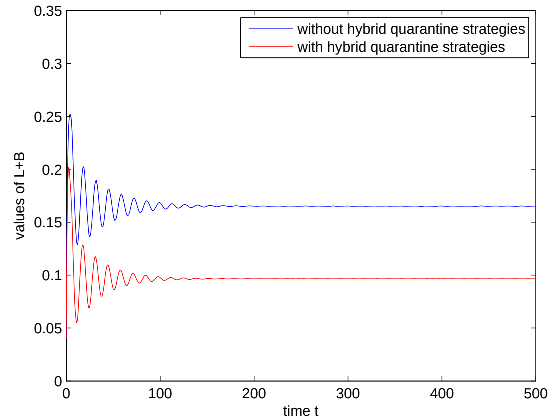

FIGURE 9. Evolutions of $L+B$ with or without hybrid quarantine strategies versus time t.

probabilities. (4) In real-world situations, systems are inevitably affected by environmental noise. To describe this phenomenon, a more complicated stochastic delayed $SLBQRS$ model should be introduced and studied [32,33]. (5) This model should be adapted to scale-free networks [34].

Appendix A

Note added in proof . Suppose that $(S_0, 0, 0, Q_0, R_0)$ is a virus-free equilibrium of system (2.1) where $S_0 \geqslant 0$, $Q_0 \geqslant 0$, $R_0 \geqslant 0$. Then we can get the algebraic system

$$\begin{cases} \mu_1 + \gamma_3 R_0 - \beta_3 S_0 - (\delta_1 + \eta + \mu) S_0 = 0, \\ \beta_3 S_0 = 0, \\ \delta_1 S_0 - (\eta + \mu) Q_0 = 0, \\ \mu_2 + \eta(1 - R_0) - (\mu + \gamma_3) R_0 = 0. \end{cases} \tag{.1}$$

From the second equation of system (.1), we get $S_0 = 0$. Then, solving the first equation of system (.1), it is easy to get $R_0 = -\frac{\mu_1}{\gamma_3}$. It is not consistent with $R_0 \geqslant 0$. Then, we can get that system (2.1) has no virus-free equilibrium. □

Appendix B

Note added in proof . Suppose that $(\overline{S}, \overline{L}, \overline{B}, \overline{Q}, \overline{R})$ is an equilibrium of system (2.1). Then we can get the algebraic system

$$\begin{cases} \mu_1 + \gamma_1 \overline{L} + \gamma_2 \overline{B} + \gamma_3 \overline{R} - (\beta_1 \overline{L} + \beta_2 \overline{B} + \beta_3) \overline{S} - (\delta_1 + \eta + \mu) \overline{S} = 0, \\ (\beta_1 \overline{L} + \beta_2 \overline{B} + \beta_3) \overline{S} - (\gamma_1 + \mu + \delta_2 + \eta) \overline{L} + \overline{L}(t - \tau) = 0, \\ \alpha \overline{L}(t - \tau) - (\gamma_2 + \mu + \delta_3 + \eta) \overline{B} = 0, \\ \delta_1 \overline{S} + \delta_2 \overline{L} + \delta_3 \overline{B} - (\eta + \mu) \overline{Q} = 0, \\ \mu_2 + \eta(1 - \overline{R}) - (\mu + \gamma_3) \overline{R} = 0. \end{cases} \tag{.1}$$

From the fifth equation of system (.1), we get $\overline{R} = R^* = \dfrac{\mu_2+\eta}{\mu+\gamma_3+\eta}$. Then, solving the third and the fourth equations of system (.1), it is easy to get $\overline{L} = \dfrac{\gamma_2+\mu+\delta_3+\eta}{\alpha}\overline{B}$ and $\overline{Q} = \dfrac{\delta_1(1-R^*)}{\eta+\mu+\delta_1} + d\overline{B}$. Replacing them into the second equation of system (.1), we can gain

$$m\overline{B}^2 - n\overline{B} - \beta_3 b = 0. \tag{.2}$$

It is clear that $m > 0$, $b > 0$, $\overline{L} \geq 0, \overline{B} \geq 0$. We get $\overline{B} = B^*$ as the unique root of Eq. (.2). Hence, $\overline{S} = S^*$, $\overline{L} = L^*$ and $\overline{Q} = Q^*$. Thus, the assertion holds. □

Appendix C

Note added in proof . Let $u_1(t) = S(t) - S^*$, $u_2(t) = L(t) - L^*$, $u_3(t) = B(t) - B^*$, $u_4(t) = Q(t) - Q^*$, and $\tau = \tau_0 + \mu, \mu \in R$, and normalize the delay by $t \to \frac{t}{\tau}$. Then, system (2.2) can be transformed into a functional differential equation (FDE) as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{.1}$$

where

$$\begin{aligned} &u_t = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C = C([-1,0], R^4),\\ &L_\mu\phi = (\tau_0+\mu)(A'\phi(0) + B'\phi(-1)),\\ &F(\mu, u_t) = (\tau_0+\mu)\begin{pmatrix} -\beta_1\phi_1(0)\phi_2(0) - \beta_2\phi_1(0)\phi_3(0) \\ \beta_1\phi_1(0)\phi_2(0) + \beta_2\phi_1(0)\phi_3(0) \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{.2}$$

and

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & 0 \\ 0 & 0 & a_8 & 0 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{.3}$$

By the Riesz representation theorem, there exists a 4×4 matrix function $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ of bounded variation components such that

$$L_\mu\phi = \textstyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C. \tag{.4}$$

In fact, we choose

$$\eta(\theta,\mu) = (\tau_0+\mu)(A'\delta(\theta) + B'\delta(\theta+1)), \tag{.5}$$

where $\delta(\theta)$ is Dirac delta function.

For $\phi \in C([-1,0], R^4)$, we define

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\rho)\phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases} \end{aligned} \tag{.6}$$

Then, system (.1) is equivalent to the following form:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{.7}$$

Next, we define the adjoint operator A^* of A as

$$A^*(\mu)s = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,\mu)\varphi(-s), & s = 0, \end{cases} \tag{.8}$$

where η^T is the transpose of the matrix η.

We define a bilinear inner product as

$$\langle \varphi \, , \, \phi \rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \overline{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{.9}$$

where $\eta(\theta) = \eta(\theta, 0)$, $\phi \in [-1, 0]$, $\varphi \in [-1, 0]$.

Let $q(\theta) = (1, q_2, q_3, q_4)^T e^{i\tau_0\omega_0\theta}$ be the eigenvector of A corresponding to the eigenvalue $i\tau_0\omega_0$ and let $q^*(\theta) = D(1, q_2^*, q_3^*, q_4^*)e^{i\tau_0\omega_0 s}$ be the eigenvector of A^* corresponding to the eigenvalue $-i\tau_0\omega_0$. It is easy to get

$$\begin{aligned} &A(0)q(0) = i\tau_0\omega_0 q(0), \\ &A^*(0)q(0) = -i\tau_0\omega_0 q^*(0). \end{aligned} \tag{.10}$$

Then, we obtain

$$\begin{aligned} &q_2 = \frac{a_5 + a_7 q_3}{i\omega_0 - a_6 - b_1 e^{-i\tau_0\omega_0}}, \\ &q_3 = \frac{a_5 b_2 e^{-i\tau_0\omega_0}}{(i\omega_0 - a_6 - b_1 e^{-i\tau_0\omega_0})(i\omega_0 - a_8) - a_7 b_2 e^{-i\tau_0\omega_0}}, \\ &q_4 = -\frac{a_9 + a_{10} q_2 + a_{11} q_3}{(i\omega_0 - a_{12})}, \qquad q_2^* = -\frac{i\omega_0 + a_1 + a_9 q_4^*}{a_5}, \\ &q_3^* = -\frac{a_1 + (i\omega_0 + a_6 + b_1 e^{i\tau_0\omega_0}) q_2^* + a_{10} q_4^*}{b_2 e^{i\tau_0\omega_0}}, \qquad q_4^* = -\frac{a_4}{i\omega_0 + a_{12}}. \end{aligned} \tag{.11}$$

From (.9), we can get

$$\langle q^*(\theta), q(\theta) \rangle = \overline{D}[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + q_2\tau_0 e^{-i\tau_0\omega_0}(b_1\overline{q}_2^* + b_2\overline{q}_3^*)]. \tag{.12}$$

Then, we choose

$$\overline{D} = [1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* + q_2\tau_0 e^{-i\tau_0\omega_0}(b_1\overline{q}_2^* + b_2\overline{q}_3^*)]^{-1}. \tag{.13}$$

Then, $\langle q^*, q \rangle = 1, \langle q^*, \overline{q} \rangle = 0$.

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