Math. Model. Nat. Phenom. 13 (2018) 38
https://doi.org/10.1051/mmnp/2018030

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

HEAT KERNELS AND GREEN FUNCTIONS OF SUB-LAPLACIANS ON HEISENBERG GROUPS WITH MULTI-DIMENSIONAL CENTER[☆]

Shahla Molahajloo[1,*] and M.W. Wong[2]

Abstract. We compute the sub-Laplacian on the Heisenberg group with multi-dimensional center. By taking the inverse Fourier transform with respect to the center, we get the parametrized twisted Laplacians. Then by means of the special Hermite functions, we find the eigenfunctions and the eigenvalues of the twisted Laplacians. The explicit formulas for the heat kernels and Green functions of the twisted Laplacians can then be obtained. Then we give an explicit formula for the heat kernal and Green function of the sub-Laplacian on the Heisenberg group with multi-dimensional center.

Mathematics Subject Classification. 47G30

Received November 5, 2017. Accepted January 31, 2018.

1. Introduction

The Heisenberg group is the simplest non-commutative nilpotent Lie group. Analysis on the Heisenberg group is a subject of continuing interest in various areas of mathematics from partial differential equations to geometry to number theory.

The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ equipped with the binary operation $\cdot$ given by

$$(z,t)\cdot(z',t') = \left(z+z', t+t' + \frac{1}{2}\sum_{j=1}^{n}(x_j y_j' - x_j' y_j)\right),$$

for all $z=(x,y)$, $z'=(x',y')$ in $\mathbb{R}^n \times \mathbb{R}^n$ and t,t' in $\mathbb{R}$. The center Z of the Heisenberg group $\mathbb{H}^n$ is the one-dimensional subgroup given by

$$Z = \{(0,0,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : t \in \mathbb{R}\}.$$

[☆] This research has been supported by the Natural Sciences and Engineering Research Council of Canada under Discovery Grant 0008562.

Keywords and phrases: Heisenberg group with multi-dimensional center, sub-Laplacian, twisted Laplacians, λ-Wigner transforms, λ-Weyl transforms, heat kernels, Green functions.

[1] Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Gava Zang P.O. Box 45195-1159, Zanjan 45137-66731, Iran.
[2] Department of Mathematics and Statistics, York University, 4700 Keele Street, Ontario, Toronto M3J 1P3, Canada.

[*] Corresponding author: smollaha@gmail.com

In this paper, we look at a class of Heisenberg groups with multi-dimensional center related to partial differential equations. The Heisenberg group with a one-dimensional center, well known as a model for a hypoelliptic partial differential operator, suggests that we can envisage Heisenberg groups with multi-dimensional center to be models for hypoelliptic partial differential operators with higher complexities. To do this, we consider $n \times n$ orthogonal matrices $B_1, B_2, \ldots, B_m$ such that

$$B_j^{-1} B_k = -B_k^{-1} B_j, \quad j \neq k.$$

Then we define the Heisenberg group $\mathbb{G}$ with multi-dimensional center $\mathbb{G}$ to be the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ with the binary operation $\cdot$ defined by

$$(z, t) \cdot (z', t') = \left(z + z', t + t' + \frac{1}{2}[z, z'] \right)$$

for all (z, t) and (z', t') in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, where $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $z' = (x', y') \in \mathbb{R}^n \times \mathbb{R}^n$, $t, t' \in \mathbb{R}^m$ and $[z, z'] \in \mathbb{R}^m$ is defined by

$$[z, z']_j = x' \cdot B_j y - x \cdot B_j y', \quad j = 1, 2, \ldots, m.$$

The center of the Heisenberg group $\mathbb{G}$ with multi-dimensional center is of dimension m and is given by $\{(0, 0, t) : t \in \mathbb{R}^m\}$.

In fact, $\mathbb{G}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $\mathrm{d}z\, \mathrm{d}t$. For more details, see [7]. Moreover, Heisenberg groups with multi-dimensional center are special cases of the so-called Heisenberg type groups or H-type groups in [2, 4, 5]. The geometric properties of the H-type group are in, *e.g.*, [5]. Note that if we let $m = 1$ and $B_1 = -I_n$, where I_n is the $n \times n$ identity matrix, then we get the ordinary Heisenberg group $\mathbb{H}^n$.

It is well known from [8] that Weyl transforms have intimate connections with analysis on the Heisenberg group and with the so-called twisted Laplacian. We begin with a recall of the basic definitions and properties of Weyl transforms and Wigner transforms in, for instance, the book [8].

Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi)\, \mathrm{d}x\, \mathrm{d}\xi, \quad f, g \in L^2(\mathbb{R}^n),$$

where $W(f, g)$ is the Wigner transform of f and g defined by

$$W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} \mathrm{d}p, \quad x, \xi \in \mathbb{R}^n.$$

Closely related to the Wigner transform $W(f, g)$ of f and g in $L^2(\mathbb{R}^n)$ is the Fourier–Wigner transform $V(f, g)$ given by

$$V(f, g)(q, p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq \cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} \mathrm{d}y, \quad q, p \in \mathbb{R}^n.$$

It is easy to see that that

$$W(f, g) = V(f, g)^\wedge$$

for all f and g in $L^2(\mathbb{R}^n)$, where $\wedge$ denotes the Fourier transform given by

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^n,$$

for all F in $L^1(\mathbb{R}^n)$.

In Section 2, We recall some of the results from [7]. We define the Schrödinger representations of the Heisenberg group $\mathbb{G}$ with multi-dimensional center. Then we define the λ-Wigner and λ-Weyl transform related to the Heisenberg group $\mathbb{G}$. The Moyal identity for the λ-Wigner transform and Hilbert–Schmidt properties of the λ-Weyl transform are given [1]. In Section 3, the sub-Laplacian on the Heisenberg group $\mathbb{G}$ is computed. Then by taking the inverse Fourier transform with respect to the center, we get the parametrized twisted Laplacians. In Sections 4–7, the heat kernels and the Green functions of the parametrized twisted Laplacians and the sub-Laplacian on the Heisenberg group $\mathbb{G}$ with multi-dimensional center are obtained.

This paper extends the results in [3, 9] and Chapters 17–23 in [10] from the ordinary Heisenberg group to Heisenberg groups with multi-dimensional center.

2. Schrödinger representations of Heisenberg groups with multi-dimensional center and λ-Weyl transforms

Let

$$\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$$

and let $\lambda \in \mathbb{R}^{m*}$. We define the Schrödinger representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ by

$$(\pi_\lambda(q,p,t)\varphi)(x) = e^{i\lambda\cdot t} e^{iq\cdot B_\lambda(x+p/2)} \varphi(x+p), \quad x \in \mathbb{R}^n,$$

for all $\varphi \in L^2(\mathbb{R}^n)$ and $(q,p,t) \in \mathbb{G}$, where $z = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $B_\lambda = \sum_{j=1}^m \lambda_j B_j$. For all $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\varphi \in L^2(\mathbb{R}^n)$, if we define $\rho(q,p)\varphi$ to be the function on $\mathbb{R}^n$ by

$$(\pi_\lambda(q,p)\varphi)(x) = e^{iq\cdot B_\lambda(x+p/2)} \varphi(x+p), \quad x \in \mathbb{R}^n,$$

then

$$\pi_\lambda(q,p,t) = e^{i\lambda\cdot t} \pi_\lambda(q,p), \quad t \in \mathbb{R}^m.$$

We have the Stone–von Neumann theorem stating that any irreducible and unitary representation of $\mathbb{G}$ on a Hilbert space that is non-trivial on the center is equivalent to some π_λ. More precisely, we have the following result.

Theorem 2.1. *Let Π_λ be an irreducible and unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ such that $\Pi_\lambda(0,0,t) = e^{i\lambda\cdot t} I$, for some $\lambda \in \mathbb{R}^m$, where I is the identity operator on $\mathcal{H}$. Then Π_λ is unitarily equivalent to π_λ.*

We define the λ-Fourier–Wigner transform $V^\lambda(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ to be the function on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \left(\pi_\lambda(q,p)f, g\right), \quad q,p \in \mathbb{R}^n.$$

In fact,

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(B_\lambda^t q)\cdot x} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}\,\mathrm{d}x, \quad q,p\in\mathbb{R}^n.$$

It is easy to see that the λ-Fourier–Wigner transform is related to the ordinary Fourier–Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p), \quad q,p\in\mathbb{R}^n.$$

Note that

$$V^\lambda(f,g)(q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p\in\mathbb{R}^n.$$

Now, we define the λ-Wigner transform $W^\lambda(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ to be the Fourier transform of $V^\lambda(f,g)$. In fact, the λ-Wigner transform has the form

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ip\cdot\xi} f\left(\frac{B_\lambda^t x}{|\lambda|^2}+\frac{p}{2}\right)\overline{g\left(\frac{B_\lambda^t x}{|\lambda|^2}-\frac{p}{2}\right)}\,\mathrm{d}p$$

for all x and ξ in $\mathbb{R}^n$, and and it is related to the ordinary Wigner transform by

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n} W(f,g)\left(\frac{B_\lambda^t x}{|\lambda|^2},\xi\right)$$

for all x,ξ in $\mathbb{R}^n$. Moreover,

$$W^\lambda(f,g) = \overline{W^\lambda(g,f)}, \quad f,g\in L^2(\mathbb{R}^n).$$

Let $\sigma\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ and $f\in\mathcal{S}(\mathbb{R}^n)$. Then we define the λ-Weyl transform $W_\sigma^\lambda f$ of f corresponding to the symbol σ by

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(x,\xi)W^\lambda(f,g)(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi,$$

for all $g\in\mathcal{S}(\mathbb{R}^n)$. Therefore using Parseval's identity, we have

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{\sigma}(q,p)V^\lambda(f,g)(q,p)\,\mathrm{d}q\,\mathrm{d}p.$$

Hence, formally, we can write

$$\left(W_\sigma^\lambda f\right)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{\sigma}(q,p)\left(\pi_\lambda(q,p)f\right)(x)\,\mathrm{d}q\,\mathrm{d}p, \quad x\in\mathbb{R}^n.$$

Proposition 2.2. *Let $\sigma\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is given by*

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where W_{σ_λ} is the ordinary Weyl transform corresponding to the symbol σ_λ given by

$$\sigma_\lambda(x,\xi) = \sigma(B_\lambda x,\xi), \quad x,\xi\in\mathbb{R}^n.$$

Proposition 2.3. *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is a Hilbert–Schmidt operator with kernel*

$$k_\sigma^\lambda(x,p) = (\mathcal{F}_2\sigma)\left(B_\lambda\left(\frac{x+p}{2}\right), p-x\right), \quad x,p \in \mathbb{R}^n,$$

where $\mathcal{F}_2\sigma$ is the ordinary Fourier transform of σ with respect to the second variable, i.e.,

$$(\mathcal{F}_2\sigma)(x,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\sigma(x,\xi)\,\mathrm{d}\xi, \quad x,p \in \mathbb{R}^n.$$

Moreover,

$$\|W_\sigma^\lambda\|_{HS} = |\lambda|^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)},$$

where $\|W_\sigma^\lambda\|_{HS}$ is the Hilbert–Schmidt norm of W_σ^λ.

Let F and G be functions in $L^2(\mathbb{R}^{2n})$. Then the λ-twisted convolution $F *_\lambda G$ of F and G is the function on $\mathbb{R}^{2n}$ defined by

$$(F *_\lambda G)(z) = \int_{\mathbb{R}^{2n}} F(z-w)G(w)e^{\frac{i}{2}\lambda\cdot[z,w]}\,\mathrm{d}w, \quad z \in \mathbb{R}^{2n},$$

provided that the integral exists.

Theorem 2.4. *Let σ and τ be in $L^2(\mathbb{R}^{2n})$. Then*

$$W_\sigma^\lambda W_\tau^\lambda = W_\omega^\lambda,$$

where $\omega \in L^2(\mathbb{R}^{2n})$ and $\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_\lambda \hat{\tau})$.

We have the following Moyal identity for the λ-Wigner transform and λ-Fourier–Wigner transform.

Proposition 2.5. *For all f_1, f_2, g_1, g_2 in $L^2(\mathbb{R}^n)$,*

$$\left(W^\lambda(f_1,g_1), W^\lambda(f_2,g_2)\right) = |\lambda|^{-n}\,(f_1,f_2)\,\overline{(g_1,g_2)}$$

and

$$\left(V^\lambda(f_1,g_1), V^\lambda(f_2,g_2)\right) = |\lambda|^{-n}\,(f_1,f_2)\,\overline{(g_1,g_2)}.$$

3. Sub-Laplacians on Heisenberg groups with multi-dimensional center

Let $\mathfrak{g}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{G}$. For $j = 1,2,\ldots,m$, let $\gamma_{1,j} : \mathbb{R} \to \mathbb{G}$ and $\gamma_{2,j} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{1,j}(r) = (re_j, 0, 0)$$

and

$$\gamma_{2,j}(r) = (0, re_j, 0)$$

for all $r \in \mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^n$. For all $k = 1, 2, \ldots, m$, let $\gamma_{3,k} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{3,k}(r) = (0, 0, re_k)$$

for all $r \in \mathbb{R}$, where e_k is the standard unit vector in $\mathbb{R}^m$. Then we define the left-invariant vector fields X_j, Y_j and T_k, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$, on $\mathbb{G}$ as follows. Let $f \in C^\infty(\mathbb{G})$. Then for all $j = 1, 2, \ldots, n$, we define X_j and Y_j by

$$\begin{aligned} (X_j f)(x,y,t) &= \frac{\mathrm{d}}{\mathrm{d}s} f((x,y,t)\cdot\gamma_{1j}(s))\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} f\left(x + se_j, y, \left(t_k + \frac{1}{2}(B_k y, se_k)\right)_{k=1}^{m}\right)\Big|_{s=0} \\ &= \frac{\partial f}{\partial x_j}(x,y,t) + \frac{1}{2}\sum_{k=1}^{m}(B_k y, e_j)\frac{\partial f}{\partial t_k}(x,y,t) \end{aligned}$$

and

$$\begin{aligned} (Y_j f)(x,y,t) &= \frac{\mathrm{d}}{\mathrm{d}s} f((x,y,t)\cdot\gamma_{2j}(s))\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} f\left(x, y + se_j, \left(t_k - \frac{1}{2}(x, sB_k e_j)\right)_{k=1}^{m}\right)\Big|_{s=0} \\ &= \frac{\partial f}{\partial y_j}(x,y,t) - \frac{1}{2}\sum_{k=1}^{m}(x, B_k e_j)\frac{\partial f}{\partial t_k}(x,y,t) \end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$. Similarly, for $k = 1, 2, \ldots, m$, the function $T_k f$ is defined by

$$\begin{aligned} (T_k f)(x,y,t) &= \frac{\mathrm{d}}{\mathrm{d}s} f((x,y,t)\cdot\gamma_{3k}(s))\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} f(x, y, t + se_k)\Big|_{s=0} \\ &= \frac{\partial f}{\partial t_k}(x,y,t) \end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$. We can easily check that

$$[X_i, Y_j] = -\sum_{k=1}^{m}(B_k)_{ij} T_k, \quad i, j = 1, 2, \ldots, n,$$

and the other commutators are zero. Therefore $\mathbb{G}$ is a nipotenet Lie group of step two.

Theorem 3.1. *The Lie algebra* $\mathfrak{g}$ *is generated by* $\{X_i, Y_j, [X_i, Y_j] : i, j = 1, 2, \ldots, n\}$.

Proof. It is enough to show that

$$\mathrm{span}\{T_1, T_2, \ldots, T_m\} = \mathrm{span}\{[X_i, X_j] : i, j = 1, 2, \ldots, n\}.$$

Let

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_m \end{pmatrix}$$

and

$$Z = \begin{pmatrix} [X_1, Y_1] \\ [X_1, Y_2] \\ \vdots \\ [X_1, Y_n] \\ [X_2, Y_1] \\ [X_2, Y_2] \\ \vdots \\ [X_n, Y_n]. \end{pmatrix}$$

For $1 \leq k \leq m$ and $1 \leq i, j \leq n$, let $(B_k)_{ij}$ be the entry of the matrix B_k in the ith row and jth column. Consider the $n^2 \times m$ matrix

$$C = \begin{bmatrix} (B_1)_{11} & (B_2)_{11} & \dots & (B_m)_{11} \\ (B_1)_{12} & (B_2)_{12} & \dots & (B_m)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (B_1)_{1n} & (B_2)_{1n} & \dots & (B_m)_{1n} \\ (B_1)_{21} & (B_2)_{21} & \dots & (B_m)_{21} \\ (B_1)_{22} & (B_2)_{22} & \dots & (B_m)_{22} \\ \vdots & \vdots & \vdots & \vdots \\ (B_1)_{nn} & (B_2)_{nn} & \dots & (B_m)_{nn} \end{bmatrix}.$$

Then $CT = -Z$. Since C has full column rank, it follows that there exists a an $m \times n^2$ matrix (left inverse) D such that

$$DC = I$$

where I is the $m \times m$ identity matrix. Therefore

$$T = DY.$$

□

We can now define the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ by

$$\mathcal{L} = -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

Explicitly,

$$\mathcal{L} = -\Delta_x - \Delta_y - \frac{1}{4}\left(|x|^2+|y|^2\right)\Delta_t + \sum_{k=1}^{m}\sum_{j=1}^{n}\left\{-(B_k y, e_j)\frac{\partial}{\partial x_j} + (x, B_k e_j)\frac{\partial}{\partial y_j}\right\}\frac{\partial}{\partial t_k}.$$

By taking the inverse Fourier transform of the sub-laplacian $\mathcal{L}$ with respect to t, we get parametrized twisted Laplacians L^λ, $\lambda \in \mathbb{R}^m$, given by

$$L^\lambda = -\Delta_x - \Delta_y + \frac{1}{4}\left(|x|^2+|y|^2\right)|\lambda|^2 - i\sum_{j=1}^{n}\left\{-(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j}\right\}.$$

4. Spectral analysis of λ-twisted Laplacians

For $k = 0, 1, 2, \dots,$ the Hermite function of order k is the function e_k on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

For any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, we define the function e_α on $\mathbb{R}^n$ by $e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}$. We fix a nonzero vector $\lambda \in \mathbb{R}^m$. Let α and β be multi-indices in $(\mathbb{N}\cup\{0\})^n$. Then we define the special Hermite function $e^\lambda_{\alpha,\beta}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$e^\lambda_{\alpha,\beta}(q,p) = |\lambda|^{n/2} V_\lambda(e_\alpha, e_\beta)\left(\frac{q}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p\right), \quad q, p \in \mathbb{R}^n.$$

In fact, $e^\lambda_{\alpha,\beta}$ is given by

$$e^\lambda_{\alpha,\beta}(q,p) = |\lambda|^{n/2} V(e^\lambda_\alpha, e^\lambda_\beta)(q, \sqrt{|\lambda|}p), \quad q, p \in \mathbb{R}^n,$$

where

$$e^\lambda_\alpha(x) = |\lambda|^{n/4} e_\alpha(\sqrt{|\lambda|}x), \quad x \in \mathbb{R}^n.$$

Using the fact that $\{e_k : k = 0, 1, \dots\}$ is an orthonormal basis for $L^2(\mathbb{R})$ and the Moyal identity for the λ-Fourier–Wigner transform, we get the following result.

Proposition 4.1. *$\{e^\lambda_{\alpha,\beta} : \alpha, \beta \in (\mathbb{N}\cup\{0\})^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$.*

For $l = 1, 2, \dots, n$, we define the linear partial differential operators Z^λ_l and $\bar{Z}^\lambda_l$ by

$$Z^\lambda_l = \frac{1}{|\lambda|}\sum_{j=1}^{n}(B_\lambda)_{jl}\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_l} + \frac{1}{2}(B^t_\lambda x)_l - \frac{i|\lambda|}{2}y_l$$

and

$$\bar{Z}_l^\lambda = \frac{1}{|\lambda|}\sum_{l=1}^{n}(B_\lambda)_{jl}\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_l} - \frac{1}{2}(B_\lambda^t x)_l - \frac{i|\lambda|}{2}y_l.$$

Then

$$L^\lambda = -\frac{1}{2}\sum_{l=1}^{n}(Z_l^\lambda \bar{Z}_l^\lambda + \bar{Z}_l^\lambda Z_l^\lambda).$$

In the following lemma, e_l is the standard unit vector in $\mathbb{R}^n$.

Lemma 4.2. *For all* $l = 1, 2, \ldots, n$*, and for all multi-indices* α *and* β*,*

(i) $Z_l^\lambda e_{\alpha,\beta}^\lambda = i|\lambda|^{n/2}(2\beta_l)^{1/2}e_{\alpha,\beta-e_l}^\lambda, \quad \beta_l \neq 0,$
(ii) $\bar{Z}_l^\lambda e_{\alpha,\beta}^\lambda = i|\lambda|^{n/2}(2\beta_l + 2)^{1/2}e_{\alpha,\beta+e_l}^\lambda.$

Theorem 4.3. *For all multi-indices* α *and* β *in* $(\mathbb{N}\cup\{0\})^n$*,*

$$L^\lambda e_{\alpha,\beta}^\lambda = |\lambda|^n(2|\beta| + n)e_{\alpha,\beta}^\lambda.$$

5. Heat kernels of the λ-twisted Laplacians

In this section, we are interested in finding the heat kernel of L^λ, which is the kernel of the integral operator $e^{-\tau L^\lambda}$, $\tau > 0$. We need the following theorem that follows from the Moyal identity and Theorem 2.4.

Theorem 5.1. *For all multi-indices* α, β, μ *and* ν

$$e_{\alpha,\gamma}^\lambda *_\lambda e_{\beta,\nu}^\lambda = (2\pi)^{n/2}|\lambda|^{-n/2}\delta_{\beta,\gamma}e_{\alpha,\nu}^\lambda,$$

where $\delta_{\beta,\alpha}$ *is the Kronecker delta function.*

Theorem 5.2. *For all* $f \in L^2(\mathbb{R}^{2n})$ *and all* $\tau > 0$*,*

$$e^{-\tau L^\lambda} f = k_\tau *_{-\lambda} f,$$

where

$$k_\tau^\lambda(z) = (2\pi)^{-n}\frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n}e^{-\frac{1}{4}|\lambda||z|^2\coth(|\lambda|^n\tau)}$$

for all $z \in \mathbb{R}^{2n}$*.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^{2n})$ and $\tau > 0$. Then by Theorem 4.3,

$$e^{-\tau L^\lambda} f = \sum_\beta e^{-\tau|\lambda|^n(2|\beta|+n)}\sum_\alpha \left(f, e_{\alpha,\beta}^\lambda\right)_{L^2(\mathbb{R}^{2n})} e_{\alpha,\beta}^\lambda.$$

By Theorem 5.1

$$f *_\lambda e_{\beta,\beta}^\lambda = \sum_\alpha\sum_\gamma \left(f, e_{\alpha,\gamma}^\lambda\right)_{L^2(\mathbb{R}^{2n})} e_{\alpha,\gamma}^\lambda *_\lambda e_{\beta,\beta}^\lambda$$

$$= (2\pi)^{n/2}|\lambda|^{-n/2}\sum_{\alpha}\sum_{\gamma}\left(f, e^{\lambda}_{\alpha,\gamma}\right)_{L^2(\mathbb{R}^{2n})}\delta_{\gamma,\beta}e^{\lambda}_{\alpha,\beta}$$
$$= (2\pi)^{n/2}|\lambda|^{-n/2}\sum_{\alpha}\left(f, e^{\lambda}_{\alpha,\beta}\right)_{L^2(\mathbb{R}^{2n})}e^{\lambda}_{\alpha,\beta}$$

for all $\beta \in (\mathbb{N}\cup\{0\})^n$. Thus,

$$e^{-\tau L^{\lambda}}f = (2\pi)^{-n/2}|\lambda|^{n/2}\sum_{\beta}e^{-\tau|\lambda|^n(2|\beta|+n)}f *_{\lambda} e^{\lambda}_{\beta,\beta}$$
$$= (2\pi)^{-n/2}|\lambda|^{n/2}\sum_{\beta}e^{-\tau|\lambda|^n(2|\beta|+n)}e^{\lambda}_{\beta,\beta} *_{-\lambda} f.$$

To compute $\sum_{\beta}e^{-t|\lambda|^n(2|\beta|+n)}e^{\lambda}_{\beta,\beta}$, we use Mehler's formula. In fact,

$$e^{\lambda}_{\beta,\beta}(q,p) = |\lambda|^{n/2}\prod_{j=1}^{n}e_{\beta_j,\beta_j}\left(\frac{(B^t_{\lambda}q)_j}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p_j\right), \quad q,p\in\mathbb{R}^n,$$

where e_{β_j,β_j} is the ordinary Fourier–Wigner transform of the Hermite functions e_{β_j}. Hence for all $(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$\sum_{\beta}e^{-\tau|\lambda|^n(2|\beta|+n)}e^{\lambda}_{\beta,\beta}(q,p) = |\lambda|^{n/2}\prod_{j=1}^{n}\left(\sum_{\beta_j=0}^{\infty}e^{-(2\beta_j+1)|\lambda|^n\tau}e_{\beta_j,\beta_j}\left(\frac{(B^t_{\lambda}q)_j}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p_j\right)\right).$$

Now, by (23.7) in [8],

$$\sum_{\beta_j=0}^{\infty}e^{-\tau|\lambda|^n(2\beta_j+1)}e_{\beta_j,\beta_j}(q_j,p_j) = \frac{1}{\sqrt{2\pi}}\frac{1}{2\sinh(|\lambda|^n\tau)}e^{-\frac{1}{4}(|q_j|^2+|p_j|^2)\coth(\tau|\lambda|^n)}$$

for all (q_j,p_j) in $\mathbb{R}\times\mathbb{R}$. So,

$$\sum_{\beta}e^{-\tau|\lambda|^n(2|\beta|+n)}e^{\lambda}_{\beta,\beta}(q,p) = |\lambda|^{n/2}(2\pi)^{-n/2}\frac{1}{[2\sinh(|\lambda|^n\tau)]^n}e^{-\frac{1}{4}|\lambda||z|^2\coth(\tau|\lambda|^n)}.$$

□

Therefore the heat kernel κ^{λ}_{τ} of the λ-twisted Laplacian $L^{\lambda}$for allis given by

$$\kappa^{\lambda}_{\tau}(z,w) = k^{\lambda}_{\tau}(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}$$
$$= (2\pi)^{-n}\frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n}e^{-\frac{1}{4}|\lambda||z-w|^2\coth(\tau|\lambda|^n)}e^{-\frac{i}{2}\lambda.[z,w]}$$

for all z and w in $\mathbb{R}^{2n}$.

6. Green functions of λ-twisted Laplacians

In this section, we compute for all $\lambda \in \mathbb{R}^{m*}$ the kernel G^λ of the inverse $(L^\lambda)^{-1}$ of L^λ, which is known as the Green function of the L^λ. The Green function G^λ is related to the heat kernel κ_τ^λ of L^λ by

$$G^\lambda(z,w) = \int_0^\infty \kappa_\tau^\lambda(z,w)\,\mathrm{d}\tau, \quad z,w \in \mathbb{R}^{2n}.$$

Let

$$g^\lambda(z) = \int_0^\infty k_\tau^\lambda(z)\,\mathrm{d}\tau, \quad z \in \mathbb{R}^{2n}.$$

Then the Green funciton G^λ of L^λ is given by

$$G^\lambda(z,w) = e^{-\frac{i}{2}\lambda\cdot[z,w]} g^\lambda(z-w)$$

for all z, w in $\mathbb{R}^{2n}$.

Lemma 6.1. *For all z in $\mathbb{R}^{2n}$,*

$$g^\lambda(z) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{n-1}}\; K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right),$$

where $K_{(n-1)/2}$ is the modified Bessel function of order $\frac{n-1}{2}$ given by

$$K_{(n-1)/2}(x) = \int_0^\infty e^{-x\cosh\delta}\cosh((n-1)\delta/2)\,\mathrm{d}\delta, \quad x > 0.$$

Proof. Let $z \in \mathbb{R}^{2n}$. Then

$$g^\lambda(z) = (2\pi)^{-n}\left(\frac{|\lambda|}{2}\right)^n \int_0^\infty \frac{1}{\sinh^n(|\lambda|^n\tau)} e^{-\frac{1}{4}|\lambda||z|^2\coth(|\lambda|^n\tau)}\,\mathrm{d}\tau.$$

By a change of variable from τ to u where $u = \coth(|\lambda|^n\tau)$, we get

$$g^\lambda(z) = (4\pi)^{-n}\int_1^\infty (u^2-1)^{n/2-1} e^{-\frac{1}{4}|\lambda||z|^2 u}\,\mathrm{d}u, \quad z \in \mathbb{R}^{2n}.$$

By the formula at page 250 of the book [6] to the effect that

$$\int_1^\infty (u^2-1)^{\gamma-1} e^{-\mu u}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\gamma-\frac{1}{2}}\;\Gamma(\gamma)\;K_{\gamma-\frac{1}{2}}(\mu),$$

where K_ν is the modified Bessel function of order ν given by

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)\,\mathrm{d}t, \quad x > 0.$$

So, for $\gamma = \frac{n}{2}$ and $\mu = \frac{1}{4}|\lambda||z|^2$, we have for all $z \in \mathbb{R}^{2n}$,

$$\int_1^\infty (u^2-1)^{n/2-1} e^{-\frac{1}{4}|\lambda||z|^2 u}\,\mathrm{d}u = \frac{2^{(3n-2)/2}}{\sqrt{\pi}} \frac{\Gamma(n/2)}{(|\lambda||z|^2)^{(n-1)/2}}\; K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right).$$

Hence we get the formula for the Green function, as asserted. □

Hence by (6.1), we get the following theorem.

Theorem 6.2. *The Green function G_λ of the λ-twisted Laplacian L^λ is given by*

$$G^\lambda(z,w) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} e^{-\frac{i}{2}\lambda\cdot[z,w]} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{(n-1)}}\; K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right)$$

for all $z, w \in \mathbb{R}^{2n}$.

7. Heat kernels and Green functions of sub-Laplacians

In this section, we use the heat kernel κ_τ^λ of the λ-twisted Laplacian to find the heat kernel of the sub-Laplacian by taking the Fourier transform with respect to the parameter λ. To do this, we need some preparation. The group convolution of two measurable functions f and g on $\mathbb{G}$ is defined by

$$(f *_{\mathbb{G}} g)(z,t) = \int_{\mathbb{G}} f((z,t)\cdot_{\mathbb{G}}(w,s)^{-1}) g(w,s)\,\mathrm{d}w\,\mathrm{d}s, \quad z \in \mathbb{R}^{2n}, t \in \mathbb{R}^m,$$

if the integral exists. Moreover, we denote by f_λ the ordinary Fourier transform of f with respect to the t variable evaluated at the point $\lambda \in \mathbb{R}^m$. More precisely,

$$f_\lambda(z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} f(z,t)\,\mathrm{d}t, \quad z \in \mathbb{R}^{2n}.$$

We need the following theorem.

Theorem 7.1. *Let f and g be functions in $L^1(\mathbb{G})$. Then for all nonzero $\lambda \in \mathbb{R}^m$,*

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{m/2} f_\lambda *_{-\lambda} g_\lambda.$$

Proof. For all $z \in \mathbb{R}^{2n}$,

$$\begin{aligned}(f *_{\mathbb{G}} g)_\lambda &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} (f *_\lambda g)(z,t)\mathrm{d}t \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} \left(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} f\left(z-w, t-s-\frac{1}{2}[z,w]\right) g(w,s)\,\mathrm{d}w\,\mathrm{d}s \right) \mathrm{d}t.\end{aligned}$$

Let $t' = t - \frac{1}{2}[z,w]$. Then

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} e^{-it'\cdot\lambda} f(z-w, t'-s) g(w,s) e^{-\frac{i}{2}\lambda\cdot[z,w]}\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}t'.$$

On the other hand, for all z in $\mathbb{R}^{2n}$, we get

$$\begin{aligned}(f_\lambda *_{-\lambda} g_\lambda)(z) &= \int_{\mathbb{R}^{2n}} f_\lambda(z-w) g_\lambda(w) e^{-\frac{i}{2}\lambda\cdot[z,w]}\,\mathrm{d}w \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^{2n}} \left\{ \int_{\mathbb{R}^m} f(z-w, \cdot - s) g(w,s)\,\mathrm{d}s \right\}^{\wedge} (\lambda)\; e^{-\frac{i}{2}\lambda\cdot[z,w]}\,\mathrm{d}w \\ &= (2\pi)^{-m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} e^{-it\cdot\lambda} f(z-w, t-s) g(w,s) e^{-\frac{i}{2}\lambda\cdot[z,w]}\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

and the proof is complete. □

Now, we consider the initial-value problem given by

$$\begin{cases} \dfrac{\partial u}{\partial \tau}(z,t,\tau) = -\left(\mathcal{L}u\right)(z,t,\tau), & z\in\mathbb{R}^{2n}, t\in\mathbb{R}^m, \tau>0, \\ u(z,t,0) = f(z,t), & z\in\mathbb{R}^{2n}, t\in\mathbb{R}^m. \end{cases}$$

By taking the inverse Fourier transform with respect to t and evaluated at λ, we get an initial-value problem for the heat equation governed by the λ-twisted Laplacian L^λ, *i.e.*,

$$\begin{cases} \dfrac{\partial u_\lambda}{\partial \tau}(z,\tau) = -\left(L^\lambda u_\lambda\right)(z,\tau), \\ u_\lambda(z,0) = f_\lambda(z), \end{cases}$$

for all $z\in\mathbb{R}^{2n}$, $\tau>0$ and $\lambda\in\mathbb{R}^{m*}$. By Theorem 5.2,

$$u_\lambda(z,\tau) = \left(k_\tau^\lambda *_{-\lambda} f_\lambda\right)(z), \quad z\in\mathbb{R}^{2n}, \tau>0,$$

for all $\lambda\in\mathbb{R}^{m*}$. Therfore by taking the Fourier transform with respect to λ and evaluated at t, and using Theorem 7.1, we get the solution of the initial-value problem governed by the sub-Laplacian given by

$$u(z,t,\tau) = (2\pi)^{-m/2}\left(f *_{\mathbb{G}} K_\tau\right)(z,t), \quad z\in\mathbb{R}^{2n}, t\in\mathbb{R}^m, \tau>0,$$

where K_τ is the Fourier transform of the heat kernel of k_τ^λ with respect to λ and evaluated at t. So, the heat kernel of $\mathcal{L}$ is given in the following theorem.

Theorem 7.2. *For all f in $L^2(\mathbb{G})$, $e^{-\tau\mathcal{L}}f = f *_{\mathbb{G}} K_\tau$, where*

$$K_\tau(z,t) = (2\pi)^{-(n+m)} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n} e^{-\frac{1}{4}|\lambda||z|^2\coth(|\lambda|^n\tau)}\,\mathrm{d}\lambda$$

for all $(z,t)\in\mathbb{G}$.

Hence the heat kernel κ_τ of $\mathcal{L}$ is given by

$$\kappa_\tau\left((z,t),(w,s)\right) = K_\tau\left(z-w, t-s+\frac{1}{2}[z,w]\right)$$

for all (z,t) and (w,s) in $\mathbb{G}$.

The Green function $\mathcal{G}$ of the sub-Laplacian $\mathcal{L}$ on the Heisenebrg group $\mathbb{G}$ with multi-dimensional center is the kernel of $\mathcal{L}^{-1}$. More precisely, the Green function $\mathcal{G}$ is given by

$$\mathcal{L}^{-1}f = f *_{\mathbb{G}} \mathcal{G}$$

for all suitable functions f on $\mathbb{G}$.

As in the case of the heat kernel of the sub-Laplacian $\mathcal{L}$, the Green function $\mathcal{G}$ is obtained by taking the Fourier transform of G^λ with respect to λ and evaluated at t. Therefore by Lemma 6.1, we have the following theorem.

Theorem 7.3. *The Green function $\mathcal{G}$ of $\mathcal{L}$ is given by*

$$\mathcal{G}(z,t) = \frac{c_n}{|z|^{n-1}} \int_{\mathbb{R}^m} e^{-i\lambda\cdot t} \frac{1}{|\lambda|^{(n-1)/2}}\ K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right) \mathrm{d}\lambda,$$

where

$$c_n = (2\pi)^{-m} \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \Gamma(n/2).$$

References

[1] A. Dasgupta and M.W. Wong, Hilbert–Schmidt and trace class pseudo-differential operators on the Heisenberg group. *J. Pseudo-Differ. Oper. Appl.* **4** (2013) 345–359.

[2] A. Dasgupta and M.W. Wong, Weyl transforms for H-type groups. *J. Pseudo-Differ. Oper. Appl.* **6** (2015) 11–19.

[3] X. Duan, The heat kernel and Green function of the sub-Laplacian on the Heisenberg group, in Pseudo-Differential Operators, Generalized Functions and Asymptotics. Vol. 231 of *Operator Theory: Advances and Applications*. Birkhäuser (2013) 3–12.

[4] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Am. Math. Soc* **258** (1980) 147–153.

[5] A. Korányi, Geometric properties of Heisenberg-type groups. *Adv. Math.* **56** (1986) 28–38.

[6] W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics. Springer-Verlag, Berlin (1964).

[7] S. Molahajloo, Pseudo-differential operators on non-isotropic Heisenberg groups with multi-dimensional centers, in Pseudo-Differential Operators: Groups, Geometry and Applications. *Trends in Mathematics*. Birkhäuser (2017) 15–35.

[8] M.W. Wong, Weyl Transforms. Springer, New York (1998).

[9] M.W. Wong, Weyl transforms, the heat kernel and Green function of a degenerate elliptic operator. *Ann. Glob. Anal. Geom.* **28** (2005) 271–283.

[10] M.W. Wong, Partial Differential Equations: Topics in Fourier Analysis. CRC Press (2014).