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NONLOCAL PROBLEMS IN THE MECHANICS OF THREE-LAYER SHELLS☆

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Abstract. We consider an elastic system consisting of two coaxial cylindrical shells connected by goffered filler. Using a variational principle, this model is reduced to a boundary value problem for strongly elliptic system of differential-difference equations. It is proved the existence and uniquenesss of generalized solution of the above problem, smoothness of solution, and convergence of the Ritz method.

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1. Introduction

Modern aircraft technology is based on constructions containing sandwich shells and plates. Figure 1 shows a wing which has a panel with a goffered filler. This panel can be considered as an elastic system consisting of two parallel plates connected by two regular systems of ribs.

In [4] this discrete-continuous model was reduced to a strongly elliptic system of differential-difference equations. It were studied a connection between variational and boundary value problems, existence and uniqueness of generalized solutions, spectral properties of corresponding operator and convergence of the Ritz method, see [4–6]. It was proved that smoothness of generalized solutions can be violated inside a domain [6].

This paper is devoted to investigation of mathematical model for sandwich shell. Figure 2 shows a rocket engine, whose cooling system has the form of cylindrical three-layer shell with a goffered filler.

In Section 2, we describe an elastic model of this sandwich shell including nonlocal interactions between external and internal shells. In Section 3, we reduce the model of this system to a variational problem for the functional of total potential energy of a three-layer shell. Then we consider this variational problem and corresponding boundary value problem for a system of four differential-difference equations, see Section 4. Section 5 is devoted to energy estimates of generalized solutions. In Section 6, we prove existence and uniqueness of generalized solution to the above-mentioned boundary value problem for a system of differential-difference equations. We also prove that the spectrum of the corresponding operator is discrete and consists of real positive eigenvalues. The convergence of the Ritz method is stated. In Section 7, we prove that smoothness of generalized solutions remains in the whole of domain.

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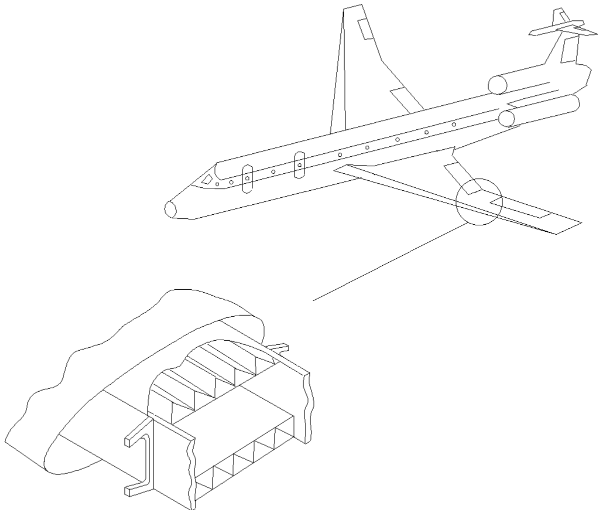

FIGURE 1.

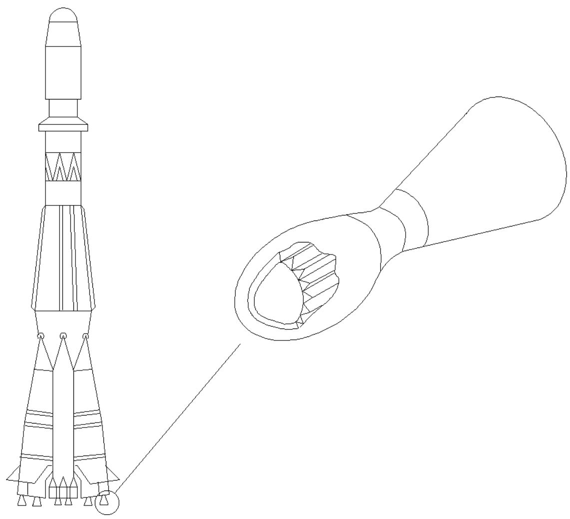

FIGURE 2.

2. The elastic model

Figure 3 shows an elastic system consisting of two coaxial cylindrical shells connected by a regular system of N radial and N slanting ribs, all oriented in the longitudinal direction. It is natural to associate with this discrete-continuous system a continuous model by "spreading out" both the radial ribs (the 0-connections) and the slanting ribs (the α-connections) in the space between the shells. For this it is necessary to introduce kinematically independent continuous fields of elastic displacements of 0-connections and α-connections which are continuously distributed in the space between the shells. As a result, we arrive at a three-layer shell with a "two-phase" model of the filler combining simultaneously the medium of 0-connections and the medium of α-connections.

We introduce a cylindrical system of coordinates x, θ, ρ such that the axis $0x$ is directed along the axis of the cylinder. Let $r = (r_1 + r_{-1})/2$ and $h = (r_1 - r_{-1})/2$, where r_1 and r_{-1} are the radiuses of curvature of

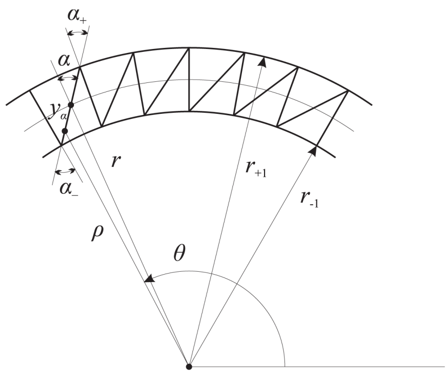

FIGURE 3.

mean surface for external and internal shells, respectively. We also introduce the local Cartesian coordinates $x_\beta, 0_\beta, y_\beta$ $(\beta = 0, \alpha)$ in the planes of the ribs, making the axis $0_\beta x_\beta$ coincide with the line of intersection of the corresponding rib and the surface $\rho = r$. For the simplicity, we shall assume that the shells themselves and the ribs, both radial and slanting, are moment free; the ribs offer no resistance to tension and compression in the longitudinal direction but are absolutely rigid in the transverse direction (in the plane of the rib).

Let us examine a single rib. By virtue of the simplifications we have made, the cross-section of the rib $x_\beta = \text{const}$, is displaced in the plane of the rib like a rigid body. Therefore the elastic displacements u_β, v_β of an arbitrary point of the rib in the direction of the axes x_β, y_β can be represented in the form

$$u_\beta(x_\beta, y_\beta, k) = \varphi_\beta(x_\beta, k) - \psi_\beta(y_\beta, k) y_\beta, \qquad v_\beta(x_\beta, y_\beta, k) = v_\beta(x_\beta, k) \qquad (\beta = 0, \alpha), \tag{2.1}$$

where $\varphi_\beta, \psi_\beta$ are the translational displacement in the direction of the axis $0_\beta x_\beta$ and the rotation in the plane $x_\beta 0_\beta y_\beta$ of the rib cross-section $x_\beta = \text{const}$, respectively. The parameter k determining the rib is equal to the angle θ, which corresponds to the line connecting the point 0 (in cylindrical system of coordinates) and 0_β (in local Cartesian coordinates of the rib).

The local Cartesian system of coordinates is connected with the cylindrical one in the following way:

$$x = x_\beta, \qquad y_\beta = r\cos\beta - \sqrt{\rho^2 - r^2\sin^2\beta} \qquad (\beta = 0, \alpha), \tag{2.2}$$

where β is the dihedral angle between the plane of the rib and the radial plane passing through the axis $0x_\beta$.

The equation for the plane of the rib has the form

$$\theta - \arcsin(\sin\beta \cdot y_\beta/\rho) = k \qquad (\beta = 0, \alpha). \tag{2.3}$$

Let $h/r \ll 1$. Then, by virtue of the Taylor formula, the equations (2.2) and (2.3) to within higher order terms take the form

$$x_\beta = x, \qquad y_\beta = (r - \rho)/\cos\beta \qquad (\beta = 0, \alpha), \tag{2.4}$$

$$\theta - \tan\beta(1 - \rho/r) = k \qquad (\beta = 0, \alpha). \tag{2.5}$$

Spreading out the ribs in the space between the plates, we must introduce the continuous fields $U_\beta(x, y, z), V_\beta(x, y, z)$ $(x \in [0, a],\ y \in [0, 2\pi],\ z \in [r_{-1}, r_{+1}])$ of the elastic displacements of the β-braces $(\beta = 0, \alpha)$. To do this, after first passing from the local coordinates to the unified system of coordinates, we must "extend" the expressions (2.1), which relate to an individual rib, in accordance with the condition

$$U_\beta(x, k + (1-\rho/r)\tan\beta, \rho) = u_\beta(x_\beta, y_\beta, k), \qquad V_\beta(x, k + (1-\rho/r)\tan\beta, \rho) = v_\beta(x_\beta, y_\beta, k) \tag{2.6}$$

to the entire space between the plates.

From (2.1), (2.4)–(2.6) it follows that

$$\begin{aligned} &U_\beta(x,\theta,\rho) = \varphi_\beta(x, \theta - (1-\rho/r)\tan\beta) - \psi_\beta(x, \theta - (1-\rho/r)\tan\beta)(r-\rho)/\cos\beta, \\ &V_\beta(x,\theta,\rho) = v_\beta(x, \theta - (1-\rho/r)\tan\beta) \qquad\qquad (\beta = 0, \alpha). \end{aligned} \tag{2.7}$$

The expressions (2.7) for $\beta = 0, \alpha$ represent kinematically independent continuous fields of elastic displacements of the two-phase filler of a three-layer shell. These fields must be subjected to the conditions of the kinematic connection of the filler with the supporting layers

$$U_\beta(x,\theta,r_{\pm 1}) = u^\pm(x,\theta), \qquad V_\beta(x,\theta,r_{\pm 1}) = v^\pm(x,\theta)\sin\beta^\pm + w^\pm(x,\theta)\cos\beta^\pm \qquad (\beta = 0,\alpha), \tag{2.8}$$

where $u^\pm, v^\pm, w^\pm$ are the displacements of the surfaces of external ($\rho = r_1$) and the internal ($\rho = r_{-1}$) shells; $\beta^\pm$ is the angle between the corresponding rib and the radial plane passing through the line of intersection of this rib with the cylindrical surface $\rho = r_{\pm 1}$, respectively.

From (2.7) and (2.8) it follows that the field of elastic displacements of the three-layer shell with a two-phase filler is determined by 12 functions of two independent variables $u^+, v^+, w^+, u^-, v^-, w^-, \varphi_0, \psi_0, v_0, \varphi_\alpha, \psi_\alpha, v_\alpha$, connected by the eight relations

$$\begin{aligned} &u^\pm(x,\theta) = \varphi_\beta(x, \theta - (1-r_{\pm 1}/r)\tan\beta) - \psi_\beta(x, \theta - (1-r_{\pm 1}/r)\tan\beta)(r - r_{\pm 1})/\cos\beta, \\ &v^\pm(x,\theta)\sin\beta^\pm + w^\pm(x,\theta)\cos\beta^\pm = v_\beta(x, \theta - (1-r_{\pm 1}/r)\tan\beta) \qquad (\beta = 0,\alpha). \end{aligned} \tag{2.9}$$

3. Potential energy of a three-layer shell

As our main unknown, we introduce the four-dimensional vector-valued function $u = (u^1, u^2, u^3, u^4)$ of two variables x and θ as following:

$$u^1 = \varphi_\alpha(x,\theta), \qquad u^2 = \psi_\alpha(x,\theta), \qquad u^3 = v_\alpha(x,\theta), \qquad u^4 = v_0(x,\theta) \quad (x \in [0,a],\ \theta \in [0, 2\pi]). \tag{3.1}$$

Then, by virtue of (2.9), we have

$$\begin{aligned} &u^\pm = u^1_{\pm\tau} \pm h\sec\alpha u^2_{\pm\tau}, \qquad v^\pm = \csc\alpha^\pm u^3_{\pm\tau} - \cot\alpha^\pm u^4, \\ &w^\pm = u^4, \qquad \varphi_0 = (u^1_\tau + u^1_{-\tau})/2 + h\sec\alpha(u^2_\tau - u^2_{-\tau})/2, \\ &\psi_0 = h^{-1}(u^1_\tau - u^1_{-\tau})/2 + \sec\alpha(u^2_\tau + u^2_{-\tau})/2, \end{aligned} \tag{3.2}$$

where $\pm\tau$ denotes the shift of the second argument by $\tau = hr^{-1}\mathrm{tg}\alpha$, *e.g.* $u^i_{\pm\tau} = u^i(x, \theta \pm \tau)$, $\alpha^\pm = \alpha \mp \tau$ (see Fig. 3). Assume that $0 < \tau < \pi/2$.

Remark 3.1. We extend the functions $u^i(x,\theta)$ ($(x,\theta) \in (0,a) \times (0,2\pi)$) to the strip $(0,a) \times \mathbb{R}$ as the 2π-periodic with respect to θ functions. Therefore the functions $u^i_{\pm\tau}$ ($i = 1,\dots,4$) are correctly defined.

The functional of the total potential energy of the three-layer shell with a two-phase filler is given by the formula

$$E(u) = \sum_{i=\pm 1} \left(E_{\Gamma^i}(u) + E_{F^i}(u)\right) + E_{V^0}(u) + E_{V^\alpha}(u). \tag{3.3}$$

Here $E_{\Gamma^i}(u)$ is the potential energy of the shell $\Gamma^i = \{(x,\theta,\rho) : 0 \le x \le a,\ 0 \le \theta \le 2\pi,\ \rho = r_i\}$ given by

$$\begin{aligned}
E_{\Gamma^i}(u) = \frac{G\delta r_i}{2}\int_0^a\int_0^{2\pi}\Big\{&2(1-\nu)^{-1}\Big[\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)^2 \\
&+r_i^{-2}\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right)^2 + r_i^{-2}(u^4)^2 + \\
&+2\nu r_i^{-1}\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right) \\
&+2\nu r_i^{-1}\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)u^4 + 2r_i^{-2}\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right)u^4\Big] \\
&+\left[r_i^{-1}(u^1_{i\tau})_\theta + ih\sec\alpha r_i^{-1}(u^2_{i\tau})_\theta + \csc(\alpha-i\tau)(u^3_{i\tau})_x - \cot(\alpha-i\tau)u^4_x\right]^2\Big\}\,\mathrm{d}x\,\mathrm{d}\theta,
\end{aligned} \tag{3.4}$$

$E_{F^i}(u)$ is the energy of the external forces given by

$$E_{F^i}(u) = -\int_0^a\int_0^{2\pi} r_i\left[X^i\left(u^1_{i\tau} + ih\sec\alpha u^2_{i\tau}\right) + Y^i\left(\csc(\alpha-i\tau)u^3_{i\tau} - \cot(\alpha-i\tau)u^4\right) + Z^i u^4\right]\mathrm{d}x\,\mathrm{d}\theta, \tag{3.5}$$

$E_{V^\beta}(u)$ is the potential energy of the β-braces

$$E_{V^\alpha}(u) = G\mu_\alpha hr\int_0^a\int_0^{2\pi}\left[r^{-1}u^2 - u^3_x\right]^2\mathrm{d}x\,\mathrm{d}\theta, \tag{3.6}$$

$$E_{V^0}(u) = G\mu_0 hr\int_0^a\int_0^{2\pi}\left[h^{-1}(u^1_\tau - u^1_{-\tau})/2 + \sec\alpha(u^2_\tau + u^2_{-\tau})/2 - u^4_x\right]^2\mathrm{d}x\,\mathrm{d}\theta; \tag{3.7}$$

G and ν are the shear modulus and the Poisson coefficient, respectively; μ_β ($\beta = 0, \alpha$) is the volumetric content of the β-braces in a unit volume V_β after spreading out; X^i, Y^i, Z^i ($i = \pm 1$) are the components of the external load on the external and internal shells in the cylindrical system of coordinates.

We assume that the three-layer shell is rigidly fixed for $x = 0, a$. Therefore there are no displacements of the β-braces ($\beta = 0, \alpha$), *i.e.* $U_\beta(x,\theta,\rho) = V_\beta(x,\theta,\rho) = 0$ for $x = 0, a$. Hence, by virtue of (2.7) and (3.1), we obtain

$$u(x,\theta) = 0 \qquad (x = 0, a;\ 0 \le \theta \le 2\pi). \tag{3.8}$$

Since we consider the cylindrical shell, we have

$$u(x,0) = u(x,2\pi) \qquad (0 \le x \le a). \tag{3.9}$$

Thus the proposed continuous model can be described on the basis of the variational problem

$$E(u) \to \min \tag{3.10}$$

with the boundary conditions (3.8) and (3.9).

4. Variational and boundary value problems

Denote by $H^k(Q)$ the Sobolev space of real functions (classes) $u \in L_2(Q)$ having all generalized derivatives $D^\alpha u \in L_2(Q)$ $(|\alpha| \le k)$ with the inner product

$$(u,v)_{H^k(Q)} = \sum_{|\alpha|\le k} \int_0^a \int_0^{2\pi} D^\alpha u(x,\theta) D^\alpha v(x,\theta)\, \mathrm{d}x\, \mathrm{d}\theta,$$

where $Q = (0,a)\times(0,2\pi)$, $k\in\mathbb{N}$.

Let $H^k_{0,2\pi}(Q) = \{u \in H^k(Q) : u|_{x=0} = u|_{x=a} = 0,\ D^j_\theta u|_{\theta=0} = D^j_\theta u|_{\theta=2\pi},\ j = 0,1,\dots,k-1\}$.

We introduce the real space of vector-valued functions

$$\begin{aligned} L_2^4(Q) &= L_2(Q)\times L_2(Q)\times L_2(Q)\times L_2(Q),\\ H^{k,4}(Q) &= H^k(Q)\times H^k(Q)\times H^k(Q)\times H^k(Q),\\ H^{k,4}_{0,2\pi}(Q) &= H^k_{0,2\pi}(Q)\times H^k_{0,2\pi}(Q)\times H^k_{0,2\pi}(Q)\times H^k_{0,2\pi}(Q). \end{aligned}$$

Suppose that $X^i, Y^i, Z^i \in L_2(Q)\ (i=\pm 1)$.

We shall find an extremum of the functional (3.10) with the boundary conditions (3.8) and (3.9) in the space $H^{1,4}_{0,2\pi}(Q)$. Assume that the vector-valued function $u \in H^{1,4}_{0,2\pi}(Q)$ yields a minimum of the functional (3.10). Then for each $v \in H^{1,4}_{0,2\pi}(Q)$

$$\frac{\mathrm{d}E(u+tv)}{\mathrm{d}t}|_{t=0} = 0. \tag{4.1}$$

Integral identity (4.1) has the following form:

$$\begin{aligned} &\int_0^a\int_0^{2\pi}\Big\{\sum_{i=\pm1}\frac{G\delta r_i}{2}\Big[\frac{2}{1-\nu}\Big[2\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)\left((v^1_{i\tau})_x + ih\sec\alpha(v^2_{i\tau})_x\right)\\ &+r_i^{-2}2\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right)\left(\csc(\alpha-i\tau)(v^3_{i\tau})_\theta - \cot(\alpha-i\tau)v^4_\theta\right)\\ &+2r_i^{-2}u^4v^4 + 2\nu r_i^{-1}\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)\left(\csc(\alpha-i\tau)(v^3_{i\tau})_\theta - \cot(\alpha-i\tau)v^4_\theta\right)\\ &+2\nu r_i^{-1}\left((v^1_{i\tau})_x + ih\sec\alpha(v^2_{i\tau})_x\right)\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right)\\ &+2\nu r_i^{-1}\left((u^1_{i\tau})_x + ih\sec\alpha(u^2_{i\tau})_x\right)v^4 + 2\nu r_i^{-1}\left((v^1_{i\tau})_x + ih\sec\alpha(v^2_{i\tau})_x\right)u^4\\ &+2r_i^{-2}\left(\csc(\alpha-i\tau)(u^3_{i\tau})_\theta - \cot(\alpha-i\tau)u^4_\theta\right)v^4 + 2r_i^{-2}\left(\csc(\alpha-i\tau)(v^3_{i\tau})_\theta - \cot(\alpha-i\tau)v^4_\theta\right)u^4\Big]\\ &+2\left(r_i^{-1}(u^1_{i\tau})_\theta + ih\sec\alpha r_i^{-1}(u^2_{i\tau})_\theta + \csc(\alpha-i\tau)(u^3_{i\tau})_x - \cot(\alpha-i\tau)u^4_x\right)\\ &\times\left(r_i^{-1}(v^1_{i\tau})_\theta + ih\sec\alpha r_i^{-1}(v^2_{i\tau})_\theta + \csc(\alpha-i\tau)(v^3_{i\tau})_x - \cot(\alpha-i\tau)v^4_x\right)\Big]\\ &+G\mu_\alpha hr2(r^{-1}u^2 - u^3_x)(r^{-1}v^2 - v^3_x) + G\mu_0 hr2(h^{-1}(u^1_\tau - u^1_{-\tau})/2 + \csc\alpha(u^2_\tau + u^2_{-\tau})/2 - u^4_x)\\ &\times(h^{-1}(v^1_\tau - v^1_{-\tau})/2 + \csc\alpha(v^2_\tau + v^2_{-\tau})/2 - v^4_x)\\ &-\sum_{i=\pm1}\left[r_i\left(X^i\left(v^1_{i\tau} + ih\sec\alpha v^2_{i\tau}\right) + Y^i\left(\csc(\alpha-i\tau)v^3_{i\tau} - \cot(\alpha-i\tau)v^4\right)\right) + Z^iv^4\right]\Big\}\,\mathrm{d}x\,\mathrm{d}\theta = 0. \end{aligned}$$

In the integrals containing the functions $v^i_{\pm\tau}$, $(v^i_{\pm\tau})_x$, $(v^i_{\pm\tau})_\theta$ $(i=1,2,3)$ we make the change of variables $x'=x$, $\theta'=\theta\pm\tau$, thereby passing from integrals over the region $(0,a)\times(0,2\pi)$ to intervals over the region $(0,a)\times(\pm\tau,2\pi\pm\tau)$. For any 2π-periodic with respect to θ' function $v(x,\theta')$, we have

$$\int\limits_0^a\int\limits_{\pm\tau}^{2\pi\pm\tau} v(x,\theta')\,\mathrm{d}x\,\mathrm{d}\theta' = \int\limits_0^a\int\limits_0^{2\pi} v(x,\theta')\,\mathrm{d}x\,\mathrm{d}\theta'.$$

Therefore in the achieved integrals we can change the domain of integration $(0,a)\times(\pm\tau,2\pi\pm\tau)$ by $(0,a)\times(0,2\pi)$. Reducing similar terms, we have

$$\int\limits_0^a\int\limits_0^{2\pi}\Big\{G\delta\Big[\frac{2}{1-\nu}(r_1+r_{-1})u^1_x+\frac{2h\sec\alpha}{1-\nu}(r_1-r_{-1})u^2_x+\frac{2\nu}{1-\nu}\left(\csc(\alpha+\tau)+\csc(\alpha-\tau)\right)u^3_\theta$$
$$-\frac{2\nu}{1-\nu}\left(\cot(\alpha-\tau)u^4_{-\tau}+\cot(\alpha+\tau)u^4_\tau\right)_\theta+\frac{2\nu}{1-\nu}\left(u^4_\tau+u^4_{-\tau}\right)\Big]v^1_x+G\delta\Big[(r_1^{-1}+r_{-1}^{-1})u^1_\theta$$
$$+h\sec\alpha(r_1^{-1}-r_{-1}^{-1})u^2_\theta+\left(\csc(\alpha+\tau)+\csc(\alpha-\tau)\right)u^3_x-\left(\cot(\alpha+\tau)u^4_\tau+\cot(\alpha-\tau)u^4_{-\tau}\right)_x\Big]v^1_\theta$$
$$+G\left[\mu_0rh^{-1}(2u^1-u^1_{2\tau}-u^1_{-2\tau})/2-\mu_0r\sec\alpha(u^2_{2\tau}-u^2_{-2\tau})/2+\mu_0r(u^4_\tau-u^4_{-\tau})_x\right]v^1\Big\}\mathrm{d}x\,\mathrm{d}\theta$$
$$+\int\limits_0^a\int\limits_0^{2\pi}\Big\{G\delta\Big[\frac{2h\sec\alpha}{1-\nu}(r_1-r_{-1})u^1_x+\frac{2h^2\sec^2\alpha}{1-\nu}(r_1+r_{-1})u^2_x-\frac{2\nu h\sec\alpha}{1-\nu}\big(\csc(\alpha+\tau)$$
$$-\csc(\alpha-\tau)\big)u^3_\theta+\frac{2\nu h\sec\alpha}{1-\nu}\left(\cot(\alpha+\tau)u^4_\tau-\cot(\alpha-\tau)u^4_{-\tau}\right)_\theta-\frac{2\nu h\sec\alpha}{1-\nu}(u^4_\tau-u^4_{-\tau})\Big]v^2_x$$
$$+G\delta\Big[h\sec\alpha(r_1^{-1}-r_{-1}^{-1})u^1_\theta+h^2\sec^2\alpha(r_1^{-1}+r_{-1}^{-1})u^2_\theta-h\sec\alpha\left(\csc(\alpha+\tau)-\csc(\alpha-\tau)\right)u^3_x$$
$$+h\sec\alpha\left(\cot(\alpha+\tau)u^4_\tau-\cot(\alpha-\tau)u^4_{-\tau}\right)_x\Big]v^2_\theta+G\Big[2\mu_\alpha r^{-1}hu^2-2\mu_\alpha hu^3_x$$
$$+\mu_0r\sec\alpha(u^1_{2\tau}-u^1_{-2\tau})/2+\mu_0hr\sec^2\alpha(2u^2+u^2_{2\tau}+u^2_{-2\tau})/2-\mu_0hr\sec\alpha(u^4_\tau+u^4_{-\tau})_x\Big]v^2\Big\}\mathrm{d}x\,\mathrm{d}\theta$$
$$+\int\limits_0^a\int\limits_0^{2\pi}\Big\{G\delta\Big[\left(\csc(\alpha+\tau)+\csc(\alpha-\tau)\right)u^1_\theta-h\sec\alpha\left(\csc(\alpha+\tau)-\csc(\alpha-\tau)\right)u^2_\theta$$
$$+\left(r_1\csc^2(\alpha-\tau)+r_{-1}\csc^2(\alpha+\tau)\right)u^3_x-\left(\frac{r_{-1}\cos(\alpha+\tau)}{\sin^2(\alpha+\tau)}u^4_\tau+\frac{r_1\cos(\alpha-\tau)}{\sin^2(\alpha-\tau)}u^4_{-\tau}\right)_x\Big]v^3_x$$
$$+G\delta\Big[\frac{2}{1-\nu}\left(\frac{\csc^2(\alpha-\tau)}{r_1}+\frac{\csc^2(\alpha+\tau)}{r_{-1}}\right)u^3_\theta-\frac{2}{1-\nu}\left(\frac{\cos(\alpha+\tau)}{r_{-1}\sin^2(\alpha+\tau)}u^4_\tau+\frac{\cos(\alpha-\tau)}{r_1\sin^2(\alpha-\tau)}u^4_{-\tau}\right)_\theta$$
$$+\frac{2\nu}{1-\nu}\left(\csc(\alpha+\tau)+\csc(\alpha-\tau)\right)u^1_x-\frac{2\nu h}{1-\nu}\sec\alpha\left(\csc(\alpha+\tau)-\csc(\alpha-\tau)\right)u^2_x$$
$$+\frac{2}{1-\nu}\left(\frac{\csc(\alpha+\tau)}{r_{-1}}u^4_\tau+\frac{\csc(\alpha-\tau)}{r_1}u^4_{-\tau}\right)\Big]v^3_\theta+G\left[2\mu_\alpha hru^3_x-2\mu_\alpha hu^2\right]v^3_x\Big\}\mathrm{d}x\,\mathrm{d}\theta$$
$$+\int\limits_0^a\int\limits_0^{2\pi}\Big\{G\delta\Big[-\left(\cot(\alpha-\tau)u^1_\tau+\cot(\alpha+\tau)u^1_{-\tau}\right)_\theta-\frac{h}{\cos\alpha}\left(\cot(\alpha-\tau)u^2_\tau-\cot(\alpha+\tau)u^2_{-\tau}\right)_\theta$$

$$
\begin{aligned}
&-\left(\frac{r_1\cos(\alpha-\tau)}{\sin^2(\alpha-\tau)}u^3_\tau+\frac{r_{-1}\cos(\alpha+\tau)}{\sin^2(\alpha+\tau)}u^3_{-\tau}\right)_x+\left(r_1\cot^2(\alpha-\tau)+r_{-1}\cot^2(\alpha+\tau)\right)u^4_x\Bigg]v^4_x\\
&+G\delta\Bigg[-\frac{2}{1-\nu}\left(\frac{\cos(\alpha-\tau)}{r_1\sin^2(\alpha-\tau)}u^3_\tau+\frac{\cos(\alpha+\tau)}{r_{-1}\sin^2(\alpha+\tau)}u^3_{-\tau}\right)_\theta+\frac{2}{1-\nu}\left(\frac{\cot^2(\alpha-\tau)}{r_1}+\frac{\cot^2(\alpha+\tau)}{r_{-1}}\right)u^4_\theta\\
&-\frac{2\nu}{1-\nu}\left(\cot(\alpha-\tau)u^1_\tau+\cot(\alpha+\tau)u^1_{-\tau}\right)_x-\frac{2\nu h}{(1-\nu)\cos\alpha}\left(\cot(\alpha-\tau)u^2_\tau-\cot(\alpha+\tau)u^2_{-\tau}\right)_x\Bigg]v^4_\theta\\
&+G\delta\Bigg[\frac{2\nu}{1-\nu}(u^1_\tau+u^1_{-\tau})_x+\frac{2\nu h}{(1-\nu)\cos\alpha}(u^2_\tau+u^2_{-\tau})_x+\frac{2}{1-\nu}\left(\frac{\csc(\alpha-\tau)}{r_1}u^3_\tau+\frac{\csc(\alpha+\tau)}{r_{-1}}u^3_{-\tau}\right)\\
&+\frac{2}{1-\nu}(r_1^{-1}+r_{-1}^{-1})u^4\Bigg]v^4+G\mu_0\left[-r(u^1_\tau-u^1_{-\tau})-hr\sec\alpha(u^2_\tau+u^2_{-\tau})+2hru^4_x\right]v^4_x\Bigg\}\mathrm{d}x\,\mathrm{d}\theta\\
&=\sum_{i=1}^{4}\int_0^a\int_0^{2\pi}f^iv^i\,\mathrm{d}x\,\mathrm{d}\theta, \qquad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
f^1&=(r_1X^1_{-\tau}+r_{-1}X^{-1}_\tau),\\
f^2&=h\sec\alpha(r_1X^1_{-\tau}-r_{-1}X^{-1}_\tau),\\
f^3&=(r_1\csc(\alpha-\tau)Y^1_{-\tau}+r_{-1}\csc(\alpha+\tau)Y^{-1}_\tau),\\
f^4&=(r_1Z^1+r_{-1}Z^{-1}_\tau)-(r_1\cot(\alpha-\tau)Y^1+r_{-1}\cot(\alpha+\tau)Y^{-1}).
\end{aligned}
$$

Let us make some formal calculations. Integrating by parts in the left part of (4.2), by virtue of the Du Bois-Reymond Lemma, we make sure that the vector-valued function u satisfies the following system of differential-difference equations:

$$
\begin{aligned}
&-\frac{2G\delta}{1-\nu}(r_1+r_{-1})u^1_{xx}-G\delta\left(\frac{1}{r_1}+\frac{1}{r_{-1}}\right)u^1_{\theta\theta}-\frac{2G\delta h}{(1-\nu)\cos\alpha}(r_1-r_{-1})u^2_{xx}-\frac{G\delta h}{\cos\alpha}\left(\frac{1}{r_1}-\frac{1}{r_{-1}}\right)u^2_{\theta\theta}\\
&-G\delta\frac{1+\nu}{1-\nu}\left(\frac{1}{\sin(\alpha+\tau)}+\frac{1}{\sin(\alpha-\tau)}\right)u^3_{x\theta}+G\delta\frac{1+\nu}{1-\nu}\left(\cot(\alpha+\tau)u^4_\tau+\cot(\alpha-\tau)u^4_{-\tau}\right)_{x\theta}\\
&-\frac{2G\delta\nu}{1-\nu}\left(u^4_\tau+u^4_{-\tau}\right)_x+G\mu_0r\left(u^4_\tau-u^4_{-\tau}\right)_x+\frac{G\mu_0r}{2h}\left(2u^1-u^1_{2\tau}-u^1_{-2\tau}\right)\\
&-\frac{G\mu_0r}{2\cos\alpha}\left(u^2_{2\tau}-u^2_{-2\tau}\right)=f^1, \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2G\delta h}{(1-\nu)\cos\alpha}(r_1-r_{-1})u^1_{xx}-\frac{G\delta h}{\cos\alpha}\left(\frac{1}{r_1}-\frac{1}{r_{-1}}\right)u^1_{\theta\theta}-\frac{2G\delta h^2}{(1-\nu)\cos^2\alpha}(r_1+r_{-1})u^2_{xx}\\
&-\frac{G\delta h^2}{\cos^2\alpha}\left(\frac{1}{r_1}+\frac{1}{r_{-1}}\right)u^2_{\theta\theta}-\frac{G\delta(1+\nu)h}{(1-\nu)\cos\alpha}\left(\frac{1}{\sin(\alpha+\tau)}-\frac{1}{\sin(\alpha-\tau)}\right)u^3_{x\theta}\\
&+\frac{G\delta(1+\nu)h}{(1-\nu)\cos\alpha}\left(\cot(\alpha+\tau)u^4_\tau-\cot(\alpha-\tau)u^4_{-\tau}\right)_{x\theta}+\frac{2G\delta\nu h}{(1-\nu)\cos\alpha}\left(u^4_\tau-u^4_{-\tau}\right)_x\\
&+\frac{2G\delta\nu h}{(1-\nu)\cos\alpha}\left(u^4_\tau-u^4_{-\tau}\right)_x-2G\mu_\alpha hu^3_x+\frac{2G\mu_\alpha h}{r}u^2+\frac{G\mu_0r}{2\cos\alpha}\left(u^1_{2\tau}-u^1_{-2\tau}\right)\\
&+\frac{G\mu_0hr}{2\cos^2\alpha}\left(2u^2+u^2_{2\tau}+u^2_{-2\tau}\right)-\frac{G\mu_0hr}{\cos\alpha}\left(u^4_\tau+u^4_{-\tau}\right)_x=f^2, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{G\delta(1+\nu)}{1-\nu}\left(\frac{1}{\sin(\alpha+\tau)}+\frac{1}{\sin(\alpha-\tau)}\right)u^1_{x\theta}+\frac{G\delta(1+\nu)h}{(1-\nu)\cos\alpha}\left(\frac{1}{\sin(\alpha+\tau)}-\frac{1}{\sin(\alpha-\tau)}\right)u^2_{x\theta}\\
&-G\delta\left(\frac{r_1}{\sin^2(\alpha-\tau)}+\frac{r_{-1}}{\sin^2(\alpha+\tau)}\right)u^3_{xx}-\frac{2G\delta}{1-\nu}\left(\frac{1}{r_1\sin^2(\alpha-\tau)}+\frac{1}{r_{-1}\sin^2(\alpha+\tau)}\right)u^3_{\theta\theta}\\
&+G\delta\left(\frac{r_{-1}\cos(\alpha+\tau)}{\sin^2(\alpha+\tau)}u^4_\tau+\frac{r_1\cos(\alpha-\tau)}{\sin^2(\alpha-\tau)}u^4_{-\tau}\right)_{xx}\\
&+\frac{2G\delta}{1-\nu}\left(\frac{\cos(\alpha+\tau)}{r_{-1}\sin^2(\alpha+\tau)}u^4_\tau+\frac{\cos(\alpha-\tau)}{r_1\sin^2(\alpha-\tau)}u^4_{-\tau}\right)_{\theta\theta}\\
&-\frac{2G\delta}{1-\nu}\left(\frac{1}{r_{-1}\sin(\alpha+\tau)}u^4_\tau+\frac{1}{r_1\sin(\alpha-\tau)}u^4_{-\tau}\right)_\theta-2G\mu_\alpha hru^3_{xx}+2G\mu_\alpha hu^2_x=f^3, \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{G\delta(1+\nu)}{1-\nu}\left(\cot(\alpha-\tau)u^1_\tau+\cot(\alpha+\tau)u^1_{-\tau}\right)_{x\theta}+\frac{G\delta(1+\nu)h}{(1-\nu)\cos\alpha}\left(\cot(\alpha-\tau)u^2_\tau-\cot(\alpha+\tau)u^2_{-\tau}\right)_{x\theta}\\
&+G\delta\left(\frac{r_1\cos(\alpha-\tau)}{\sin^2(\alpha-\tau)}u^3_\tau+\frac{r_{-1}\cos(\alpha+\tau)}{\sin^2(\alpha+\tau)}u^3_{-\tau}\right)_{xx}+\frac{2G\delta}{1-\nu}\left(\frac{\cos(\alpha-\tau)}{r_1\sin^2(\alpha-\tau)}u^3_\tau+\frac{\cos(\alpha+\tau)}{r_{-1}\sin^2(\alpha+\tau)}u^3_{-\tau}\right)_{\theta\theta}\\
&-G\delta\left(r_1\cot^2(\alpha-\tau)+r_{-1}\cot^2(\alpha+\tau)\right)u^4_{xx}-\frac{2G\delta}{1-\nu}\left(\frac{\cot^2(\alpha-\tau)}{r_1}+\frac{\cot^2(\alpha+\tau)}{r_{-1}}\right)u^4_{\theta\theta}\\
&+\frac{2G\delta\nu}{1-\nu}\left(u^1_\tau+u^1_{-\tau}\right)_x+\frac{2G\delta\nu h}{(1-\nu)\cos\alpha}\left(u^2_\tau-u^2_{-\tau}\right)_x+\frac{2G\delta}{1-\nu}\left(\frac{1}{r_1\sin(\alpha-\tau)}u^3_\tau+\frac{1}{r_{-1}\sin(\alpha+\tau)}u^3_{-\tau}\right)_\theta\\
&+\frac{2G\delta}{1-\nu}\left(\frac{1}{r_1}+\frac{1}{r_{-1}}\right)u^4-2G\mu_0hru^4_{xx}+G\mu_0r\left(u^1_\tau-u^1_{-\tau}\right)_x+\frac{G\mu_0hr}{\cos\alpha}\left(u^2_\tau+u^2_{-\tau}\right)_x=f^4. \qquad (4.6)
\end{aligned}
$$

Remark 4.1. Thus, if u gives a minimum to functional (3.3) with boundary conditions (3.8) and (3.9), then in some sense it satisfies the boundary value problem (4.3)–(4.6), (3.8), (3.9). In Section 6, we shall show that u is a generalized solution of the boundary value problem (4.3)–(4.6), (3.8), (3.9). In order to give a correct definition of generalized solutions and to study a relation between variational problems and boundary value problems we need some auxiliary statements.

5. Energy estimates

We introduce linear operators $R_i : L_2(Q) \to L_2(Q)$ by the formula

$$R_iu = u_{i\tau} \qquad (i=\pm1,\pm2). \qquad (5.1)$$

In order to define a function $u(\cdot)$ at the point $(x,\theta+i\tau)$ for $\theta+i\tau\notin(0,2\pi)$, we extend it from $(0,a)\times(0,2\pi)$ to the strip $(0,a)\times\mathbb{R}$ as a 2π-periodic function. The following statement is evident.

Lemma 5.1. *The operators $R_i : L_2(Q)\to L_2(Q)$ are bounded, and $R_i^{-1}=R_i^*=R_{-i}$ $(i=\pm1,\pm2)$.*

Lemma 5.2. *The operators $R_i : H^1_{0,2\pi}(Q)\to H^1_{0,2\pi}(Q)$ are bounded, and $(R_iu)_x=R_iu_x$, $(R_iu)_\theta=R_iu_\theta$ $(i=\pm1,\pm2)$.*

Proof. Let $C^\infty_{0,2\pi}(\tilde{Q}) = \{u \in C^\infty(\overline{Q}) : D^k_\theta|_{\theta=0} = D^k_\theta|_{\theta=2\pi}, k = 0,1,2,\dots; \operatorname{supp} u \cap (\{x=0\} \cup \{x=a\}) = \emptyset\}$, where $\tilde{Q} = (0,a) \times [0,2\pi]$. For any $u \in C^\infty_{0,2\pi}(\tilde{Q})$, equalities in Lemma 5.2 hold, and

$$\|R_i u\|_{H^1_{0,2\pi}(Q)} \le \|u\|_{H^1_{0,2\pi}(Q)}. \tag{5.2}$$

Since $C^\infty_{0,2\pi}(\tilde{Q})$ is dense in $H^1_{0,2\pi}(Q)$, the operator $R_i : H^1_{0,2\pi}(Q) \to H^1_{0,2\pi}(Q)$ is bounded, and equalities $(R_i u)_x = R_i u_x$, $(R_i u)_\theta = R_i u_\theta$ are satisfied for all $u \in H^1_{0,2\pi}(Q)$. □

Lemma 5.3. *Let $R : L_2(Q) \to L_2(Q)$ be a bounded linear operator given by the formula $R = \beta_1 R_1 + \beta_{-1} R_{-1}$, where $\beta_1, \beta_{-1} \in \mathbb{R}$, $\beta_1^2 + \beta_{-1}^2 \ne 0$, and $|\beta_1| \ne |\beta_{-1}|$. Then the operator $R : L_2(Q) \to L_2(Q)$ has a bounded inverse $R^{-1} : L_2(Q) \to L_2(Q)$.*

Proof. Without loss of generality, we assume that $|\beta_{-1}| > |\beta_1|$. By virtue of Lemma 5.1, $R_{-1} = R_1^{-1}$. Therefore $R = \beta_{-1} R_1^{-1}\left(I + \dfrac{\beta_1}{\beta_{-1}} R_1^2\right)$. Clearly, $R_1^2 u = u_{2\tau}$. Hence $\|R_1^2\| = 1$. Then $\left\|\dfrac{\beta_1}{\beta_{-1}} R_1^2\right\| < 1$. Thus, the operator R has a bounded inverse $R^{-1} = \dfrac{1}{\beta_{-1}}\left(I + \dfrac{\beta_1}{\beta_{-1}} R_1^2\right)^{-1} R_1$. □

We define a quadratic functional on $H^{1,4}_{0,2\pi}(Q)$ by the formula

$$J(u) = \sum_{i=\pm 1} E_{\Gamma^i}(u) + \sum_{\beta=0,\alpha} E_{V^\beta}(u). \tag{5.3}$$

Lemma 5.4. *For any $u \in H^{1,4}_{0,2\pi}(Q)$*

$$J(u) \ge c_1 \|u\|^2_{H^{1,4}_{0,2\pi}(Q)}, \tag{5.4}$$

where $c_1 > 0$ does not depend on u.

Proof. Let $u \in H^{1,4}_{0,2\pi}(Q)$ be an arbitrary function. From (3.3) and (3.4) it follows that

$$J(u) = \sum_{i=\pm 1} \sum_{k=1}^{4} J_{ik}(u) + \sum_{\beta=0,\alpha} E_{V^\beta}(u), \tag{5.5}$$

where

$$J_{i1}(u) = G\delta r_i(1+\nu) \int_0^a \int_0^{2\pi} \left((u^1_{i\tau})_x + \frac{ih}{\cos\alpha}(u^2_{i\tau})_x\right)^2 \mathrm{d}x\,\mathrm{d}\theta, \tag{5.6}$$

$$J_{i2}(u) = \frac{G\delta r_i}{1-\nu} \int_0^a \int_0^{2\pi} \left[\left((u^1_{i\tau})_x + \frac{ih}{\cos\alpha}(u^2_{i\tau})_x\right) + \frac{(u^3_{i\tau})_\theta}{r_i \sin(\alpha - i\tau)} - \frac{\cot(\alpha - i\tau)}{r_i} u^4_\theta + \frac{u^4}{r_i}\right]^2 \mathrm{d}x\,\mathrm{d}\theta, \tag{5.7}$$

$$J_{i3}(u) = \frac{G\delta r_i}{2} \int_0^a \int_0^{2\pi} \left[\frac{(u^1_{i\tau})_\theta}{r_i} + \frac{ih}{r_i \cos\alpha}(u^2_{i\tau})_\theta + \frac{(u^3_{i\tau})_x}{\sin(\alpha - i\tau)} - \cot(\alpha - i\tau) u^4_x\right]^2 \mathrm{d}x\,\mathrm{d}\theta. \tag{5.8}$$

Assume to the contrary that inequality (5.4) does not hold. Then for any $n > 0$, there exists a function $u_n \in H^{1,4}_{0,2\pi}(Q)$ such that $J(u_n) < \dfrac{1}{n}\|u_n\|^2_{H^{1,4}_{0,2\pi}(Q)}$. Without loss of generality, we suppose that $\|u_n\|^2_{H^{1,4}_{0,2\pi}(Q)} = 1$.

Since the imbedding of $H^{1,4}_{0,2\pi}(Q)$ into $L^4_2(Q)$ is compact, one can choose a subsequence $\{v_n\}$ of the sequence $\{u_n\}$ such that v_n converges in $L^4_2(Q)$ to some function $v \in L^4_2(Q)$, and

$$J(v_n) < \frac{1}{n}. \tag{5.9}$$

From (5.5) to (5.8) and periodicity of a vector-valued function $u \in H^{1,4}_{0,2\pi}(Q)$ with respect to θ it follows that

$$\begin{aligned} J(u) &\geq G\delta(1+\nu)\sum_{i=\pm} r_i \int_0^a \int_0^{2\pi} \left[(u^1_{i\tau})_x + \frac{ih}{\cos\alpha}(u^2_{i\tau})_x\right]^2 \mathrm{d}x\,\mathrm{d}\theta \\ &= G\delta(1+\nu)\sum_{i=\pm} r_i \int_0^a \int_0^{2\pi} \left[(u^1)_x + \frac{ih}{\cos\alpha}(u^2)_x\right]^2 \mathrm{d}x\,\mathrm{d}\theta \\ &\geq 2G\delta(1+\nu)r_{-1} \int_0^a \int_0^{2\pi} \left[(u^1)_x + \frac{h^2}{\cos^2\alpha}(u^2)_x\right]^2 \mathrm{d}x\,\mathrm{d}\theta, \end{aligned} \tag{5.10}$$

$$J(u) \geq E_{V^\alpha}(u) + E_{V^0}(u). \tag{5.11}$$

By virtue of (5.9) and (5.10), we have

$$\lim_{n\to\infty} v^1_{nx} = \lim_{n\to\infty} v^2_{nx} = 0. \tag{5.12}$$

From the formulas (5.11), (3.6), and (3.7) and continuity of the operators $R_j : L_2(Q) \to L_2(Q)\,(j = \pm 1)$ we have

$$\lim_{n\to\infty} v^3_{nx} = \frac{v^2}{r}, \qquad \lim_{n\to\infty} v^4_{nx} = \frac{1}{2h}(v^1_\tau - v^1_{-\tau}) + \frac{1}{2\cos\alpha}(v^2_\tau + v^2_{-\tau}). \tag{5.13}$$

Clearly, $\lim\limits_{n\to\infty} J_{i3}(v_n) = 0$. On the other hand, Lemma 5.2 and formula (5.13) imply that the sequence $\{(v^4_n)_{-i\tau,x}\}\,(i = \pm 1)$ converges in $L_2(Q)$. Therefore, by virtue of (5.8) we have

$$\lim_{n\to\infty}\left(v^1_{n\theta} + \frac{ih}{\cos\alpha}v^2_{n\theta}\right) = p_i,$$

where $p_i = -\dfrac{r_i}{\sin(\alpha - i\tau)}\lim\limits_{n\to\infty} v^3_{nx} + r_i\cot(\alpha - i\tau)\lim\limits_{n\to\infty}(v^4_n)_{-i\tau,x}$ $(i = \pm 1)$. Hence $\lim\limits_{n\to\infty} v^1_{n\theta} = \dfrac{p_1 + p_{-1}}{2}$, $\lim\limits_{n\to\infty} v^2_{n\theta} = \dfrac{\cos\alpha(p_1 - p_{-1})}{2h}$. From here, using a convergence of the sequences $\{v^k_n\}$ and $\{v^k_{nx}\}$ in $L_2(Q)$, we make sure that the sequence $\{v^k_n\}$ converges in $H^1_{0,2\pi}(Q)$ to an element v^k, in addition, by virtue of (5.12), $v^k_x = 0\,(k = 1, 2)$. Since $v^k|_{x=0} = v^k|_{x=a} = 0$, we have $v^k = 0\,(k = 1, 2)$. Thus, by virtue of (5.13),

$$\lim_{n\to\infty} v^3_{nx} = \lim_{n\to\infty} v^4_{nx} = 0. \tag{5.14}$$

From (5.5) and (5.9) it follows that $\lim\limits_{n\to\infty} J_{i2}(v_n) = 0$. Therefore the equalities (5.7), (5.12) and Lemma 5.2 imply that

$$\lim_{n\to\infty}\left((v^3_n)_{i\tau,\theta} - \cos(\alpha - i\tau)v^4_{n\theta}\right) = -\sin(\alpha - i\tau)v^4 \quad (i = \pm 1).$$

Hence

$$\lim_{n\to\infty} \left(\cos(\alpha+\tau)R_1 - \cos(\alpha-\tau)R_{-1}\right) v_{n\theta}^3 = \sin 2\tau v^4,$$
$$\lim_{n\to\infty} \left(\cos(\alpha+\tau)R_1 - \cos(\alpha-\tau)R_{-1}\right) v_{n\theta}^4 = \left(\sin(\alpha+\tau)R_1 - \sin(\alpha-\tau)R_{-1}\right) v^4.$$

Since

$$|\cos(\alpha+\tau)| \neq |\cos(\alpha-\tau)|, \tag{5.15}$$

then the sequence $\{v_{n\theta}^s\}$ converges in $L_2(Q)$ $(s=3,4)$ by Lemma 5.3. Therefore from (5.14) and equalities $v|_{x=0} = v|_{x=a} = 0$ $(s=3,4)$ it follows that $\lim\limits_{n\to\infty} \|v_n^s\|_{H^1_{0,2\pi}(Q)} = 0$ $(s=3,4)$.

Thus $\lim\limits_{n\to\infty} \|v_n\|_{H^{1,4}_{0,2\pi}(Q)} = 0$. However this contradicts to our assumption that $\|v_n\|_{H^1_{0,2\pi}(Q)} = 1$. It remains to note that since $0<\alpha<\pi/2$ and $0<\tau<\pi/2$, inequality (5.15) holds. □

6. Generalized solutions to boundary value problem

In order to explain, in which sense a vector-valued function $u(x,\theta)$ satisfies the boundary value problem (4.3)–(4.6), (3.8) and (3.9), we give some definitions. Clearly, $C^\infty_{0,2\pi}(\tilde{Q})$ is dense in $H^1_{0,2\pi}(Q)$. We identify the points (x_0,θ_0), $(x_0,\theta_1) \in \tilde{Q}$, if $|\theta_0-\theta_1| = 0$ or $|\theta_0-\theta_1| = 2\pi$. Let us define a neighborhood of a point $(x_0,\theta_0) \in \tilde{Q}$ as intersection of $\tilde{Q}$ with an open set in $\mathbb{R}^2$, containing a point (x_0,θ_0) and a point, with which (x_0,θ_0) is identified. Thus $\tilde{Q}$ is topologically equivalent to a direct product of the interval $(0,a)$ and the unit circle. In that sense we understand the term "open set", "closed set", and "compact set". We write that $\varphi_k \rightrightarrows \varphi$ in $C^\infty_{0,2\pi}(\tilde{Q})$ if $\varphi_k, \varphi \in C^\infty_{0,2\pi}(\tilde{Q})$, supp$\varphi_k$, supp$\varphi$ belong to the same compact set in $\tilde{Q}$, and $\sup\limits_{(x,\theta)\in\tilde{Q}} |D^\alpha(\varphi_k(x,\theta) - \varphi(x,\theta))| \to 0$ as $k\to\infty$ for any $\alpha = (\alpha_1,\alpha_2)$.

Definition 6.1. *Let $D_{0,2\pi}(\tilde{Q})$ be a space of all linear functionals F on $C^\infty_{0,2\pi}(\tilde{Q})$ such that $F(\varphi_k) \to F(\varphi)$ if $\varphi_k \rightrightarrows \varphi$ in $C^\infty_{0,2\pi}(\tilde{Q})$. An element F in the space $D_{0,2\pi}(\tilde{Q})$ is called a distribution on $\tilde{Q}$.*

A distribution F given by fornula

$$F(\varphi) = \int\limits_Q f(x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\theta \qquad (\varphi \in C^\infty_{0,2\pi}(\tilde{Q}))$$

is called the distribution associated with the function $f \in L_2(Q)$. We shall identify such distribution with the function f.

Definition 6.2. A function $D^\alpha F$ is called a generalized derivative of distribution $F \in D_{0,2\pi}(\tilde{Q})$ if for all $\varphi \in C^\infty_{0,2\pi}(\tilde{Q})$ we have

$$D^\alpha F(\varphi) = (-1)^{|\alpha|} F(D^\alpha \varphi).$$

Clearly, $D^\alpha F \in D_{0,2\pi}(\tilde{Q})$.

In Section 4, we have proved that if a vector-valued function $u \in H^{1,4}_{0,2\pi}(Q)$ yields a minimum of functional (3.3) with the boundary conditions (3.8), (3.9), then u satisfies integral identity (4.2) for all $v \in H^{1,4}_{0,2\pi}(Q)$. This means that the function u satisfies system (4.3)–(4.6) in the space $D_{0,2\pi}(\tilde{Q})$.

We can rewrite the system of differential-difference equations (4.3)–(4.6) with boundary conditions (3.8), (3.9) in the form

$$\mathcal{L}u = f, \tag{6.1}$$

where $f = (f^1, f^2, f^3, f^4)^T$, the unbounded operator $\mathcal{L} : L_2^4(Q) \supset D(\mathcal{L}) \to L_2^4(Q)$ with domain $D(\mathcal{L}) = \left\{u \in H^{1,4}_{0,2\pi}(Q) : \mathcal{L}u \in L_2^4(Q)\right\}$ acts in the space of distributions $D_{0,2\pi}(\tilde{Q})$ as following. First, we apply difference operators $R_i\,(i = \pm1, \pm2)$ to the functions $u^j \in H^1_{0,2\pi}(Q)\,(j = 1, \dots, 4)$. By virtue of Lemma 5.2, $R_i u^j \in H^1_{0,2\pi}(Q)$. Then we apply the differential operators $D^\alpha\,(|\alpha| \le 2)$ to the functions $R_i u^j$ in the sense of distributions.

Definition 6.3. A vector-valued function $u \in D(\mathcal{L})$ is called a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9) if it satisfies operator equation (6.1).

The following definition is equivalent to definition 6.3.

Definition 6.4. A vector-valued function $u \in H^{1,4}_{0,2\pi}(Q)$ is said to be a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9) if it satisfies the integral identity (4.2) for all $v \in H^{1,4}_{0,2\pi}(Q)$.

Denote by $B(u, v)$ the left hand side of integral identity (4.2).

Theorem 6.5. *A vector-valued function $u \in H^{1,4}_{0,2\pi}(Q)$ gives a minimum to functional (3.3) with the boundary conditions (3.8) and (3.9) if and only if it is a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9).*

Proof. The necessity was proved in Section 4. Let us prove the sufficiency. Let $u \in H^{1,4}_{0,2\pi}(Q)$ be a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9). The bilinear form $B(u, v)$ in $H^{1,4}_{0,2\pi}(Q)$ is the Gâteaux differential of the functional $J(u)$. Therefore, by virtue of Lemma 5.4, for any $v \in H^{1,4}_{0,2\pi}(Q)$ we have $E(u + v) = E(u) + B(u, v) - \sum\limits_{i=1}^{4} \int\limits_0^a \int\limits_0^{2\pi} f^i v^i \,\mathrm{d}x\,\mathrm{d}\theta + J(v) \ge E(u)$. □

Lemma 6.6. *In the space $H^{1,4}_{0,2\pi}(Q)$ one can define an equivalent inner product by the formula*

$$(u, v)'_{H^{1,4}_{0,2\pi}(Q)} = B(u, v), \tag{6.2}$$

i.e. $B(u, v)$ is an inner product in $H^{1,4}_{0,2\pi}(Q)$ such that for any $u \in H^{1,4}_{0,2\pi}(Q)$

$$c_1\|u\|^2_{H^{1,4}_{0,2\pi}(Q)} \le B(u, u) \le c_2\|u\|^2_{H^{1,4}_{0,2\pi}(Q)},$$

where $c_1, c_2 > 0$ do not depend on u.

Proof. Since $C^{\infty,4}_{0,2\pi}(\tilde{Q})$ is dense in $H^{1,4}_{0,2\pi}(Q)$, it is sufficient to show that for all $u, v \in C^{\infty,4}_{0,2\pi}(\tilde{Q})$

$$B(u, v) = B(v, u), \tag{6.3}$$

$$c_1\|u\|^2_{H^{1,4}_{0,2\pi}(Q)} \le B(u, v) \le c_2\|u\|^2_{H^{1,4}_{0,2\pi}(Q)}, \tag{6.4}$$

where $c_1, c_2 > 0$ do not depend on u.

Integrating by parts, we have

$$B(u, v) = (\mathcal{L}u, v)_{L_2^4(Q)}, \qquad B(v, u) = (u, \mathcal{L}v)_{L_2^4(Q)}. \tag{6.5}$$

On the other hand, by virtue of Lemma 5.1, we obtain

$$(\mathcal{L}u, v)_{L_2^4(Q)} = (u, \mathcal{L}v)_{L_2^4(Q)} \tag{6.6}$$

for any $u, v \in C_{0,2\pi}^{\infty,4}(\tilde{Q})$. From (6.5) and (6.6) we derive (6.3).

The right hand side of (6.4) follows from the Schwarz inequality and Lemma 5.2. Since $J(u)$ is a quadratic functional, for any $u, v \in H_{0,2\pi}^{1,4}(Q)$ we have $J(u+v) = J(u) + B(u,v) + J(v)$, *i.e.* $4J(u) = J(2u) = 2J(u) + B(u,u)$. The equality $B(u,u) = 2J(u)$ and Lemma 5.4 imply a correctness of the left hand side of (6.4). □

Theorem 6.7. *For every $f \in L_2^4(Q)$, the boundary value problem (4.3)–(4.6), (3.8) and (3.9) has a unique generalized solution $u \in D(\mathcal{L})$. Moreover,*

$$\|u\|^2_{H_{0,2\pi}^{1,4}(Q)} \le c_0 \|f\|_{L_2^4(Q)}, \tag{6.7}$$

where $c_0 > 0$ does not depend on f.

Proof. The integral identity (4.2) has the form

$$B(u,v) = (f,v)_{L_2^4(Q)} \tag{6.8}$$

for any $v \in H_{0,2\pi}^{1,4}(Q)$.

By virtue of the Riesz theorem on a general form of a linear functional in a Hilbert space and Lemma 6.6, there exists a linear bounded operator $G : L_2^4(Q) \to H_{0,2\pi}^{1,4}(Q)$ such that

$$(f,v)_{L_2^4(Q)} = (Gf, v)'_{H_{0,2\pi}^{1,4}(Q)}$$

for any $v \in H_{0,2\pi}^{1,4}(Q)$.

Therefore identity (6.8) takes the form

$$(u,v)'_{H_{0,2\pi}^{1,4}(Q)} = (Gf, v)'_{H_{0,2\pi}^{1,4}(Q)}. \tag{6.9}$$

Hence, for any $f \in L_2^4(Q)$, there exists a unique function $u = Gf \in H_{0,2\pi}^{1,4}(Q)$ satisfying (6.8) for any $v \in H_{0,2\pi}^{1,4}(Q)$. Moreover,

$$\|u\|'_{H_{0,2\pi}^{1,4}(Q)} \le \|G\| \cdot \|f\|_{L_2^4(Q)}.$$

□

We consider an application of variational methods to finding an approximate solution to the boundary value problem (4.3)–(4.6), (3.8) and (3.9).

By virtue of (3.3)–(3.7), the equality $J(v) = \dfrac{1}{2}B(v,v)$, and Lemma 6.6, for any $v \in H_{0,2\pi}^{1,4}(Q)$ and $f \in L_2^4(Q)$ we have

$$E(v) = J(v) - (f,v)_{L_2^4(Q)} = \frac{1}{2}\left((v,v)'_{H_{0,2\pi}^{1,4}(Q)} - 2(f,v)_{L_2^4(Q)}\right). \tag{6.10}$$

From (6.10) it follows that $d = \inf\limits_{v \in H_{0,2\pi}^{1,4}(Q)} E(v) > -\infty$. A sequence $\{v_m\} \subset H_{0,2\pi}^{1,4}(Q)\,(m = 1,2,\dots)$ is called minimizing sequence for the functional $E(\cdot)$ on $H_{0,2\pi}^{1,4}(Q)$ if $\lim\limits_{m\to\infty} E(v_m) = d$. Let $\{e_k\}\,(k = 1,2,\dots)$ be a linearly independent system in $H^{1,4}_{0,2\pi}(Q)$ such that its linear span is dense in $H^{1,4}_{0,2\pi}(Q)$. Denote by $w_k \in H^{1,4}_{0,2\pi}(Q)$ an element giving a minimum to the functional $E(\cdot)$ on a linear manifold spanned upon the elements $e_1, \dots, e_k$. It is known that there exists a unique element $w_k = c_{k1}e_1 + \dots + c_{kk}e_k$. A sequence $\{w_k\}$ $(k = 1, 2, \dots)$ is called a Ritz sequence for the functional $E(\cdot)$ over the system $e_1, e_2, \dots$.

Theorem 6.8. *A Ritz sequence for a functional $E(\cdot)$, constructed over an arbitrary linearly independent system of vector-valued functions having a linear span that is dense in $H^{1,4}_{0,2\pi}(Q)$, converges in $H^{1,4}_{0,2\pi}(Q)$ to a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9).*

A proof is based on representation (6.10), see [3], Chapter I, Section 8.

We note that Lemmas 5.1–5.4, 6.6 and Theorem 6.7 remain true in the case of complex spaces. We only need to do the following. In Lemma 6.6 instead of bilinear form $B(u, v)$ we must use the left part of identity (4.2), in which we substitute complex conjugate functions $\overline{v^1}, \dots, \overline{v^4}$ instead of functions $v^1, \dots, v^4$. In formulas (5.4)–(5.8) and (3.6), (3.7) defining the functional $J(u)$ instead of squares of corresponding functions we have to write the squares of their modulas. Further we consider only complex spaces.

Theorem 6.9. *The operator $\mathcal{L} : L^4_2(Q) \supset D(\mathcal{L}) \to L^4_2(Q)$ is self-adjoint, the spectrum $\sigma(\mathcal{L})$ consists of real isolated eigenvalues $0 < \lambda_s$ on finite multiplicity. There exists an orthonormal basis in $L^4_2(Q)$ consisting of eigenfunctions v_s of the operator $\mathcal{L}$. Moreover, the functions $v_s/\sqrt{\lambda_s}$ form an orthonormal basis in $H^{1,4}_{0,2\pi}(Q)$ with inner product given by (6.2).*

A proof is similar to the proof of Theorem 10.2 in [6], Chapter 2, Section 10.

7. Smoothness of generalized solutions

In [4–6], it were considered different boundary value problems for a system of differential-difference equations corresponding to elastic equilibrium of a three-layer plate with a goffered filler. It was proved that smoothness of generalized solutions can be violated in the whole of the domain and preserved in some subdomains. This phenomenon is connected with properties of differential-difference operators generated by slanting ribs and absolutely rigid fixing. However in the case of three-layer shell, considered in this paper, generalized solutions $u \in H^{1,4}_{0,2\pi}(Q)$ have corresponding smoothness in the whole of the domain up to the boundary, *i.e.* $u \in H^{k+2,4}_{0,2\pi}(Q)$ if $f \in H^{k,4}_{0,2\pi}(Q)$.

Theorem 7.1. *Let $f \in H^{k,4}_{0,2\pi}(Q)$, and let $u \in H^{1,4}_{0,2\pi}(Q)$ be a generalized solution of the boundary value problem (4.3)–(4.6), (3.8) and (3.9).*

Then $u \in H^{k+2,4}(Q)$.

Proof. 1. Let $Q_l = (0, a) \times ((l-1)\,\tau, l\tau)$ and $Q_{l\varepsilon} = (0, a) \times (\varepsilon + (l-1)\,\tau, l\tau - \varepsilon)$, where $l = 1, \dots, 2N$, $0 < \varepsilon < \tau/4$, the numbers N and τ were defined in the beginning of Sections 2 and 3, respectively. For any $u \in L^4_2(Q)$, we introduce a vector-valued function $Uu \in L^{8N}_2(Q_1)$ by the formula

$$(Uu)_j(x,\theta) = \begin{cases} u^1(x, \theta + (j-1)\tau), & j = 1, \dots, 2N; \\ u^2(x, \theta + (j-2N-1)\tau), & j = 2N+1, \dots, 4N; \\ u^3(x, \theta + (j-4N-1)\tau), & j = 4N+1, \dots, 6N; \\ u^4(x, \theta + (j-6N-1)\tau), & j = 6N+1, \dots, 8N \end{cases} \tag{7.1}$$

for $(x, \theta) \in Q_1$.

Similarly to the proof of Theorem 11.2 in [6], Chapter II, Section 11 one can prove that the vector-valued function $Uu(x, \theta)$ is a generalized solution to some system of $8N$ partial differential equations with constant coefficients (without shifts of arguments) and the right hand side $Uf \in H^{k,8N}(Q_1)$. By virtue of Lemma 5.4 and Theorem 1B in [8], §2, this system of differential equations is strongly elliptic. Moreover, $(Uu)|_{x=0} =$

$(Uu)|_{x=a} = 0$. Therefore, from a theorem on smoothness of generalized solutions to strongly elliptic systems of partial differential equations near the smooth part of the boundary (see [2]) it follows that $Uu \in H^{k+2,8N}(Q_{1\varepsilon})$. Hence

$$u \in H^{k+2,4}(Q_{l\varepsilon}) \qquad (l = 1, \dots, 2N). \tag{7.2}$$

2. We extend the vector-valued functions $f \in H^{k,4}_{0,2\pi}(Q)$ and $u \in H^{1,4}_{0,2\pi}(Q)$ as 2π-periodic functions with respect to θ to the whole strip $(0,a) \times (-\infty, \infty)$. We preserve the same notations f and u for these extensions.

Let

$$\begin{aligned} Q^{\pm} &= (0,a) \times (\pm\tau/2, \pm\tau/2 + 2\pi), \\ Q_l^{\pm} &= (0,a) \times (\pm\tau/2 + (l-1)\tau, \pm\tau/2 + l\tau), \\ Q_{l\varepsilon}^{\pm} &= (0,a) \times (\varepsilon \pm \tau/2 + (l-1)\tau, -\varepsilon \pm \tau/2 + l\tau), \end{aligned}$$

where $l = 1, \dots, 2N$, $0 < \varepsilon < \tau/4$.

Clearly,

$$Q = \bigcup_{l=1}^{2N} \left(Q_{l\varepsilon} \cup \left(Q_{l\varepsilon}^{+} \cap Q \right) \cup \left(Q_{l\varepsilon}^{-} \cap Q \right) \right). \tag{7.3}$$

It is easy to see that the vector-valued function $u \in H^{1,4}_{0,2\pi}(Q^{\pm})$ satisfies the integral identity (4.2) in the domain $Q^{\pm}$ instead of Q and the boundary conditions

$$u(x,\theta) = 0 \qquad (x = 0, a;\ \pm\tau/2 \le \theta \le \pm\tau/2 + 2\pi).$$

Repeating the arguments from the first part of the proof, we have

$$u \in H^{k+2,4}(Q_{l\varepsilon}^{\pm}) \qquad (l = 1, \dots, 2N). \tag{7.4}$$

From (7.2) to (7.4) it follows that $u \in H^{k+2,4}(Q)$. □

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