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STATIONARY SOLUTIONS OF THE VLASOV–POISSON SYSTEM FOR TWO-COMPONENT PLASMA UNDER AN EXTERNAL MAGNETIC FIELD IN A HALF-SPACE

Yu. Belyaeva[a]

Abstract. We consider the first mixed problem for the Vlasov–Poisson equations with an external magnetic field in a half-space. This problem describes the evolution of the density distributions of ions and electrons in a high temperature plasma with a fixed potential of electric field on a boundary. For a sufficiently large induction of external magnetic field there are constructed stationary solutions supported on some distance from the hyperplane $x_1 = 0$.

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1. Introduction

In recent years, considerable attention has been given to the investigation of the Vlasov equations. It can be explained by a huge applications of Vlasov equations to modern natural sciences and technique. These equations are connected with the thermonuclear fusion, modeling and studying of high-temperature rarefied plasma, radiation, neutral particles transportation, atmosphere optics, etc. Depending on the initial physical models, one distinguishes the Vlasov–Poisson equations, the Vlasov–Maxwell equations, the Vlasov–Einstein equations, the generalized Vlasov equations, and so on. There is an extensive literature on the Vlasov equations (see [1–14] and the references given there).

Mixed problem for the Vlasov system in a bounded domain describes a mathematical model of kinetics of high-temperature plasma. To provide plasma confinement in thermonuclear fusion reactor it is used an external magnetic field. From mathematical point of view this means that the external magnetic field ensures the existence of solutions to Vlasov–Poisson equations for which the supports of the charged-particle density distributions do not intersect the boundary.

We consider Vlasov–Poisson system in a half-space, which describes the evolution of the density distributions of ions and electrons in a high temperature plasma with a fixed potential of electric field on a boundary:

$$-\Delta\varphi(x,t) = 4\pi e \int_{\mathbb{R}^3} \sum_\beta \beta f^\beta(x,p,t)\,\mathrm{d}p \quad (x \in R_+,\ 0 < t < T), \tag{1.1}$$

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RUDN University, 6 Miklukho-Maklaya St., 117198 Moscow, Russia. yilia-b@yandex.ru

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$$\frac{\partial f^\beta}{\partial t} + \frac{1}{m_\beta}(p, \nabla_x f^\beta) + \beta e \left(-\nabla_x \varphi + \frac{1}{m_\beta c}\,[p, B], \nabla_p f^\beta \right) = 0$$
$$(x \in R_+,\ p \in \mathbb{R}^3,\ 0 < t < T,\ \beta = \pm 1), \tag{1.2}$$

with the initial conditions

$$f^\beta(x,p,t)\big|_{t=0} = f_0^\beta(x,p) \quad (x \in R_+,\ p \in \mathbb{R}^3,\ \beta = \pm 1), \tag{1.3}$$

and the Dirichlet boundary condition

$$\varphi(x,t)|_{x_1=0} = 0 \quad (x \in R_+,\ 0 \le t < T). \tag{1.4}$$

Here $\varphi = \varphi(x,t)$ is the potential of the self-consistent electric field, $f^\beta = f^\beta(x,p)$ is the density distribution function of positively charged ions (for $\beta = +1$), or of electrons (for $\beta = -1$), at a point x with the impulse p at the moment t ; ∇_x and ∇_p are, respectively, the gradients with respect to x and p; m_{+1} and m_{-1} are the ion and electron masses; e is the electron charge; c is the velocity of light; B is the external magnetic field induction; $(\,\cdot,\cdot\,)$ is the inner product in $\mathbb{R}^3$; $[\,\cdot,\cdot\,]$ is the vector product in $\mathbb{R}^3$ and $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3 \in \mathbb{R}^3) : x_1 > 0\}$.

Stationary solutions of the Vlasov–Poisson equations play a significant role. They describe possible equilibrium positions. Boundary value problems for a stationary Vlasov equations were studied by Vedenyapin in [12, 13]. The existence of stationary solutions of the relativistic Vlasov–Maxwell system was proved by Batt and Fabian in [2]. The initial value problem for the Vlasov–Poisson system of stellar dynamics was investigated for the case of stationary, spherically symmetric solutions by Batt *et al.* in [3].

Stationary solutions in a cylindrical domain were studied in [5]. Mixed problem for the Vlasov–Poisson equations for high-temperature two-component plasma in a cylindrical domain and in a half-space were studied in [6–11]. Note that in [6–11] the influence of an external magnetic field on the trajectories of particles was considered. In [4, 8] there was constructed a stationary solution for which the charged-particle density distributions were supported in a strictly interior cylinder. It was shown that a classical solution for which the supports of the charged-particle density distributions are at a distance from the cylindrical boundary exists and is unique in some neighbourhood of the stationary solution [8].

In [9] it was proved the existence and uniqueness of classical solution for the Vlasov–Poisson equations with nonlocal conditions in an infinite cylinder for sufficiently small initial data.

In the case of a half-space and for sufficiently small initial densities with compact supports and large strength of an external magnetic field, it was proved the existence and uniqueness of classical solutions for initial-boundary value problems with different boundary conditions for the electric potential: the Dirichlet conditions, the Neumann conditions, and nonlocal conditions [6, 7].

The paper is organized as follows. In Section 2 we introduce the notation, pose the problem, and formulate the main results. Section 3 is devoted to the construction of the stationary solution of the Vlasov–Poisson system with trivial potential of electric field which supported on some distance from the boundary. In this section we construct stationary solutions as product of two cut-off functions which arguments are characteristics of Vlasov equations as a linear form and it functional combination. Section 4 is devoted to the construction of the stationary solution of the Vlasov–Poisson system with trivial potential of electric field which supported on some distance from the hyperplane $x_1 = 0$ and depends on variable x_3.

We would like to emphasize that solutions constructed in [4, 8] and in Section 3 of this paper do not depend on x_3. This fact can be interpreted in such a way that the plasma has an infinite mass. But unlike the above solutions, one constructed in Section 4 depends on x_3 which means that a mass is finite.

2. Statement of the problem. Main result

We consider the stationary Vlasov–Poisson system in a half-space:

$$-\Delta\varphi(x) = 4\pi e \int_{\mathbb{R}^3} \sum_{\beta} \beta f^{\beta}(x,p)\,\mathrm{d}p, \quad x \in \mathbb{R}^3_+, \tag{2.1}$$

$$\frac{1}{m_\beta}(p, \nabla_x f^\beta) + \beta e \left(-\nabla_x \varphi + \frac{1}{m_\beta c}[p, B(x)], \nabla_p f^\beta\right) = 0,$$
$$x \in \mathbb{R}^3_+,\ p \in \mathbb{R}^3,\ \beta = \pm 1, \tag{2.2}$$

with the Dirichlet boundary condition

$$\varphi(x)|_{x_1=0} = 0, \quad x' = (x_2, x_3) \in \mathbb{R}^2. \tag{2.3}$$

Here $\varphi = \varphi(x,t)$ is the potential of the self-consistent electric field, $f^\beta = f^\beta(x,p)$ is the density distribution function of positively charged ions (for $\beta = +1$), or of electrons (for $\beta = -1$), at a point x with the impulse p; ∇_x and ∇_p are, respectively, the gradients with respect to x and p; m_{+1} and m_{-1} are the ion and electron masses; e is the electron charge; c is the velocity of light; B is the external magnetic field induction; $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^3$; $[\cdot,\cdot]$ is the vector product in $\mathbb{R}^3$ and $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3 \in \mathbb{R}^3) : x_1 > 0\}$.

Let $B_\rho(x_0) = \{x \in \mathbb{R}^3 \colon |x - x_0| < \rho\}$, and let $B_\rho = B_\rho(0)$. We denote $\mathbb{R}^3_\delta = \{x = (x_1, x_2, x_3 \in \mathbb{R}^3) \colon x_1 > \delta\}$. We now introduce some function spaces. Denote by $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}}^n_+))$, $s \geq 0$, $n \in \mathbb{N}$ the Hölder space of continuous functions on $\mathbb{R}^n$ $(\overline{\mathbb{R}}^n_+)$ that have continuous derivatives in $\mathbb{R}^n$ $(\overline{\mathbb{R}}^n_+)$ up to order k, $k = [s]$, equipped with the finite norm

$$\|u\|_s = \max_{|\alpha| \leq k} \sup_x |\mathcal{D}^\alpha u(x)|,$$

for $s = k \in \mathbb{Z}$, $0 \leq k$,

$$\|u\|_s = \|u\|_k + \max_{|\alpha| = k} \sup_{x \neq y} |x - y|^{-\sigma} |\mathcal{D}^\alpha u(x) - \mathcal{D}^\alpha u(y)|,$$

for $s = k + \sigma$, $0 \leq k \in \mathbb{Z}$, $0 < \sigma < 1$, $\mathcal{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For any $s \geq 0$ the spaces $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}}^n_+))$ are Banach spaces. If $s = k$, $0 \leq k \in \mathbb{Z}$ then the space $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}}^n_+))$ is separable. If $s = k + \sigma$, $0 \leq k \in \mathbb{Z}$, $0 < \sigma < 1$ then the space $C^s(\mathbb{R}^n)$ $(C^s(\overline{\mathbb{R}}^n_+))$ is not separable.

Let $C^s_0(\overline{\mathbb{R}}^n_+)$, $s \geq 0$ denote the closure of the set of functions in $C^s(\overline{\mathbb{R}}^n_+)$ with compact support in $\overline{\mathbb{R}}^n_+$. In the study of the Vlasov equations an important role play stationary solutions.

Definition 2.1. A vector function $\{\mathring{\varphi}, \mathring{f}^\beta\}$, $\mathring{\varphi} \in C^{2+\sigma}_0(\overline{\mathbb{R}}^n_+)$, $\mathring{f}^\beta \in C^1(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^3)$ is called a stationary solution of equations (2.1) and (2.2) with the boundary condition (2.3) if $\{\mathring{\varphi}, \mathring{f}^\beta\}$ satisfies the equations

$$-\Delta\mathring{\varphi}(x) = 4\pi e \int_{\mathbb{R}^3} \sum_\beta \beta \mathring{f}^\beta(x,p)\,\mathrm{d}p \qquad (x \in \mathbb{R}^n_+), \tag{2.4}$$

$$(v, \nabla_x \mathring{f}^\beta) + \beta e \left(-\nabla_x \mathring{\varphi} + \frac{1}{m_\beta c}[v, B], \nabla_v \mathring{f}^\beta\right) = 0 \quad (x \in \mathbb{R}^n_+,\ p \in \mathbb{R}^3,\ \beta = \pm 1), \tag{2.5}$$

and the boundary condition (2.3).

We now formulate the conditions which the magnetic field B must satisfy.

Condition 2.2. *Let $B=(0,0,h)$ for $x\in\mathbb{R}_+$, where $h>0$ is independent of x and*

$$\frac{c\rho}{e\delta}<h. \tag{2.6}$$

This paper is devoted to the construction of stationary solutions of the Vlasov–Poisson system in a half-space. The following results will be proved in Sects. 3 and 4, respectively:

Theorem 2.3. *Let $\delta>0$ be such that $x_1>\delta$. Assume that Condition 2.2 is satisfied for such δ and some $h,\rho>0$. Then for any $\alpha>0$ there are stationary solutions $\{0,\mathring{f}^\beta\}$ of equations (2.1) and (2.2) with the boundary condition (2.3) such that:*

$$\mathring{f}^\beta(x,p)=\psi_1^\beta(|p|^2)\psi_2^\beta\left(\frac{eh}{c}x_1+\beta p_2\right), \tag{2.7}$$

$$\mathring{f}^\beta(x,p)=\psi_1^\beta(|p|^2)\psi_3^\beta\left(\frac{1}{x_1+\frac{c}{\beta eh}p_2}\right), \tag{2.8}$$

$$\mathring{f}^\beta(x,p)=\psi_1^\beta(|p|^2)\psi_4^\beta\left(\frac{1}{(x_1+\frac{c}{\beta eh}p_2)^m}\right), \tag{2.9}$$

with the following properties: $\mathring{f}^\beta\in C^\infty(R_\delta\times\mathbb{R}^3)$, $\operatorname{supp}\mathring{f}^\beta\subset R_\delta\times B_{\rho/4}$ and $\sup\limits_{x,p}\mathring{f}^\beta(x,p)>\alpha$. Here $\psi_i^\beta(\cdot)$ $(i=1,2,3,4)$ are non-negative functions satisfying the conditions: $\psi_i^\beta\in\dot{C}^\infty(\mathbb{R})$ $(i=1,2,3,4)$ such that $\psi_i^{-1}(0)=2\alpha>0$, $\psi_i^{-1}(0)=1$, $\psi_i^\beta(\tau)\geq 0$, $\psi_i^{+1}(\tau)=\psi_i^{-1}(\tau)$ $(\tau\in\mathbb{R})$, $\operatorname{supp}\psi_1^\beta\subset(-\rho_1^2/16,\rho_1^2/16)$, where $0<\rho_1<\rho$ and $\psi_i^{+1}(0)>2\alpha$, $\operatorname{supp}\psi_2^{-1}\subset\big(0,\frac{4\delta_0-\delta}{4\delta}\big)$, where $4\delta_0>\delta$, $\operatorname{supp}\psi_3^{-1}\subset\big(0,\frac{4}{5\delta}\big)$, $\operatorname{supp}\psi_4^{-1}\subset\big(0,\big(\frac{4}{5\delta}\big)^m\big)$.

Theorem 2.4. *Let $\delta>0$ be such that $x_1>\delta$. Assume that Condition 2.2 is satisfied for such δ and some $h,\rho>0$. Then for any $\alpha>0$ there is a stationary solution $\{0,\mathring{f}^\beta\}$ of equations (2.1) and (2.2) with the boundary condition (2.3) such that:*

$$\mathring{f}^\beta(x,p)=\Psi_1^\beta\big(p_1^2+p_2^2\big)\,\Psi_2^\beta\big(p_3^2\big)\,\Psi_3^\beta\left(\frac{\beta eh}{c}x_1+p_2\right)\Psi_4^\beta\left(\frac{\beta eh}{c}x_2-p_2\right)\Psi_5^\beta\left(\frac{eh}{c}x_3-\beta p_3\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right),$$

with the following properties: $\mathring{f}^\beta\in C^\infty(R_\delta\times\mathbb{R}^3)$, $\operatorname{supp}\mathring{f}^\beta\subset(\mathbb{R}_{\delta_1}\cap\{x\in\mathbb{R}\colon|x_2|\leq\delta_2,|x_3|\leq\delta_3\})\times B_{\rho/4}$ and $\sup\limits_{x,p}\mathring{f}^\beta(x,p)>\alpha$. Here $\Psi_1^\beta,\Psi_2^\beta,\Psi_3^\beta,\Psi_4^\beta,\Psi_5^\beta\in\dot{C}^\infty(\mathbb{R})$ are non-negative functions such that $\int\limits_{\mathbb{R}^3}\Psi_i^{+1}(\tau)\mathrm{d}\tau=\int\limits_{\mathbb{R}^3}\Psi_i^{-1}(\tau)\mathrm{d}\tau$ $(i=1,2,3,4,5)$, $\operatorname{supp}\Psi_1^\beta\subset\Big(-\frac{\rho_1^2}{16},\frac{\rho_1^2}{16}\Big)$, $\operatorname{supp}\Psi_2^\beta\subset\Big(-\frac{\rho_1^2}{16},\frac{\rho_1^2}{16}\Big)$, $\operatorname{supp}\Psi_3^\beta\subset\Big(\delta,\frac{\rho(4\delta_0-\delta)}{4\delta}\Big)$, $\operatorname{supp}\Psi_4^\beta\subset\Big(\delta,\frac{\rho(4\delta_2-\delta)}{4\delta}\Big)$, $\operatorname{supp}\Psi_5^\beta\subset\Big(\delta,\frac{\rho(8\delta_3-\pi\delta)}{8\delta}\Big)$, where $\frac{\delta}{4}<\delta_1<\delta$, $\frac{\delta}{2}<\delta_2<\delta$, $\frac{\delta\pi}{8}<\delta_3<\delta$ and $\Psi_1^\beta(0)=0$, $\Psi_i^{-1}(0)=2\alpha>0$, $\Psi_i^{-1}(0)=1$ and $\Psi_i^{+1}(0)>2\alpha$ $(i=1,2,3,4)$.

3. Construction of stationary solution as a linear form

1) We now construct a stationary solution (2.7) of equations (2.4) and (2.5) such that:

$$\mathring{f}^\beta\in C^\infty(\mathbb{R}^3_\delta\times\mathbb{R}^3),\quad \operatorname{supp}\mathring{f}^\beta\subset\mathbb{R}^3_\delta\times B_{\rho/4}\text{ and }\sup_{x,p}\mathring{f}^\beta(x,p)>\alpha.$$

Let $\mathring{\varphi}(x) \equiv 0$ ($x \in \mathbb{R}^3_\delta$). Then the system (2.5) assumes the form

$$\Big(p, \nabla_x \mathring{f}^\beta\Big) + \frac{\beta e}{c}\Big([p, B], \nabla_p \mathring{f}^\beta\Big) = 0 \quad (\mathbb{R}^3_\delta,\ p \in \mathbb{R}^3,\ \beta = \pm 1). \tag{3.1}$$

a. We shall find a solution of equation (2.1) and (2.2) as a product of two cut-off functions whose arguments are particular solutions of the Vlasov system. Different particular solutions of equation (3.1) will be denoted by $\mathring{f}^\beta_i$ $(i = 0, 2, \dots, 7)$.

Clearly, the function $\mathring{f}^\beta_0(x, v) = |p|^2$ is a solution of (3.1) for any $x \in \mathbb{R}^3_\delta$, $p \in \mathbb{R}^3$ and $\beta = \pm 1$. We consider non-negative functions $\psi^\beta_1 \in \dot{C}^\infty(\mathbb{R})$ such that $\psi^{-1}_1(0) = 2\alpha > 0$, $\psi^\beta_1(\tau) \geq 0$, $\psi^{+1}_1(\tau) = \psi^{-1}_1(\tau)$ $(\tau \in \mathbb{R})$, $\operatorname{supp} \psi^\beta_1 \subset (-\rho_1^2/16, \rho_1^2/16)$, where $0 < \rho_1 < \rho$ and $\psi^{+1}_1(0) > 2\alpha$. The function $\mathring{f}^\beta_1(x, v) = \psi^\beta_1(|p|^2)$ is a solution of (3.1).

We now look for a solution of (3.1) as a following linear form with undetermined coefficients:

$$\mathring{f}^\beta_2(x, p) = a_1 x_1 + b_1 p_1 + b_2 p_2 + b_3 p_3. \tag{3.2}$$

Substituting (3.2) in (3.1) we get:

$$a_1 p_1 + \frac{\beta e h}{c}(b_1 p_2 - b_2 p_1) = 0.$$

According to the indefinite coefficients method we get equalities:

$$a_1 - \frac{\beta e h}{c} b_2, \quad b_1 = 0. \tag{3.3}$$

Let $b_3 = 0$ and $b_2 = \beta$, then $a_1 = \frac{eh}{c}$. Returning to the form of solution we construct it as follows:

$$\mathring{f}^\beta_2(x, p) = \frac{eh}{c} x_1 + \beta p_2.$$

We consider non-negative functions $\psi^\beta_2 \in \dot{C}^\infty(\mathbb{R})$ such that $\psi^{-1}_2(0) = 1$, $\psi^\beta_2(\tau) \geq 0$, $\operatorname{supp} \psi^\beta_1 \subset (-\rho_1^2/16, \rho_1^2/16)$; $(\tau \in \mathbb{R})$, $\operatorname{supp} \psi^{-1}_2 \subset \big(0, \rho\left(\frac{4\delta_0 - \delta}{4\delta}\right)\big)$ and $\psi^{+1}_2(0) > 1$. The function

$$\mathring{f}^\beta_3(x, p) = \psi^\beta_2\left(\frac{eh}{c} x_1 + \beta p_2\right),$$

is a solution of (3.1).

b. We prove that the vector function $\{0, \mathring{f}^\beta_1 \mathring{f}^\beta_3\}$ is a stationary solution of the problem (2.1)–(2.3) satisfying the assumptions of Theorem 2.3.

By construction, the function $\mathring{f}^\beta(x, v) = \mathring{f}^\beta_1(x, p) \times \mathring{f}^\beta_3(x, p)$ satisfies equation (3.1) and $\sup\limits_{x,p} \mathring{f}^\beta(x, p) \geq \mathring{f}^\beta(0, 0) \geq 2\alpha > 0$. It suffices to show that the right-hand side of (2.4) is identically zero and $\operatorname{supp} \mathring{f}^\beta \subset \mathbb{R}_\delta \times B_{\rho/4}$. Let us show that

$$\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x, p)\, \mathrm{d}p = \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x, p)\, \mathrm{d}p.$$

We make the change of variables $y = \frac{eh}{c}x$ and $w = (p_2, p_1, p_3)$. Then using the equalities $\psi_1^{+1}(\tau) = \psi_1^{-1}(\tau)$, $\psi_2^{+1}(\tau) = \psi_2^{-1}(\tau)$ and defining $x = \frac{c}{eh}y$ and $p = (w_2, -w_1, w_3)$, we find that

$$\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p = \int\limits_{\mathbb{R}^3} \psi_1^{+1}(|p|^2)\psi_2^{+1}\left(\frac{eh}{c}x_1 + p_2\right)\mathrm{d}p = \int\limits_{\mathbb{R}^3} \psi_1^{+1}\left(|w|^2\right)\psi_2^{+1}\left(y_1 + w_1\right)\mathrm{d}w$$
$$= \int\limits_{\mathbb{R}^3} \psi_1^{-1}\left(|w|^2\right)\psi_2^{-1}\left(y_1 + w_1\right)\mathrm{d}w = \int\limits_{\mathbb{R}^3} \psi_1^{-1}\left(|p|^2\right)\psi_2^{-1}\left(\frac{eh}{c}x_1 - p_2\right)dv = \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.$$

We now assert that $\operatorname{supp} \mathring{f}^{\beta} \subset R_\delta \times B_{\rho/4}$. Indeed, if $|p| > \rho_1/4$ then $f_1^{\beta}(x,p) = \psi_1^{\beta}(|p|^2) = 0$ by construction. Hence, $\mathring{f}^{\beta}(x,p) = 0$ for $|p| > \rho_1/4$. If $\delta_0 \le x_1 \le \delta$, where $\delta_0 < \delta/4$ and $|p| \le \rho_1/4$, then from Condition 2.2 it follows that:

$$\frac{eh}{c}x_1 + \beta p_2 \ge \frac{eh}{c}x_1 - \rho_1/4 \ge \frac{\rho\delta_0}{\delta} - \rho/4 \ge \rho\left(\frac{4\delta_0 - \delta}{4\delta}\right).$$

Consequently,

$$f_3^{\beta}(x,p) = \psi_2^{\beta}\left(\frac{eh}{c}x_1 + \beta p_2\right) = 0.$$

Hence, $\mathring{f}^{\beta}(x,p) = 0$ for $x < \delta,\ \ |p| \le \rho/4$.

2) We now construct a stationary solution (2.8) of equations (2.4) and (2.5) such that:

$$\mathring{f}^{\beta} \in C^{\infty}(\mathbb{R}^3_\delta \times \mathbb{R}^3), \quad \operatorname{supp} \mathring{f}^{\beta} \subset \mathbb{R}^3_\delta \times B_{\rho/4} \text{ and } \sup_{x,p} \mathring{f}^{\beta}(x,p) > \alpha.$$

a. We now look for a solution of (3.1) in the form:

$$\mathring{f}_4^{\beta}(x,p) = \frac{1}{a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3}. \tag{3.4}$$

Then, gradients with respect to x and p have a form:

$$\nabla_x \mathring{f}_4^{\beta}(x,p) = \left(\frac{-a_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}, 0, 0\right),$$
$$\nabla_p \mathring{f}_4^{\beta}(x,p) = \left(\frac{-b_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}, \frac{-b_2}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}, \frac{-b_3}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}\right).$$

Substituting (3.4) in (3.1) we get:

$$\frac{1}{m_\beta}\frac{a_1p_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}$$
$$+ \frac{\beta eh}{m_\beta c}\left(\frac{b_1p_2}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2} - \frac{b_2p_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^2}\right) = 0.$$

According to the undetermined coefficients method we get equalities:

$$\frac{1}{m_\beta}a_1 - \frac{\beta eh}{m_\beta c}b_2 = 0, \quad b_1 = 0.$$

We put $b_3 = 0$, $a_1 = 1$ then $b_2 = \frac{c}{\beta eh}$. Returning to the form of solution we get:

$$\mathring{f}_4^\beta(x,p) = \frac{1}{x_1 + \frac{c}{\beta eh}p_2}.$$

We consider non-negative functions $\psi_3^\beta \in \dot{C}^\infty(\mathbb{R})$ such that $\psi_3^{-1}(0) = 1$, $\psi_2^\beta(\tau) \geq 0$, $\psi_3^{+1}(\tau) = \psi_3^{-1}(\tau)$ $(\tau \in \mathbb{R})$, $\operatorname{supp}\psi_3^{-1} \subset (0, \frac{4}{5\delta})$ and $\psi_3^{+1}(0) > 1$. The function

$$\mathring{f}_5^\beta(x,p) = \psi_3^\beta\left(\frac{1}{x_1 + \frac{c}{\beta eh}p_2}\right),$$

is a solution of (3.1).

b. We prove that the vector function $\{0, \mathring{f}_1^\beta f_5^\beta\}$ is a stationary solution of the problem (2.1)–(2.3) satisfying the assumptions of Theorem 2.3.

By construction, the function $\mathring{f}^\beta(x,p) = \mathring{f}_1^\beta(x,p) \times \mathring{f}_5^\beta(x,p)$ satisfies equation (3.1) and $\sup_{x,p}\mathring{f}^\beta(x,p) \geq \mathring{f}^\beta(0,0) \geq 2\alpha > 0$. It suffices to show that the right-hand side of (2.4) is identically zero and $\operatorname{supp}\mathring{f}^\beta \subset R_\delta \times B_{\rho/4}$. Let us show that

$$\int_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p = \int_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.$$

We make the change of variables $y = x$ and $w = \dfrac{eh}{c}(p_2, p_1, p_3)$. Then using the equalities $\psi_1^{+1}(\tau) = \psi_1^{-1}(\tau)$, $\psi_3^{+1}(\tau) = \psi_2^{-1}(\tau)$, defining $x = y$ and $p = \frac{c}{eh}(w_2, -w_1, w_3)$, we find that

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p &= \int_{\mathbb{R}^3} \psi_1^{+1}(|p|^2)\psi_3^{+1}\left(\frac{1}{x_1 + \frac{c}{eh}p_2}\right)\mathrm{d}p \\
&= \int_{\mathbb{R}^3} \psi_1^{+1}\left(|w|^2\right)\psi_3^{+1}\left(\frac{1}{y_1 + w_1}\right)\mathrm{d}w = \int_{\mathbb{R}^3} \psi_1^{-1}\left(|w|^2\right)\psi_3^{-1}\left(\frac{1}{y_1 + w_1}\right)\mathrm{d}w \\
&= \int_{\mathbb{R}^3} \psi_1^{-1}\left(|p|^2\right)\psi_3^{-1}\left(\frac{1}{x_1 - \frac{c}{eh}p_2}\right)\mathrm{d}p = \int_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.
\end{aligned}$$

We now assert that $\operatorname{supp}\mathring{f}^\beta \subset R_\delta \times B_{\rho/4}$. Indeed, if $|p| > \rho_1/4$ then $f_1^\beta(x,p) = \psi_1^\beta(|p|^2) = 0$ by construction. Hence, $\mathring{f}^\beta(x,p) = 0$ for $|p| > \rho_1/4$. If $x_1 < \delta_0$ where $\delta_0 < \delta$ and $|p| \leq \rho_1/4$, where $\rho_1 < \rho$, then from Condition 2.2 and the inequalities $\delta_0 < \delta$ and $\frac{\rho_1}{\rho} < 1$ it follows that:

$$\frac{1}{x_1 + \frac{c}{\beta eh}p_2} > \frac{1}{\delta_0 + \frac{\delta\rho_1}{4\rho}} > \frac{1}{\delta_0 + \delta/4} > \frac{1}{\delta + \delta/4} = \frac{4}{5\delta}.$$

Consequently,

$$f_5^\beta(x,p) = \psi_3^\beta\left(\frac{1}{x_1 + \frac{c}{\beta eh}p_2}\right) = 0.$$

Hence, $\mathring{f}^\beta(x,p) = 0$ for $x_1 > \delta_0,\ |p| \le \rho/4$.

3) We now construct a stationary solution (2.9) of equations (2.4) and (2.5) such that:

$$\mathring{f}^\beta \in C^\infty(\mathbb{R}^3_\delta \times \mathbb{R}^3), \quad \operatorname{supp} \mathring{f}^\beta \subset \mathbb{R}^3_\delta \times B_{\rho/4} \text{ and } \sup_{x,p} \mathring{f}^\beta(x,p) > \alpha.$$

a. We now look for a solution of (3.1) as a form with undetermined coefficients:

$$\mathring{f}_6^\beta(x,p) = \frac{1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^m}, \tag{3.5}$$

where $m > 0$. Then gradients with respect to x and p have a form:

$$\nabla_x \mathring{f}_6^\beta(x,p) = \left(\frac{-ma_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}}, 0, 0\right),$$

$$\nabla_p \mathring{f}_6^\beta(x,p) = \left(\frac{-mb_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}}, \frac{-mb_2}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}}, \frac{-mb_3}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}}\right).$$

Substituting (3.4) in (3.1) we get:

$$\frac{1}{m_\beta}\frac{ma_1p_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}} + \frac{\beta eh}{m_\beta c}\left(\frac{mb_1p_2}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}} - \frac{mb_1p_1}{(a_1x_1 + b_1p_1 + b_2p_2 + b_3p_3)^{m+1}}\right) = 0.$$

According to the undetermined coefficients method we get equalities:

$$\frac{1}{m_\beta}a_1 - \frac{\beta eh}{m_\beta c}b_2 = 0, \quad b_1 = 0.$$

We put $b_3 = 0$, $a_1 = 1$ then $b_2 = \frac{c}{\beta eh}$. Returning to the form of solution we get:

$$\mathring{f}_6^\beta(x,p) = \frac{1}{\left(x_1 + \frac{c}{\beta eh}p_2\right)^m}.$$

We consider non-negative functions $\psi_4^\beta \in \dot{C}^\infty(\mathbb{R})$ such that $\psi_4^{-1}(0) = 1$, $\psi_4^\beta(\tau) \ge 0$, $\psi_4^{+1}(\tau) = \psi_4^{-1}(\tau)\,(\tau \in \mathbb{R})$, $\operatorname{supp}\psi_4^{-1} \subset (0, \left(\frac{4}{5\delta}\right)^m)$ and $\psi_4^{+1}(0) > 1$. The function

$$\mathring{f}_7^\beta(x,p) = \psi_4^\beta\left(\frac{1}{(x_1 + \frac{c}{\beta eh}p_2)^m}\right),$$

is a solution of (3.1).
b. We prove that the vector function $\{0, \mathring{f}_1^{\beta} f_7^{\beta}\}$ is a stationary solution of the problem (2.1)–(2.3) satisfying the assumptions of Theorem 2.3.

By construction, the function $\mathring{f}^{\beta}(x,p) = \mathring{f}_1^{\beta}(x,p) \times \mathring{f}_7^{\beta}(x,p)$ satisfies equation (3.1) and $\sup\limits_{x,p} \mathring{f}^{\beta}(x,p) \geq \mathring{f}^{\beta}(0,0) \geq 2\alpha > 0$. It suffices to show that the right-hand side of (2.4) is identically zero and $\operatorname{supp} \mathring{f}^{\beta} \subset R_\delta \times B_{\rho/4}$. Let us show that

$$\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p = \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.$$

We make the change of variables $y = x$ and $w = \dfrac{eh}{c}(p_2, p_1, p_3)$. Then using the equalities $\psi_1^{+1}(\tau) = \psi_1^{-1}(\tau)$, $\psi_4^{+1}(\tau) = \psi_4^{-1}(\tau)$, defining $x = y$ and $p = \frac{c}{eh}(w_2, -w_1, w_3)$, we find that

$$\begin{aligned}
\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p &= \int\limits_{\mathbb{R}^3} \psi_1^{+1}(|p|^2)\psi_4^{+1}\left(\frac{1}{(x_1 + \frac{c}{eh}p_2)^m}\right)\mathrm{d}p \\
&= \int\limits_{\mathbb{R}^3} \psi_1^{+1}\left(|w|^2\right)\psi_4^{+1}\left(\frac{1}{(y_1 + w_1)^m}\right)\mathrm{d}w = \int\limits_{\mathbb{R}^3} \psi_1^{-1}\left(|w|^2\right)\psi_4^{-1}\left(\frac{1}{(y_1 + w_1)^m}\right)\mathrm{d}w \\
&= \int\limits_{\mathbb{R}^3} \psi_1^{-1}\left(|p|^2\right)\psi_4^{-1}\left(\frac{1}{(x_1 - \frac{c}{eh}p_2)^m}\right)\mathrm{d}p = \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.
\end{aligned}$$

We now assert that $\operatorname{supp} \mathring{f}^{\beta} \subset R_\delta \times B_{\rho/4}$. Indeed, if $|p| > \rho_1/4$ then $f_1^{\beta}(x,p) = \psi_1^{\beta}(|p|^2) = 0$ by construction. Hence, $\mathring{f}^{\beta}(x,p) = 0$ for $|p| > \rho_1/4$. If $x_1 < \delta_0$ where $< \delta_0 < \delta$ and $|p| \leq \rho_1/4$ where $\rho_1 < \rho$, then from Condition 2.2 and the inequalities $< \delta_0 < \delta$ and $\frac{\rho_1}{\rho} < 1$ it follows that:

$$\frac{1}{(x_1 + \frac{c}{\beta eh}p_2)^m} > \frac{1}{(\delta_0 + \frac{\delta\rho_1}{4\rho})^m} > \frac{1}{(\delta_0 + \delta/4)^m} > \left(\frac{1}{\delta + \delta/4}\right)^m = \left(\frac{4}{5\delta}\right)^m.$$

Consequently

$$f_7^{\beta}(x,p) = \psi_4^{\beta}\left(\frac{1}{(x_1 + \frac{c}{\beta eh}p_2)^m}\right) = 0.$$

Hence, $\mathring{f}^{\beta}(x,p) = 0$ for $x_1 < \delta,\ |p| \leq \rho/4$.

4. Construction of stationary solution which depends on x_3

1) We now construct a general solution of equations (2.2) with zero potential. Transform the system (2.1) and (2.2) to the following form:

$$\int\limits_{\mathbb{R}^3} \mathring{f}^{1}(x,p) - \mathring{f}^{-1}(x,p)\mathrm{d}p = 0, \quad x \in R_\delta,\ p \in \mathbb{R}^3, \tag{4.1}$$

$$p_1 \frac{\partial \mathring{f}^\beta(x,p)}{\partial x_1} + p_2 \frac{\partial \mathring{f}^\beta(x,p)}{\partial x_2} + p_3 \frac{\partial \mathring{f}^\beta(x,p)}{\partial x_3} + \frac{\beta e h}{c} \left(p_2 \frac{\partial \mathring{f}^\beta(x,p)}{\partial p_1} - p_1 \frac{\partial \mathring{f}^\beta(x,p)}{\partial p_2} \right) = 0,$$
$$x \in R_\delta,\ p \in \mathbb{R}^3,\ \beta = \pm 1. \tag{4.2}$$

Equation (4.2) is a first order partial differential equation, which can be solved by using the method of characteristics. To solve equation (4.2) we consider the following system of ordinary differential equations:

$$\frac{\mathrm{d}x_1}{p_1} = \frac{\mathrm{d}x_2}{p_2} = \frac{\mathrm{d}x_3}{p_3} = \frac{\mathrm{d}p_1}{\frac{\beta e h}{c} p_2} = \frac{-\mathrm{d}p_2}{\frac{\beta e h}{c} p_1} = \frac{\mathrm{d}p_3}{0} = \frac{\mathring{f}^\beta}{0}. \tag{4.3}$$

Integrating

$$\frac{\mathring{f}^\beta}{0} = \frac{\mathrm{d}x_3}{p_3}, \quad \frac{\mathrm{d}p_1}{\frac{\beta e h}{c} p_2} = \frac{-\mathrm{d}p_2}{\frac{\beta e h}{c} p_1} \quad \text{and} \quad \frac{\mathrm{d}p_3}{0} = \frac{\mathrm{d}x_3}{p_3},$$

we obtain

$$\mathring{f}^\beta = C_0, \quad p_1^2 + p_2^2 = C_1 \quad \text{and} \quad p_3 = C_2.$$

Further, integrating pairs

$$\frac{\mathrm{d}x_1}{p_1} = \frac{-\mathrm{d}p_2}{\frac{\beta e h}{c} p_1}, \quad \frac{\mathrm{d}x_2}{p_2} = \frac{\mathrm{d}p_1}{\frac{\beta e h}{c} p_2},$$

we get

$$\frac{\beta e h}{c} x_1 + p_2 = C_3, \quad \frac{\beta e h}{c} x_2 - p_1 = C_4.$$

Substituting $p_3 = C_2$ and $p_2 = \pm\sqrt{C_1 - p_1^2}$ in $\frac{\mathrm{d}x_3}{p_3} = \frac{\mathrm{d}p_1}{\frac{\beta e h}{c} p_2}$ we get the following integrating combination:

$$\frac{\frac{\beta e h}{c} \mathrm{d}x_3}{C_2} = \frac{\mathrm{d}p_1}{\pm\sqrt{C_1 - p_1^2}}.$$

From the latter if follows that:

$$\frac{e h}{c} x_3 - \beta p_3 \arcsin \frac{p_1}{\sqrt{p_1^2 + p_2^2}} = C_5.$$

We put

$$\varphi_0(x,p) = \mathring{f}^\beta, \quad \varphi_1(x,p) = p_1^2 + p_2^2, \quad \varphi_2(x,p) = p_3, \quad \varphi_3(x,p) = \frac{\beta e h}{c} x_1 + p_2,$$
$$\varphi_4(x,p) = \frac{\beta e h}{c} x_2 - p_1, \quad \varphi_5(x,p) = \frac{e h}{c} x_3 - \beta p_3 \arcsin \frac{p_1}{\sqrt{p_1^2 + p_2^2}}. \tag{4.4}$$

Functions $\varphi_0(x,p) = C_0,\ \varphi_1(x,p) = C_1,\ \varphi_2(x,p) = C_2,\ \varphi_3(x,p) = C_3,\ \varphi_4(x,p) = C_4,\ \varphi_5(x,p) = C_5$ are the first integrals of the system (4.3). According to the method of characteristics, the solution of equation (4.2) has the form:

$$\varPhi\left(\mathring{f}^{\beta}, p_1^2+p_2^2, p_3, \frac{\beta eh}{c}x_1+p_2, \frac{\beta eh}{c}x_2-p_1, \frac{eh}{c}x_3-\beta p_3\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right)=0, \tag{4.5}$$

where $\varPhi$ is an arbitrary function. Since only one first integral of the system (4.3) depends on $\mathring{f}^{\beta}$, (4.5) can be transformed as follows:

$$\mathring{f}^{\beta} = F\left(p_1^2+p_2^2, p_3, \frac{\beta eh}{c}x_1+p_2, \frac{\beta eh}{c}x_2-p_1, \frac{eh}{c}x_3-\beta p_3\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right), \tag{4.6}$$

where F is an arbitrary function. To investigate functional independence of the first integrals of the system (4.3) we construct the Jacobi matrix J:

$$J=\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial p_3}\\ & \cdots & \\ \frac{\partial\varphi_5}{\partial x_1} & \cdots & \frac{\partial\varphi_5}{\partial p_3}\end{pmatrix}=\begin{pmatrix}0&0&0&2p_1&2p_2&0\\0&0&0&0&0&1\\\frac{\beta eh}{c}&0&0&0&1&0\\0&\frac{\beta eh}{c}&0&-1&0&0\\0&0&\frac{eh}{c}&-\beta\frac{p_2p_3}{p_1^2+p_2^2}&\beta\frac{p_1p_3}{p_1^2+p_2^2}&-\beta\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\end{pmatrix}.$$

Since

$$\begin{aligned}\det J' &= \det\begin{pmatrix}0&0&0&2p_1&0\\0&0&0&0&1\\\frac{\beta eh}{c}&0&0&0&0\\0&\frac{\beta eh}{c}&0&-1&0\\0&0&\frac{eh}{c}&-\beta\frac{p_2p_3}{p_1^2+p_2^2}&-\beta\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\end{pmatrix}\\
&=(-1)^{1+3}\frac{\beta eh}{c}\det\begin{pmatrix}0&0&2p_1&0\\0&0&0&1\\\frac{\beta eh}{c}&0&-1&0\\0&\frac{eh}{c}&-\beta\frac{p_2p_3}{p_1^2+p_2^2}&-\beta\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\end{pmatrix}\\
&=(-1)^{1+3}(-1)^{1+3}\left(\frac{\beta eh}{c}\right)^2\det\begin{pmatrix}0&2p_1&0\\0&0&1\\\frac{eh}{c}&-\beta\frac{p_2p_3}{p_1^2+p_2^2}&-\beta\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\end{pmatrix}\\
&=(-1)^{1+3}(-1)^{1+3}(-1)^{1+3}\left(\frac{eh}{c}\right)^3\det\begin{pmatrix}2p_1&0\\0&1\end{pmatrix}=2\left(\frac{eh}{c}\right)^3p_1\neq 0\end{aligned}$$

if $p_1\neq 0$, then $\operatorname{rank} J = 5$ if $p_1\neq 0$.

If $p_1 = 0$ then the Jacobi matrix J has the form

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 2p_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\beta eh}{c} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\beta eh}{c} & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{eh}{c} & -\beta\frac{p_3}{p_2} & 0 & 0 \end{pmatrix}.$$

Then

$$\det J' = \det \begin{pmatrix} 0 & 0 & 0 & 2p_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\beta eh}{c} & 0 & 0 & 1 & 0 \\ 0 & \frac{\beta eh}{c} & 0 & 0 & 0 \\ 0 & 0 & \frac{eh}{c} & 0 & 0 \end{pmatrix} = (-1)^{1+3}(-1)^{1+3}(-1)^{1+3} 2\left(\frac{eh}{c}\right)^3 p_2$$

$$= 2\left(\frac{eh}{c}\right)^3 p_2 \neq 0 \text{ since } p_1 = 0 \text{ and } p_1^2 + p_2^2 \neq 0.$$

Since the rank of the matrix J equals 5 then functions (4.6) are functionally independent. According to the method of characteristics function (4.6) is a general solution of equation (2.2) with zero potential.

2) We consider non-negative functions $\Psi_1^\beta, \Psi_2^\beta, \Psi_3^\beta, \Psi_4^\beta, \Psi_5^\beta \in \dot{C}^\infty(\mathbb{R})$ such that $\int\limits_{\mathbb{R}^3} \Psi_i^{+1}(\tau)\mathrm{d}\tau = \int\limits_{\mathbb{R}^3} \Psi_i^{-1}(\tau)\mathrm{d}\tau$ $(i = 1,2,3,4,5)$, $\operatorname{supp}\Psi_1^\beta \subset \left(-\frac{\rho_1^2}{16}, \frac{\rho_1^2}{16}\right)$, $\operatorname{supp}\Psi_2^\beta \subset \left(-\frac{\rho_1^2}{16}, \frac{\rho_1^2}{16}\right)$, $\operatorname{supp}\Psi_3^\beta \subset \left(\delta, \frac{\rho(4\delta_0-\delta)}{4\delta}\right)$, $\operatorname{supp}\Psi_4^\beta \subset \left(\delta, \frac{\rho(4\delta_2-\delta)}{4\delta}\right)$, $\operatorname{supp}\Psi_5^\beta \subset \left(\delta, \frac{\rho(8\delta_3-\pi\delta)}{8\delta}\right)$, where $\frac{\delta}{4} < \delta_0 < \delta$, $\frac{\delta}{2} < \delta_2 < \delta$, $\frac{\delta\pi}{8} < \delta_3 < \delta$ and $\Psi_1^\beta(0) = 0$, $\Psi_i^{-1}(0) = 2\alpha > 0$, $\Psi_i^{-1}(0) = 1$ and $\Psi_i^{+1}(0) > 2\alpha$ $(i = 1,2,3,4)$. The functions

$$\Psi_1^\beta(p_1^2+p_2^2), \quad \Psi_2^\beta(p_3^2), \quad \Psi_3^\beta\left(\frac{\beta eh}{c}x_1 + p_2\right), \quad \Psi_4^\beta\left(\frac{\beta eh}{c}x_2 - p_1\right), \quad \Psi_5^\beta\left(\frac{eh}{c}x_3 - \beta p_3 \arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right),$$

are solutions of (2.2).

3) We prove that the vector function $\{0, \mathring{f}^\beta\}$, where

$$\mathring{f}^\beta(x,p) = \Psi_1^\beta(p_1^2+p_2^2)\,\Psi_2^\beta(p_3^2)\,\Psi_3^\beta\left(\frac{\beta eh}{c}x_1 + p_2\right)\Psi_4^\beta\left(\frac{\beta eh}{c}x_2 - p_2\right)\Psi_5^\beta\left(\frac{eh}{c}x_3 - \beta p_3 \arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right),$$

is a stationary solution of the problem (2.1)–(2.3) satisfying the assumptions of Theorem 2.4.

By construction, the function $\mathring{f}^\beta(x,p)$ satisfies equation (3.1) and $\sup\limits_{x,p} \mathring{f}^\beta(x,p) \le \mathring{f}^\beta(0,0) \le 2\alpha > 0$. It suffices to show that the right-hand side of (2.4) is identically zero and $\operatorname{supp}\mathring{f}^\beta \subset (\mathbb{R}_{\delta_1} \cap \{x \in \mathbb{R} \colon |x_2| \le \delta_2, |x_3| \le \delta_3\}) \times B_{\rho/4}$. Let us show that

$$\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p = \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.$$

We make the change of variables $y = \dfrac{eh}{c}x$ and $w = \dfrac{eh}{c}(p_2, -p_1, -p_3)$. Then using the equalities $\int\limits_{\mathbb{R}^3} \Psi_i^{+1}(\tau)\mathrm{d}\tau = \int\limits_{\mathbb{R}^3} \Psi_i^{-1}(\tau)\mathrm{d}\tau$ $(i = 1, 2, 3, 4, 5)$ and defining $x = \frac{c}{eh}(-y_1, -y_2, y_3)$ and $p = \frac{c}{eh}(-w_2, w_1, -w_3)$, we find that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3} \mathring{f}^{+1}(x,p)\,\mathrm{d}p \\
&\quad= \int\limits_{\mathbb{R}^3} \Psi_1^{+1}\big(p_1^2+p_2^2\big)\,\Psi_2^{+1}\big(p_3^2\big)\Psi_3^{+1}\left(\frac{eh}{c}x_1+p_2\right)\Psi_4^{+1}\left(\frac{eh}{c}x_2-p_1\right)\Psi_5^{+1}\left(\frac{eh}{c}x_3-p_3\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right)\mathrm{d}p \\
&\quad= \int\limits_{\mathbb{R}^3} \Psi_1^{+1}\big(w_1^2+w_2^2\big)\,\Psi_2^{+1}\big(w_3^2\big)\Psi_3^{+1}(y_1+w_1)\Psi_4^{+1}(y_2+w_2)\Psi_5^{+1}\left(y_3+w_3\arcsin\frac{-w_2}{\sqrt{w_1^2+w_2^2}}\right)\mathrm{d}p \\
&\quad= \int\limits_{\mathbb{R}^3} \Psi_1^{-1}\big(w_1^2+w_2^2\big)\,\Psi_2^{-1}\big(w_3^2\big)\Psi_3^{-1}(y_1+w_1)\Psi_4^{-1}(y_2+w_2)\Psi_5^{-1}\left(y_3+w_3\arcsin\frac{-w_2}{\sqrt{w_1^2+w_2^2}}\right)\mathrm{d}p \\
&\quad= \int\limits_{\mathbb{R}^3} \Psi_1^{-1}\big(p_1^2+p_2^2\big)\,\Psi_2^{-1}\big(p_3^2\big)\Psi_3^{-1}\left(\frac{-eh}{c}x_1+p_2\right)\Psi_4^{-1}\left(\frac{-eh}{c}x_2-p_1\right)\Psi_5^{-1}\left(\frac{eh}{c}x_3+p_3\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right)\mathrm{d}p \\
&\quad= \int\limits_{\mathbb{R}^3} \mathring{f}^{-1}(x,p)\,\mathrm{d}p.
\end{aligned}$$

We now assert that $\operatorname{supp}\mathring{f}^\beta \subset (\mathbb{R}_{\delta_1} \cap \{x \in \mathbb{R}\colon |x_2| \le \delta_2, |x_3| \le \delta_3\}) \times B_{\rho/4}$. Indeed, if $|p| > \rho_1/4$ then $\Psi_1^{+1}\big(p_1^2+p_2^2\big) = 0$ and $\Psi_2^{+1}\big(p_3^2\big) = 0$ by construction. Hence, $\mathring{f}^\beta(x,p) = 0$ for $|p| > \rho_1/4$. Furthermore, if $p_1^2+p_2^2 = 0$, then $\mathring{f}^\beta(x,p) = 0$, since $\Psi_1^\beta(0) = 0$.

If $\delta_0 < x_1 < \delta_1$ where $\delta/4 < \delta_0 < \delta_1 < \delta$ and $|p| \le \rho_1/4$, where $\rho_1 < \rho$, then from Condition 2.2 it follows that:

$$\left|\frac{\beta eh}{c}x_1 + p_2\right| > \frac{eh}{c}|x_1| - |p_2| > \frac{\delta_0\rho}{\delta} - \frac{\rho_1}{4} > \frac{\delta_0\rho}{\delta} - \frac{\rho}{4} > \frac{\rho(4\delta_0-\delta)}{4\delta}.$$

Consequently,

$$\Psi_3^\beta\left(\frac{\beta eh}{c}x_1 + p_2\right) = 0.$$

If $|x_2| \ge \delta_2$, where $\delta/4 < \delta_2 < \delta$, then

$$\left|\left(\frac{eh}{c}x_2 - p_1\right)\right| \ge \frac{\rho\delta_2}{\delta} - \frac{\rho_1}{4} > \frac{eh}{c}|x_2| - |p_1| > \frac{\rho\delta_2}{\delta} - \frac{\rho}{4} = \frac{\rho(4\delta_2-\delta)}{4\delta}.$$

Consequently,

$$\Psi_4^\beta\left(\frac{\beta eh}{c}x_2 + p_1\right) = 0.$$

If $|x_3| \geq \delta_3$, where $\frac{\delta\pi}{8} < \delta_2 < \delta$, then

$$\left|\left(\frac{eh}{c}x_3 - \beta p_3 \arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right)\right| \geq \frac{eh}{c}|x_3| - |p_3|\left|\arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right|$$
$$> \frac{\rho\delta_3}{\delta} - \frac{\pi}{2}\frac{\rho_1}{4} > \frac{\rho\delta_2}{\delta} - \frac{\pi\rho}{8} = \frac{\rho(8\delta_3 - \pi\delta)}{8\delta}.$$

Consequently,

$$\Psi_5^\beta\left(\frac{eh}{c}x_3 - \beta p_3 \arcsin\frac{p_1}{\sqrt{p_1^2+p_2^2}}\right) = 0.$$

Hence, $\mathring{f}^\beta(x,p) = 0$ for $x_1 < \delta,\ \ |x_2| \geq \delta_2,\ \ |x_3| \geq \delta_3$ and $|p| \leq \rho/4$.

In conclusion we would like to emphasize that from the physical point of view it is important to study solutions with charged-particle density distributions supported on some distance from the boundary. Construction of stationary solutions with compactly supported charged-particle density distributions is more natural from the viewpoint of plasma physics. It can be explained in such a way that if solutions are independent of x_3 than plasma has an infinite mass. Construction of stationary solutions with compactly supported charged-particle density distributions and non-zero potential in a half-space and in an infinite cylinder is an open problem. There is also a certain interest in the study of construction of stationary solutions of the Vlasov–Poisson equations in arbitrary bounded domains.

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