Math. Model. Nat. Phenom. 13 (2018) 33
https://doi.org/10.1051/mmnp/2018019

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

CHARACTERIZATIONS OF NUCLEAR PSEUDO-DIFFERENTIAL OPERATORS ON $\mathbb{Z}$ WITH SOME APPLICATIONS

Majid JamalpourBirgani*

Abstract. In this paper, we give necessary and sufficient conditions on the symbols σ, such that the corresponding pseudo-differential operators T_σ from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$, $1 \le p_1, p_2 < \infty$, be nuclear. We show that the adjoint operators of the nuclear pseudo-differential operators from $L^{p_2'}(\mathbb{Z})$ into $L^{p_1'}(\mathbb{Z})$ are nuclear and present a necessary and sufficient condition on the symbols of the nuclear pseudo-differential operators from $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ to be self-adjoint. As applications, We get the symbol of the product of the nuclear operators with the bounded operators, and a necessary and sufficient condition on the symbols of nuclear operators is given to be normal.

Mathematics Subject Classification. 47G30, 47G10

Received August 21, 2017. Accepted January 21, 2018.

1. Introduction

Pseudo-differential operators are understood in a very broad sense and include such topics as harmonic analysis, PDE, geometry, mathematical physics, micro-local analysis, time-frequency analysis, imaging and computations. Modern trends and novel applications in mathematics, natural sciences, medicine, scientific computing, and engineering are highlighted [15].

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{S}^1$ be the unit circle centered at the origin. The aim of this paper is to present necessary and sufficient conditions on the symbols σ such that the corresponding pseudo-differential operators T_σ from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$ for $1 \le p_1, p_2 < \infty$, be nuclear. We also give necessary and sufficient conditions on σ to guarantee that the adjoints and products of pseudo-differential operators are nuclear. As applications, we give some necessary and sufficient conditions on σ so that the corresponding nuclear operator T_σ be self-adjoint or normal.

Although all results in this paper are presented for $\mathbb{Z}$ only, one can extend the results to the lattice $\mathbb{Z}^n$.

Suppose that σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$. Then for every sequence $\{a(n)\}_{n\in\mathbb{Z}}$ in $L^p(\mathbb{Z})$, $1 \le p < \infty$, the pseudo-differential operator T_σ is defined as follows:

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-in\theta} \sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}} a)(\theta) \mathrm{d}\theta, \quad n \in \mathbb{Z},$$

Keywords and phrases: Pseudo-differential operators, bounded operators, nuclear operators, adjoint, product, normal.

Iran University of Science and Technology, Iran, Tehran.

* Corresponding author: m_jamalpour@mathdep.iust.ac.ir

Article published by EDP Sciences

where $\mathcal{F}_{\mathbb{Z}}a$ is the Fourier transform of a given by

$$(\mathcal{F}_{\mathbb{Z}}a)(\theta) = \sum_{n=-\infty}^{\infty} a(n)\mathrm{e}^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Function σ is called the symbol of pseudo-differential operator T_σ.
Some properties of these operators such as boundedness, compactness and self-adjointness are studied in many papers such as [3, 4, 5, 7, 8, 13, 14, 15].
Nuclear operators on Banach spaces as generalizations of trace class operators can be traced at least to Grothendieck [11, 12]. First results on nuclear integral operators and pseudo-differential operators on L^p spaces in simple settings like the unit circle centered at the origin and the discrete group of all integers can be found in [1, 2, 6].
In [6], It has been shown that if σ be a measurable function on $\mathbb{Z} \times \mathbb{S}$ such that we can find a function c in $L^1(\mathbb{Z})$ and a function w in $L^p(\mathbb{Z})$, $1 \le p < \infty$, for which

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)| \le |c(n)||w(m)|, \quad m, n \in \mathbb{Z},$$

then $T_\sigma\colon L^p(\mathbb{Z}) \longrightarrow L^p(\mathbb{Z}), \quad 1 \le p < \infty$ is nuclear.
In [9], we showed that the pseudo-differential operator $T_\sigma\colon L^{p_1}(\mathbb{S}^1) \longrightarrow L^{p_2}(\mathbb{S}^1)$, $1 \le p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)} \|g_k\|_{L^{p_1'}(\mathbb{S}^1)} < \infty$$

and for all $(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$

$$\sigma(\theta, n) = 2\pi \mathrm{e}^{-in\theta} \sum_{k=-\infty}^{\infty} h_k(\theta)(\mathcal{F}_{\mathbb{S}^1}g_k)(-n).$$

In this paper, we first give in Section 2 the definition of the nuclear operators on Banach spaces and necessary and sufficient conditions on the symbols σ so that the corresponding pseudo-differential operators T_σ from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$ for $1 \le p_1, p_2 < \infty$, be nuclear. In Section 3 are given necessary and sufficient conditions on the symbols σ for which the adjoint operators T_{σ^*} of pseudo-differential operators T_σ be nuclear. In Section 3, we also give a formula for σ^* in terms of σ and a necessary and sufficient condition on the symbols σ so that the corresponding nuclear pseudo-differential operators T_σ from $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ be self-adjoint.
The nuclearity of the product of the nuclear operators with the bounded operators and a necessary and sufficient condition on the σ such that T_σ be normal is given in Section 4.

2. Nuclearity on $L^p(\mathbb{Z})$

Let X and Y be complex Banach spaces and let $T\colon X \longrightarrow Y$ be a bounded linear operator. Suppose that we can find sequences $\{x_k'\}_{k\in\mathbb{Z}}$ in the dual space X' of X and $\{y_k\}_{k\in\mathbb{Z}}$ in Y such that

$$\sum_{k\in\mathbb{Z}} \|x_k'\|_{X'} \|y_k\|_{Y} < \infty$$

and

$$Tx = \sum_{k\in\mathbb{Z}} x_k'(x) y_k, \quad x \in X,$$

then we call $T\colon X \longrightarrow Y$ a nuclear operator. We say a Banach space X have the approximation property if for every compact subset K of X and every $\varepsilon > 0$ there exists a finite rank bounded operator B on X such that

$$\|x - Bx\|_X < \varepsilon, \quad \textit{for all } x \in K.$$

On such spaces, if $T\colon X \longrightarrow X$ is nuclear, the trace is well-defined by

$$tr(T) = \sum_{k\in\mathbb{Z}} x_k'(y_k),$$

where

$$Tx = \sum_{k\in\mathbb{Z}} x_k'(x) y_k, \quad x \in X.$$

More details about nuclear operators on Banach spaces can be found in [10]. For L^p spaces, the main tools is the following result in [1, 2, 6].

Theorem 2.1. *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces. A bounded linear operator $T\colon L^{p_1}(X_1,\mu_1) \longrightarrow L^{p_2}(X_2,\mu_2), 1 \le p_1, p_2 < \infty,$ is nuclear if and only if there exist sequences $\{g_n\}_{n=1}^{\infty}$in $L^{p_1'}(X_1,\mu_1)$and $\{h_n\}_{n=1}^{\infty}$in $L^{p_2}(X_2,\mu_2)$ such that for all $f \in L^{p_1}(X_1)$*

$$T(f)(x) = \int_{X_1} \left(\sum_{n=1}^{\infty} h_n(x) g_n(y) \right) f(y) \mathrm{d}y, \quad x \in X_2,$$

and

$$\sum_{n=1}^{\infty} \|h_n\|_{L^{p_2}(X_2,\mu_2)} \|g_n\|_{L^{p_1'}(X_1,\mu_1)} < \infty.$$

The function $K(x,y)$ defined by

$$K(x,y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x \in X_2,\ y \in X_1$$

is known as a kernel of the nuclear operator $T\colon L^{p_1}(X_1,\mu_1) \longrightarrow L^{p_2}(X_2,\mu_2)$.
If $X_1 = X_2 = X,\ p_1 = p_2$ and $\mu_1 = \mu_2 = \mu$ is a σ-finite measure, then K by applying the Hölder's inequality is integrable and

$$\int_X |K(x,x)| \mathrm{d}x \le \sum_{n=1}^{\infty} \|h_n\|_{L^p(X,\mu)} \|g_n\|_{L^q(X,\mu)}.$$

The trace $tr(T)$ *of* $T\colon L^p(X,\mu)\longrightarrow L^p(X,\mu)$ *is given by*

$$tr(T)=\int_X K(x,x)\mathrm{d}x.$$

In order to give a necessary and sufficient condition for the pseudo-differential operator $T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z}),\quad 1\le p_1,p_2<\infty$, to be nuclear, at first we need to prove the following theorem.

Theorem 2.2. *Let* σ *be a measurable function on* $\mathbb{Z}\times\mathbb{S}^1$ *such that the pseudo-differential operator* $T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z}),\quad 1\le p_1,p_2<\infty$, *is a bounded operator. Then for all* $n\in\mathbb{Z}$

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(.\ ,\ .\ -n)\in L^{p_2}(\mathbb{Z}).$$

Proof. Assume $T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z}),\quad 1\le p_1,p_2<\infty$, be a bounded operator. Then there exist a constant $C>0$ such that for all $f\in L^{p_1}(\mathbb{Z})$,

$$\begin{aligned}\|T_\sigma\|_{L^{p_2}(\mathbb{Z})}&=\left(\sum_{m=-\infty}^{\infty}|(T_\sigma f)(m)|^{p_2}\right)^{1/p_2}\\&\le C\|f\|_{L^{p_1}(\mathbb{Z})}\\&=C\left(\sum_{m=-\infty}^{\infty}|f(m)|^{p_1}\right)^{1/p_1}.\end{aligned}$$

Now, let $n\in\mathbb{Z}$ and the function f be as follow:

$$f(m)=f_n(m)=\begin{cases}0 & \text{if}\quad m\neq n,\\ 1 & \text{if}\quad m=n,\qquad m\in\mathbb{Z}.\end{cases}$$

Then,

$$\left(\sum_{m=-\infty}^{\infty}\left|\int_{-\pi}^{\pi}\mathrm{e}^{-im\theta}\sigma(m,\theta)\mathrm{e}^{in\theta}\mathrm{d}\theta\right|^{p_2}\right)^{1/p_2}\le C<\infty$$

and proof is complete. □

In the following theorem, we give a necessary and sufficient condition on the symbol σ, to make sure that the corresponding pseudo-differential operator T_σ from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z}),\ 1\le p_1,p_2<\infty$, be nuclear.

Theorem 2.3. *Let* σ *be a measurable function on* $\mathbb{Z}\times\mathbb{S}^1$. *Then the pseudo-differential operator* $T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z}),\quad 1\le p_1,p_2<\infty$, *is a nuclear operator if and only if there exist sequences* $\{g_k\}_{k=-\infty}^{\infty}$ *in* $L^{p_1'}(\mathbb{Z})$ *and* $\{h_k\}_{k=-\infty}^{\infty}$ *in* $L^{p_2}(\mathbb{Z})$ *such that*

$$\sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})}<\infty$$

and for all $(n,\theta)\in\mathbb{Z}\times\mathbb{S}^1$

$$\sigma(n,\theta)=\mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty}h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta).$$

Proof. Assume σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$. Then for every $n\in\mathbb{Z}$ and $f\in L^{p_1}(\mathbb{Z}),\ 1\le p_1<\infty$,

$$\begin{aligned}(T_\sigma f)(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-in\theta}\sigma(n,\theta)(F_Z f)(\theta)\mathrm{d}\theta\\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-i\theta(n-m)}\sigma(n,\theta)\mathrm{d}\theta f(m).\end{aligned}$$

If T_σ: $L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z}),\ 1<p_1,p_2<\infty$, be a nuclear operator. According to Theorem 2.1, there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})}<\infty$$

and

$$\begin{aligned}(T_\sigma f)(n) &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-i\theta(n-m)}\sigma(n,\theta)\mathrm{d}\theta f(m)\\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}h_k(n)g_k(m)f(m) \qquad (2.1)\end{aligned}$$

for all $f\in L^{p_1}(\mathbb{Z})$. Now, let $l\in\mathbb{Z}$ and the function f be as follow:

$$f(m)=f_l(m)=\begin{cases}0 & \text{if} \quad m\neq l,\\ 1 & \text{if} \quad m=l, \quad m\in\mathbb{Z}.\end{cases}$$

Then $f\in L^{p_1}(\mathbb{Z})$ and for all $n\in\mathbb{Z}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-i\theta(n-l)}\sigma(n,\theta)\mathrm{d}\theta=\sum_{k=-\infty}^{\infty}h_k(n)g_k(l). \qquad (2.2)$$

In fact for all $n,l\in\mathbb{Z}$,

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)=\sum_{k=-\infty}^{\infty}h_k(n)g_k(l). \qquad (2.3)$$

So, by using Theorem 2.2, we have

$$\begin{aligned}\sigma(n,\theta) &= \sum_{l=-\infty}^{\infty}\mathrm{e}^{i(n-l)\theta}(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)\\ &= \sum_{l=-\infty}^{\infty}\mathrm{e}^{i(n-l)\theta}\sum_{k=-\infty}^{\infty}h_k(n)g_k(l)\\ &= \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty}h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta). \qquad (2.4)\end{aligned}$$

Conversely, if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})} \|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for all $(n, \theta) \in \mathbb{Z} \times \mathbb{S}^1$

$$\sigma(n, \theta) = \mathrm{e}^{in\theta} \sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta),$$

then for all $f \in L^{p_1}(\mathbb{Z})$ we have

$$\begin{aligned}
(T_\sigma f)(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-in\theta} \sigma(n, \theta)(F_Z f)(\theta) \mathrm{d}\theta \\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_k(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-im\theta} (\mathcal{F}_{\mathbb{Z}} g_k)(\theta) \mathrm{d}\theta f(m) \\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_k(n) g_k(m) f(m)
\end{aligned}$$

and proof is complete. $\square$

As a result of Theorem 2.3, we can have the following theorem.

Theorem 2.4. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that the pseudo-differential operator T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 2 \le p_1 < \infty,\ 1 \le p_2 < \infty$, be a nuclear operator. Then*

$$\sum_{n=-\infty}^{\infty} \|\sigma(n, .)\|_{L^{p_1'}(\mathbb{S}^1)}^{p_2} < \infty.$$

Proof. Assume T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 2 \le p_1 < \infty,\ 1 \le p_2 < \infty$, be a nuclear operator. There exist sequences $\{g_k\}_{k=-\infty}^{\infty} \in L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})} \|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for all $(n, \theta) \in \mathbb{Z} \times \mathbb{S}^1$

$$\sigma(n, \theta) = \mathrm{e}^{in\theta} \sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta).$$

So, by using Minkowski's inequality we get

$$\begin{aligned}
\|\sigma(n,.)\|_{L^{p_1'}(\mathbb{S}^1)} &= \left(\int_{-\pi}^{\pi} |\sigma(n,\theta)|^{p_1'} \mathrm{d}\theta\right)^{\frac{1}{p_1'}} \\
&= \left(\int_{-\pi}^{\pi} \left|\sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta)\right|^{p_1'} \mathrm{d}\theta\right)^{\frac{1}{p_1'}} \\
&\le \sum_{k=-\infty}^{\infty} |h_k(n)| \left(\int_{-\pi}^{\pi} |(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta)|^{p_1'} \mathrm{d}\theta\right)^{\frac{1}{p_1'}} \\
&\le \sum_{k=-\infty}^{\infty} |h_k(n)| \left(\int_{-\pi}^{\pi} |(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta)|^{2} \mathrm{d}\theta\right)^{\frac{1}{2}} \\
&= \sum_{k=-\infty}^{\infty} |h_k(n)| \|g_k\|_{L^2(\mathbb{Z})} \\
&\le \sum_{k=-\infty}^{\infty} |h_k(n)| \|g_k\|_{L^{p_1'}(\mathbb{Z})}
\end{aligned}$$

and

$$\|\sigma(n,.)\|^{p_2}_{L^{p_1'}(\mathbb{S}^1)} \le \left(\sum_{k=-\infty}^{\infty} |h_k(n)| \|g_k\|_{L^{p_1'}(\mathbb{Z})}\right)^{p_2}.$$

Then,

$$\left(\sum_{n=-\infty}^{\infty} \|\sigma(n,.)\|^{p_2}_{L^{p_1'}(\mathbb{S}^1)}\right)^{\frac{1}{p_2}} \le \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h_k(n) \|g_k\|_{L^{p_1'}(\mathbb{Z})}\right)^{p_2}\right)^{\frac{1}{p_2}}.$$

Now by using Minkowski's inequality we have

$$\begin{aligned}
\left(\sum_{n=-\infty}^{\infty} \|\sigma(n,.)\|^{p_2}_{L^{p_1'}(\mathbb{S}^1)}\right)^{\frac{1}{p_2}} &\le \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |h_k(n)|^{p_2}\right)^{\frac{1}{p_2}} \|g_k\|_{L^{p_1'}(\mathbb{Z})} \\
&\le \sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})} \|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty
\end{aligned}$$

□

In the next theorem, we give another characterization of nuclear operators from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$, $1 \le p_1, p_2 < \infty$, in order to find a formula for the trace of nuclear operators from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$, $1 \le p < \infty$.

Theorem 2.5. *Assume σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$. Then the pseudo-differential operator T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z})$, $1 \le p_1, p_2 < \infty$, is a nuclear operator if and only if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in*

$L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for every $n, m \in \mathbb{Z}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-i(n-m)\theta}\sigma(n,\theta)\mathrm{d}\theta = \sum_{k=-\infty}^{\infty} h_k(n)g_k(m).$$

Proof. Let $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 \leq p_1, p_2 < \infty$, is a nuclear operator. According to equation (2.2), there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for every $n, m \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-i\theta(n-m)}\sigma(n,\theta)\mathrm{d}\theta = \sum_{k=-\infty}^{\infty} h_k(n)g_k(m).$$

Conversely, if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for every $n, m \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-i\theta(n-m)}\sigma(n,\theta)\mathrm{d}\theta = \sum_{k=-\infty}^{\infty} h_k(n)g_k(m),$$

then for all $n \in \mathbb{Z},\ f \in L^{p_1}(\mathbb{Z})$, we have

$$\begin{aligned}(T_\sigma f)(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-in\theta}\sigma(n,\theta)(F_Z f)(\theta)\mathrm{d}\theta \\ &= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-i\theta(n-m)}\sigma(n,\theta)\mathrm{d}\theta f(m) \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_k(n)g_k(m)f(m)\end{aligned}$$

and proof is complete. □

A consequence of Theorem 2.5 is the following results:

Theorem 2.6. *Let $T_\sigma\colon L^p(\mathbb{Z}) \longrightarrow L^p(\mathbb{Z}),\ 1 \le p < \infty$, be a nuclear operator, then*

$$tr(T_\sigma) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta$$

and

$$\left\{ \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta \right\}_{n\in\mathbb{Z}} \in L^1(\mathbb{Z}).$$

Proof. Assume $T_\sigma\colon L^p(\mathbb{Z}) \longrightarrow L^p(\mathbb{Z}),\ 1 \le p < \infty$, is a nuclear operator, then according to Theorems 2.1 and 2.5

$$tr(T_\sigma) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_k(n) g_k(n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta$$

and since

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta \right| = \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} h_k(n) g_k(n) \right|.$$

With Hölder's inequality implies that

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta \right| \le \sum_{k=-\infty}^{\infty} \|h_k\|_{L^p(\mathbb{Z})} \|g_k\|_{L^{p'}(\mathbb{Z})} < \infty$$

and this means

$$\left\{ \int_{-\pi}^{\pi} \sigma(n,\theta)\mathrm{d}\theta \right\}_{n\in\mathbb{Z}} \in L^1(\mathbb{Z}).$$

□

Theorem 2.7. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that the pseudo-differential operator T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 \le p_1, p_2 < \infty$, is a nuclear operator. Then*

$$\{\|(\mathcal{F}_{\mathbb{S}^1}\sigma)(.,.-l)\|_{L^{p_2}(\mathbb{Z})}\}_{l\in\mathbb{Z}} \in L^{p_1'}(\mathbb{Z}).$$

Proof. Assume $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 \le p_1, p_2 < \infty$, be a nuclear operator. Then according to equation (2.3) there exist sequences $\{g_k\}_{k=-\infty}^{\infty} \in L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})} \|g_k\|_{L^{p'_1}(\mathbb{Z})} < \infty$$

and for every $n, l \in \mathbb{Z}$

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-l) = \sum_{k=-\infty}^{\infty} h_k(n) g_k(l) s$$

So, by using Minkowski's inequality, we get

$$\left(\sum_{n=-\infty}^{\infty}|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)|^{p_2}\right)^{\frac{1}{p_2}} = \left(\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty}h_k(n)g_k(l)\right|^{p_2}\right)^{\frac{1}{p_2}}$$
$$\leq \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}|h_k(n)g_k(l)|^{p_2}\right)^{\frac{1}{p_2}}$$
$$\leq \sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}|g_k(l)|$$

and then

$$\left(\sum_{l=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)|^{p_2}\right)^{\frac{p_1'}{p_2}}\right)^{\frac{1}{p_1'}} \leq \left(\sum_{l=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}|g_k(l)|\right)^{p_1'}\right)^{\frac{1}{p_1'}}$$
$$\leq \sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\left(\sum_{n=-\infty}^{\infty}|g_k(l)|^{p_1'}\right)^{\frac{1}{p_1'}}$$
$$\leq \sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty.$$

□

Theorem 2.8. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that the pseudo-differential operator T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 \leq p_1, p_2 < \infty$, is a nuclear operator. Then*

$$\{\|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-.)\|_{L^{p_1'}(\mathbb{Z})}\}_{n\in\mathbb{Z}} \in L^{p_2}(\mathbb{Z}).$$

Proof. Assume T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 \leq p_1, p_2 < \infty$, be a nuclear operator. Then there exist sequences $\{g_k\}_{k=-\infty}^{\infty} \in L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty}\|h_k\|_{L^p(\mathbb{Z})}\|g_k\|_{L^{p'}(\mathbb{Z})} < \infty$$

and for all $n, l \in \mathbb{Z}$

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l) = \sum_{k=-\infty}^{\infty}h_k(n)g_k(l).$$

Then

$$\left(\sum_{l=-\infty}^{\infty}|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)|^{p_1'}\right)^{\frac{1}{p_1'}} = \left(\sum_{l=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty}h_k(n)g_k(l)\right|^{p_1'}\right)^{\frac{1}{p_1'}}$$
$$\leq \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty}|h_k(n)g_k(l)|^{p_1'}\right)^{\frac{1}{p_1'}}$$
$$\leq \sum_{k=-\infty}^{\infty}|h_k(n)|\|g_k\|_{L^{p_1'}(\mathbb{Z})}$$

and by using Minkowski's inequality, we get

$$\left(\sum_{n=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty}|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-l)|^{p_1'}\right)^{\frac{p_2}{p_1'}}\right)^{\frac{1}{p_2}} \leq \left(\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}|h_k(n)|\|g_k\|_{L^{p_1'}(\mathbb{Z})}\right)^{p_2}\right)^{\frac{1}{p_2}}$$
$$\leq \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}|h_k(n)|^{p_2}\right)^{\frac{1}{p_2}}\|g_k\|_{L^{p_2}(\mathbb{Z})}$$
$$\leq \sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty.$$

□

3. Adjoint

Let $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 < p_1, p_2 < \infty$, be a nuclear operator. In this section we show that the adjoint operator $T_{\sigma^*}\colon L^{p_2'}(\mathbb{Z}) \longrightarrow L^{p_1'}(\mathbb{Z})$ of T_σ, is nuclear and give a formula for the symbol of the adjoint σ^*, in the terms of σ. By using given formula for σ^*, we present a necessary and sufficient condition on the symbol σ so that the corresponding nuclear operator T_σ from $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ be self-adjoint.

Theorem 3.1. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such the pseudo-differential operator $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1 < p_1, p_2 < \infty$, is a nuclear operator. Then the adjoint operator, $T_{\sigma^*}\colon L^{p_2'}(\mathbb{Z}) \longrightarrow L^{p_1'}(\mathbb{Z})$ is also a nuclear operator and for every $(n,\theta) \in \mathbb{Z} \times \mathbb{S}^1$*

$$\sigma^*(n,\theta) = \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty}(\mathcal{F}_{\mathbb{Z}}\overline{h}_k)(-\theta)\overline{g}_k(n).$$

Proof. For all $f \in L^{p_1}(\mathbb{Z}), g \in L^{p_2'}(\mathbb{Z})$, we have

$$\sum_{j=-\infty}^{\infty}(T_\sigma f)(j)\overline{g(j)} = \sum_{j=-\infty}^{\infty}f(j)\overline{(T_{\sigma^*}f)(j)}.$$

In fact, for all $f \in L^{p_1}(\mathbb{Z}), g \in L^{p_2'}(\mathbb{Z})$,

$$\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-i(j-l)\theta}\sigma(j,\theta)\mathrm{d}\theta f(l)\overline{g(j)}$$
$$= \sum_{j=-\infty}^{\infty}\overline{\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-i(j-l)\theta}\sigma^*(j,\theta)\mathrm{d}\theta g(l)}f(j).$$

Now, let $m, n \in \mathbb{Z}$ and for every $k \in \mathbb{Z}$,

$$f(k) = f_n(k) = \begin{cases} 0 & \text{if} \quad k \neq n, \\ 1 & \text{if} \quad k = n, \end{cases} \quad g(k) = g_m(k) = \begin{cases} 0 & \text{if} \quad k \neq m, \\ 1 & \text{if} \quad k = m. \end{cases}$$

Then, we get

$$\int_{-\pi}^{\pi}\mathrm{e}^{-i(m-n)\theta}\sigma(m,\theta)\mathrm{d}\theta = \overline{\int_{-\pi}^{\pi}\mathrm{e}^{-i(n-m)\theta}\sigma^*(n,\theta)\mathrm{d}\theta}$$

and so,

$$\overline{(\mathcal{F}_{\mathbb{S}^1}\sigma)(m, m-n)} = (\mathcal{F}_{\mathbb{S}^1}\sigma^*)(n, n-m).$$

Since $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z})$ $1 < p_1, p_2 < \infty$, is a nuclear operator. There exist sequences $\{g_k\}_{k=-\infty}^{\infty} \in L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{Z})}\|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for every $(n,\theta) \in \mathbb{Z} \times \mathbb{S}^1$

$$\sigma(n,\theta) = \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty}h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta).$$

So, we get

$$\begin{aligned}
\sigma^*(n,\theta) &= \sum_{m=-\infty}^{\infty} \mathrm{e}^{i(n-m)\theta}(\mathcal{F}_{\mathbb{S}^1}\sigma^*)(n,n-m)\\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{i(n-m)\theta}\overline{(\mathcal{F}_{\mathbb{S}^1}\sigma)(m,m-n)}\\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{i(n-m)\theta}\overline{\int_{-\pi}^{\pi} \mathrm{e}^{-i(m-n)\theta}\sigma(m,\theta)\mathrm{d}\theta}\\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{i(n-m)\theta}\overline{\int_{-\pi}^{\pi} \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty} h_k(m)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta)\mathrm{d}\theta}\\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{i(n-m)\theta}\sum_{k=-\infty}^{\infty} \overline{h}_k(m)\overline{g}_k(n)\\
&= \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}}\overline{h}_k)(-\theta)\overline{g}_k(n)
\end{aligned}$$

and with Theorem 2.3 proof is complete. □

As a consequence of this theorem, in the next theorem a necessary and sufficient condition in the terms of sequences $\{g_k\}_k, \{h_k\}_k \in L^2(\mathbb{Z})$, is given for the nuclear operator T_σ from $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ to be self-adjoint.

Theorem 3.2. *Assume σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that $T_\sigma\colon L^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{Z})$ is a nuclear operator. Then T_σ is self-adjoint if and only if there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{Z})$, satisfying in Theorem 2.1 and for all $(n,\theta)\in\mathbb{Z}\times\mathbb{S}^1$*

$$\sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta) = \sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}}\overline{h}_k)(-\theta)\overline{g}_k(n).$$

Proof. Let $T_\sigma\colon L^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{Z})$ be a nuclear operator and there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{Z})$ for T_σ such that for every $(n,\theta)\in\mathbb{Z}\times\mathbb{S}^1$

$$\sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta) = \sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}}\overline{h}_k)(-\theta)\overline{g}_k(n),$$

then by Therorems 2.3 and 3.1, T_σ is self-adjoint.
Conversely, since T_σ is nuclear, there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^2(\mathbb{Z})$ such that for every $(n,\theta)\in\mathbb{Z}\times\mathbb{S}^1$

$$\sigma(n,\theta) = \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}}g_k)(-\theta)$$

and

$$\sigma^*(n,\theta) = \mathrm{e}^{in\theta}\sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}}\overline{h}_k)(-\theta)\overline{g}_k(n).$$

Now, by using self adjointness of T_σ, we get

$$\sum_{k=-\infty}^{\infty} h_k(n)(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta) = \sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}} \overline{h}_k)(-\theta)\overline{g}_k(n).$$

□

In the following theorem we give a formula for σ^* in the terms of σ.

Theorem 3.3. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that the pseudo-differential operator T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z})$, $1 < p_1, p_2 < \infty$, is a nuclear operator. Then for every $(n,\theta) \in \mathbb{Z} \times \mathbb{S}^1$*

$$\sigma^*(n,\theta) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-i(m-n)\theta} \int_{-\pi}^{\pi} \mathrm{e}^{im\theta'} \overline{\sigma(m,\theta')} \mathrm{e}^{-in\theta'} \mathrm{d}\theta',$$

where σ^* is the symbol of T_{σ^*}.

Proof. Assume T_σ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z})$, $1 < p_1, p_2 < \infty$, be a nuclear operator. Then there exist sequences $\{g_k\}_{k=-\infty}^{\infty} \in L^{p_1'}(\mathbb{Z})$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^{p_2}(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{Z})} \|g_k\|_{L^{p_1'}(\mathbb{Z})} < \infty$$

and for all $(m,\theta') \in \mathbb{Z} \times \mathbb{S}^1$

$$\sigma(m,\theta') = \mathrm{e}^{im\theta'} \sum_{k=-\infty}^{\infty} h_k(m)(\mathcal{F}_{\mathbb{Z}} g_k)(-\theta').$$

Now, let $n \in \mathbb{Z}$. Then,

$$\mathrm{e}^{im\theta'} \overline{\sigma(m,\theta')} \mathrm{e}^{-in\theta'} = \mathrm{e}^{-in\theta'} \sum_{k=-\infty}^{\infty} \overline{h_k}(m) \overline{(\mathcal{F}_{\mathbb{Z}} g_k)}(-\theta')$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \mathrm{e}^{im\theta'} \overline{\sigma(m,\theta')} \mathrm{e}^{-in\theta'} \mathrm{d}\theta' &= \int_{-\pi}^{\pi} \mathrm{e}^{-in\theta'} \sum_{k=-\infty}^{\infty} \overline{h_k}(m) \overline{(\mathcal{F}_{\mathbb{Z}} g_k)}(-\theta') \mathrm{d}\theta' \\ &= \sum_{k=-\infty}^{\infty} \overline{h_k}(m) \overline{g_k}(n).\end{aligned}$$

So,

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \mathrm{e}^{-im\theta} \int_{-\pi}^{\pi} \mathrm{e}^{im\theta'} \overline{\sigma(m,\theta')} \mathrm{e}^{-in\theta'} \mathrm{d}\theta' &= \sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{Z}} \overline{h}_k)(-\theta) \overline{g}_k(n) \\ &= \mathrm{e}^{-in\theta} \sigma^*(n,\theta).\end{aligned}$$

□

Theorem 3.3 makes us able to give a necessary and sufficient condition on symbol σ so that the corresponding nuclear pseudo-differential operator T_σ, from $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ be self-adjoint. We bring up this result as theorem.

Theorem 3.4. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that the pseudo-differential operator $T_\sigma\colon L^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{Z})$, is a nuclear operator. Then T_σ is self- adjoint if and only if for every $m,n\in\mathbb{Z}$,*

$$\overline{(\mathcal{F}_{\mathbb{S}^1}\sigma)(m,m-n)} = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-m).$$

Proof. See Theorem 3.3. □

4. Product

Assume σ,τ be two measurable functions on $\mathbb{Z}\times\mathbb{S}^1$ such that $T_\tau\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1\le p_1,p_2<\infty$, is a bounded operator and $T_\sigma\colon L^{p_2}(\mathbb{Z}) \longrightarrow L^{p_3}(\mathbb{Z}),\ 1\le p_3<\infty$ is a nuclear operator. In this section, first we get the symbol of the product of T_σ and T_τ, then we show that the product operator is nuclear and as consequence we give a necessary and sufficient on symbol σ so that T_σ be a normal operator.

Theorem 4.1. *Let σ,τ be two measurable functions on $\mathbb{Z}\times\mathbb{S}^1$ such that the pseudo-differential operator T_τ: $L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1\le p_1,p_2<\infty$, is bounded and $T_\sigma\colon L^{p_2}(\mathbb{Z}) \longrightarrow L^{p_3}(\mathbb{Z}),\ 1\le p_3<\infty$ is a nuclear operator. Then $T_\sigma T_\tau\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_3}(\mathbb{Z})$ is a nuclear operator and for every $(m,\theta)\in\mathbb{Z}\times\mathbb{S}^1$*

$$\lambda(m,\theta) = \mathrm{e}^{im\theta}\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\sigma(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\tau^*(l,.))(-m)}$$

where λ is the symbol of $T_\sigma T_\tau$.

Proof. Assume $T_\sigma\colon L^{p_1}(\mathbb{Z}) \longrightarrow L^{p_2}(\mathbb{Z}),\ 1\le p_1,p_2<\infty$, be a nuclear operator, $T_\tau\colon L^{p_2}(\mathbb{Z}) \longrightarrow L^{p_3}(\mathbb{Z}),\ 1\le p_3<\infty$ is a bounded operator and λ be the symbol of $T_\tau T_\sigma$. Then for all $f\in L^{p_2}(\mathbb{Z}), g\in L^{p_3'}(\mathbb{Z})$

$$\sum_{l=-\infty}^{\infty}(T_\lambda f)(l)\overline{g(l)} = \sum_{l=-\infty}^{\infty}(T_\tau T_\sigma f)(l)\overline{g(l)} = \sum_{l=-\infty}^{\infty}(T_\sigma f)(l)\overline{(T_{\tau^*}g)(l)}.$$

So,

$$\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-il\theta}\lambda(l,\theta)(\mathcal{F}_{\mathbb{Z}}f)(\theta)\mathrm{d}\theta\overline{g(l)} = \sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-il\theta}\sigma(l,\theta)(\mathcal{F}_{\mathbb{Z}}f)(\theta)\mathrm{d}\theta\overline{(T_{\tau^*}g)(l)}.$$

Now, let $m,n\in\mathbb{Z}$ and for all $l\in\mathbb{Z}$,

$$f(l)=f_n(l)=\begin{cases}0 & \text{if}\quad l\neq n,\\ 1 & \text{if}\quad l=n,\end{cases}\qquad g(l)=g_m(l)=\begin{cases}0 & \text{if}\quad l\neq m,\\ 1 & \text{if}\quad l=m.\end{cases}$$

Then we get,

$$\int_{-\pi}^{\pi}\mathrm{e}^{-im\theta}\lambda(m,\theta)\mathrm{e}^{in\theta}\mathrm{d}\theta = \sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\mathrm{e}^{-il\theta}\sigma(l,\theta)\mathrm{e}^{in\theta}\mathrm{d}\theta\overline{\int_{-\pi}^{\pi}\mathrm{e}^{-il\theta'}\tau^*(l,\theta')\mathrm{e}^{im\theta'}\mathrm{d}\theta'}$$

and

$$(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-im\cdot}\lambda(m,.))(-n)=\sum_{l=-\infty}^{\infty}(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\sigma(l,.))(-n)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\tau^*(l,.))(-m)}.$$

So,

$$\begin{aligned}\mathrm{e}^{-im\theta}\lambda(m,\theta)&=\sum_{n=-\infty}^{\infty}\mathrm{e}^{in\theta}(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-im\cdot}\lambda(m,.))(-n)\\&=\sum_{n=-\infty}^{\infty}\mathrm{e}^{in\theta}\sum_{l=-\infty}^{\infty}(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\sigma(l,.))(-n)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\tau^*(l,.))(-m)}\\&=\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\sigma(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\tau^*(l,.))(-m)}.\end{aligned}$$

Then,

$$\lambda(m,\theta)=\mathrm{e}^{im\theta}\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\sigma(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\tau^*(l,.))(-m)}.$$

Now, since T_σ is a nuclear operator and T_τ is a bounded operator, by applying Theorem 2.3, the proof will be complete. □

Corollary 4.2. *Let σ,τ be two measurable functions on $\mathbb{Z}\times\mathbb{S}^1$ such that the pseudo-differential operator $T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_2}(\mathbb{Z})$, $1\le p_1,p_2<\infty$, is a nuclear operator and $T_\sigma\colon L^{p_2}(\mathbb{Z})\longrightarrow L^{p_3}(\mathbb{Z})$, $1\le p_3<\infty$ is a bounded operator. Then $T_\tau T_\sigma\colon L^{p_1}(\mathbb{Z})\longrightarrow L^{p_3}(\mathbb{Z})$ is nuclear and for every $(m,\theta)\in\mathbb{Z}\times\mathbb{S}^1$*

$$\lambda(m,\theta)=\mathrm{e}^{im\theta}\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\tau(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\sigma^*(l,.))(-m)}$$

where λ is the symbol of $T_\tau T_\sigma$.

Corollary 4.3. *Let σ be a measurable functions on $\mathbb{Z}\times\mathbb{S}^1$ such that the pseudo-differential operator T_σ: $L^2(\mathbb{Z})\longrightarrow L^2(\mathbb{Z})$ is a nuclear operator. Then T_σ is normal if and only if for ll $(m,\theta)\in\mathbb{Z}\times\mathbb{S}^1$*

$$\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\sigma(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\sigma^*(l,.))(-m)}=\sum_{l=-\infty}^{\infty}\mathrm{e}^{-il\theta}\sigma^*(l,\theta)\overline{(\mathcal{F}_{\mathbb{S}^1}\mathrm{e}^{-il\cdot}\sigma(l,.))(-m)}.$$

Acknowledgements. The author is grateful to the referees for the comments on the first version of the paper.

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