Math. Model. Nat. Phenom. 13 (2018) 37
https://doi.org/10.1051/mmnp/2018032

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

GENERALIZED POISSON INTEGRAL AND SHARP ESTIMATES FOR HARMONIC AND BIHARMONIC FUNCTIONS IN THE HALF-SPACE

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Abstract. A representation for the sharp coefficient in a pointwise estimate for the gradient of a generalized Poisson integral of a function f on $\mathbb{R}^{n-1}$ is obtained under the assumption that f belongs to L^p. It is assumed that the kernel of the integral depends on the parameters α and β. The explicit formulas for the sharp coefficients are found for the cases $p=1$, $p=2$ and for some values of α, β in the case $p=\infty$. Conditions ensuring the validity of some analogues of the Khavinson's conjecture for the generalized Poisson integral are obtained. The sharp estimates are applied to harmonic and biharmonic functions in the half-space.

Mathematics Subject Classification. 31B10, 31B05, 31B30

Received November 25, 2017. Accepted November 25, 2017.

1. Background and main results

In the paper [2] (see also [5]) a representation for the sharp coefficient $\mathcal{C}_p(x)$ in the inequality

$$|\nabla v(x)| \le \mathcal{C}_p(x)\big\|v\big\|_p$$

was found, where v is a harmonic function in the half-space $\mathbb{R}^n_+ = \big\{x=(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\big\}$, represented by the Poisson integral with boundary values in $L^p(\mathbb{R}^{n-1})$, $||\cdot||_p$ is the norm in $L^p(\mathbb{R}^{n-1})$, $1 \le p \le \infty$, $x \in \mathbb{R}^n_+$. It was shown that

$$\mathcal{C}_p(x) = \frac{C_p}{x_n^{(n-1+p)/p}}$$

Keywords and phrases: Generalized Poisson integral, two-parametric kernel, sharp estimates, harmonic functions, biharmonic functions.

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and explicit formulas for C_1, C_2 and C_∞ were found. Namely,

$$C_1 = \frac{2(n-1)}{\omega_n}, \quad C_2 = \sqrt{\frac{(n-1)n}{2^n \omega_n}}, \quad C_\infty = \frac{4(n-1)^{(n-1)/2}\, \omega_{n-1}}{n^{n/2}\, \omega_n},$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

In [2] it was shown that the sharp coefficients in pointwise estimates for the absolute value of the normal derivative and the modulus of the gradient of a harmonic function in the half-space coincide for the cases $p = 1$, $p = 2$ as well as for the case $p = \infty$.

Similar results for the gradient and the radial derivative of a harmonic function in the multidimensional ball with boundary values from L^p for $p = 1, 2$ were obtained in [3].

We note that explicit sharp coefficients in the inequality for the first derivative of an analytic function in the half-plane with boundary values of the real-part from L^p were found in [4].

The subjects of papers [2, 3] is closely connected with D. Khavinson problem [1] and conjecture (see [5], Chap. 6) for harmonic functions in a ball. The Khavinson's problem is to find the sharp coefficient in the inequality

$$|\nabla v(x)| \le \mathcal{K}(x) \sup_{|y|<1} |v(y)|, \tag{1.1}$$

where v is a bounded harmonic function in the ball $B = \{x \in \mathbb{R}^3 : |x| < 1\}$. The Khavinson's conjecture is that the sharp coefficient $\mathcal{K}(x)$ in (1.1) and the sharp coefficient $K(x)$ in the inequality

$$\left| \frac{\partial v(x)}{\partial |x|} \right| \le K(x) \sup_{|y|<1} |v(y)|$$

coincide for any $x \in B$.

Thus, the L^1, L^2-analogues of Khavinson's problem were solved in [2, 3] for harmonic functions in the multidimensional half-space and the ball. Also, the L^∞-analogue of Khavinson's problem for harmonic functions in the multidimensional half-space was solved in [2].

In this paper we treat a generalization of some problems considered in our work [2]. Here we consider the generalized Poisson integral in the half-space

$$u(x) = k \int_{\mathbb{R}^{n-1}} \left(\frac{x_n^\alpha}{|y-x|} \right)^\beta f(y')\mathrm{d}y' \tag{1.2}$$

with two parameters, $\alpha \ge 0$ and $\beta > (n-1)(p-1)/p$, where k is a constant, $n > 2$, $f \in L^p(\mathbb{R}^{n-1})$, $1 \le p \le \infty$, $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

In the case $\alpha = 1/n, \beta = n, k = 2/\omega_n$ integral (1.2) coincides with the Poisson integral for harmonic functions in the half-space. If $k = 2/((n-2)\omega_n)$, $\alpha = 0$ and $\beta = n-2$, then integral (1.2) gives solution of the Neumann problem for the half-space. Solution of the first boundary value problem for the biharmonic equation in the half-space is represented as the sum of two integrals (1.2) with $\alpha = 3/(n+2), \beta = n+2$ and $\alpha = 2/n, \beta = n$, accordingly. Integral (1.2) with $\alpha = 0$, $\beta \in (0, n-1)$ with appropriate choice of k can be considered as a continuation on $\mathbb{R}^n_+$ of the Riesz potential in $\mathbb{R}^{n-1}$.

In the present paper we arrive at conditions for which some analogues of Khavinson's conjecture for the generalized Poisson integral in the half-space are valid.

In Section 2 we obtain a representation for the sharp coefficient $\mathcal{C}_{\alpha,\beta,p}(x)$ in the inequality

$$|\nabla u(x)| \le \mathcal{C}_{\alpha,\beta,p}(x) \big\| f \big\|_p\,, \tag{1.3}$$

where

$$\mathcal{C}_{\alpha,\beta,p}(x) = \frac{C_{\alpha,\beta,p}}{x_n^{2-n+\beta(1-\alpha)+((n-1)/p)}} \ .$$

The constant $C_{\alpha,\beta,p}$ in this section is characterized in terms of an extremal problem on the sphere $\mathbb{S}^{n-1}$.

Analyzing this extremal problem for the case $p = 1$, in Section 3 we derive the explicit formula for $C_{\alpha,\beta,1}$ with $\beta > 0$. It is shown that $C_{\alpha,\beta,1} = |k|\beta|1-\alpha|$ if α satisfies the condition

$$0 \le \alpha \le \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}+1} \quad \text{or} \quad \alpha \ge \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1} \ .$$

For these values of α, the constant $C_{\alpha,\beta,1} = |k|\beta|1-\alpha|$ is sharp also in the weaker inequality obtained from (1.3) with $p = 1$ by replacing ∇u by $\partial u/\partial x_n$. Also, it is shown that

$$C_{\alpha,\beta,1} = |k|\beta \left(\frac{\beta}{2\alpha-1}\right)^{\beta/2} \left(\frac{\alpha^2}{1+\beta}\right)^{(\beta+2)/2}$$

if α satisfies the condition

$$\frac{\sqrt{1+\beta}}{\sqrt{1+\beta}+1} < \alpha < \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1} \ .$$

In Section 4 we consider the case $\alpha = 0$ in (1.2). Solving the extremal problem on $\mathbb{S}^{n-1}$ described in Section 2, we arrive at the explicit formula for the sharp coefficient $\mathcal{C}_{0,\beta,p}(x)$ in inequality (1.3) with $\alpha = 0$. In particular, we obtain the sharp inequality

$$|\nabla u(x)| \le \frac{C_{0,\beta,p}}{x_n^{2-n+\beta+((n-1)/p)}} \left\| f \right\|_p \tag{1.4}$$

for $\beta > n-1$ and $p \in [1,\infty]$, where $C_{0,\beta,1} = |k|\beta$ and

$$C_{0,\beta,p} = |k|\beta \left\{ \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{(\beta-n+3)p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(\beta+2)p}{2(p-1)}\right)} \right\}^{\frac{p}{p-1}} \tag{1.5}$$

for $p > 1$. The constant (1.5) is sharp also in the weaker inequality obtained from (1.4) by replacing ∇u by $\partial u/\partial x_n$.

In Section 5 we reduce the extremal problem on the sphere $\mathbb{S}^{n-1}$ from Section 2 to that of finding of the supremum of a certain double integral, depending on a scalar parameter.

Using the representation for $C_{\alpha,\beta,p}$ as the supremum of the double integral with a scalar parameter from Section 5, in Section 6 we consider the case $p = 2$. Here we obtain results similar to those of Section 3.

In Section 7 we deal with the case $p = \infty$ in (1.3). First, we show that for any $\beta > n-1$ there exists $\alpha_n(\beta) > 1$ such that for $\alpha \ge \alpha_n(\beta)$ the equality holds

$$C_{\alpha,\beta,\infty} = |k| \frac{\pi^{(n-1)/2}\, \Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \big((\alpha-1)\beta + n - 1\big).$$

The number $\alpha_n(\beta)$ is a root of a transcendental equation. For instance, $\alpha_3(2.5) \approx 1.2865$, $\alpha_3(3) \approx 1.4101$, $\alpha_3(3.5) \approx 1.4788$. Second, we consider the case $\alpha = 1$ separately and show that

$$C_{1,\beta,\infty} = |k| \frac{\pi^{(n-1)/2}(n-1)\Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}$$

for $\beta \in (n-1, n]$. In each of two assertions of Section 7 we show that the absolute value of the derivative of u with respect to the normal to the boundary of the half-space at any point $x \in \mathbb{R}^n_+$ has the same supremum as $|\nabla u(x)|$.

In Section 8 we concretize the results of Sections 3, 4, 6, 7 to obtain sharp estimates for the gradient of $x_n^{n\kappa-1}v(x)$, where $\kappa \geq 0$ and v is a harmonic function in $\mathbb{R}^n_+$ which can be represented as the Poisson integral with boundary values from L^p. Here we give the explicit formulas for the sharp constant $C_{\kappa,n,p}$ in the estimate

$$\left|\nabla\left(x_n^{n\kappa-1}v(x)\right)\right| \leq C_{\kappa,n,p}\, x_n^{n\kappa-2-(n-1)/p}\left\|v\right\|_p \tag{1.6}$$

with some values of κ and p. For instance, in the case $\kappa = 0$ we derive the inequality

$$\left|\nabla\left\{\frac{v(x)}{x_n}\right\}\right| \leq C_{0,n,p}\, x_n^{-2-(n-1)/p}\left\|v\right\|_p \tag{1.7}$$

with the sharp constant

$$C_{0,n,p} = \frac{2n}{\omega_n}\left\{\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(n+2)p}{2(p-1)}\right)}\right\}^{\frac{p}{p-1}}$$

for $1 < p < \infty$. For the cases $p = 1, 2, \infty$ inequality (1.7) becomes

$$\left|\nabla\left\{\frac{v(x)}{x_n}\right\}\right| \leq \frac{2n}{\omega_n}\frac{1}{x_n^{n+1}}\left\|v\right\|_1, \quad \left|\nabla\left\{\frac{v(x)}{x_n}\right\}\right| \leq \sqrt{\frac{n(n+3)}{2^n\omega_n}}\frac{1}{x_n^{(n+3)/2}}\left\|v\right\|_2, \quad \left|\nabla\left\{\frac{v(x)}{x_n}\right\}\right| \leq \frac{1}{x_n^2}\left\|v\right\|_\infty,$$

accordingly. We note, that the constants in inequality (1.7) remain sharp also in the weaker inequalities obtained by replacing ∇ by $\partial/\partial x_n$.

We mention one more group of inequalities for harmonic functions with the sharp coefficients obtained for the case $\kappa = 1$:

$$\left|\nabla\left(x_n^{n-1}v(x)\right)\right| \leq \frac{2(n-2)}{n\omega_n}\left\{\frac{(n-1)^2}{(n-2)(n+1)}\right\}^{(n+2)/2}\frac{1}{x_n}\left\|v\right\|_1,$$

$$\left|\nabla\left(x_n^{n-1}v(x)\right)\right| \leq \left\{\frac{n(n-1)}{2^n\omega_n}\right\}^{1/2} x_n^{(n-3)/2}\left\|v\right\|_2, \quad \left|\nabla\left(x_n^{n-1}v(x)\right)\right| \leq (n-1)x_n^{n-2}\left\|v\right\|_\infty.$$

Concluding Section 8, we present the sharp estimate

$$\left|\nabla\left(x_n^{n\kappa-2}w_0(x)\right)\right| \leq C_{\kappa,n,p}\, x_n^{n\kappa-2-(n-1)/p}\left\|\frac{\partial w_0}{\partial x_n}\right\|_p, \tag{1.8}$$

where $\kappa \geq 0$ and w_0 is a biharmonic function in $\mathbb{R}^n_+$ with the boundary values

$$w_0\Big|_{x_n=0} = 0, \quad \frac{\partial w_0}{\partial x_n}\Big|_{x_n=0} \in L^p(\mathbb{R}^{n-1}).$$

The sharp constant $C_{\kappa,n,p}$ in inequality (1.8) is the same as in (1.6). For example, in the case $\kappa = 0, p = \infty$, inequality (1.8) takes the form

$$\left|\nabla\left\{\frac{w_0(x)}{x_n^2}\right\}\right| \leq \frac{1}{x_n^2}\ \left\|\frac{\partial w_0}{\partial x_n}\right\|_\infty .$$

2. Representation for the sharp constant in inequality for the gradient in terms of an extremal problem on the unit sphere

We introduce some notation used henceforth. Let $\mathbb{R}^n_+ = \big\{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\big\}$, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $\mathbb{S}^{n-1}_+ = \{x \in \mathbb{R}^n : |x| = 1,\ x_n > 0\}$ and $\mathbb{S}^{n-1}_- = \{x \in \mathbb{R}^n : |x| = 1,\ x_n < 0\}$. Let $\boldsymbol{e}_\sigma$ stand for the n-dimensional unit vector joining the origin to a point σ on the sphere $\mathbb{S}^{n-1}$. As before, by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ we denote the area of the unit sphere in $\mathbb{R}^n$. Let $\boldsymbol{e}_n$ be the unit vector of the n-th coordinate axis.

By $||\cdot||_p$ we denote the norm in the space $L^p(\mathbb{R}^{n-1})$, that is

$$||f||_p = \left\{\int_{\mathbb{R}^{n-1}} |f(x')|^p \,\mathrm{d}x'\right\}^{1/p},$$

if $1 \leq p < \infty$, and $||f||_\infty = \text{ess sup}\{|f(x')| : x' \in \mathbb{R}^{n-1}\}$.

Let the function u in $\mathbb{R}^n_+$ be represented as the generalized Poisson integral

$$u(x) = k \int_{\mathbb{R}^{n-1}} \left(\frac{x_n^\alpha}{|y - x|}\right)^\beta f(y')\mathrm{d}y' \tag{2.1}$$

with parameters $\alpha \geq 0$ and

$$\beta > (n-1)(p-1)/p, \tag{2.2}$$

where k is a constant, $f \in L^p(\mathbb{R}^{n-1})$, $1 \leq p \leq \infty$, $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

Now, we find a representation for the best coefficient $\mathcal{C}_p(x; \boldsymbol{z})$ in the inequality for the absolute value of the derivative of $u(x)$ in an arbitrary direction $\boldsymbol{z} \in \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n_+$. In particular, we obtain a formula for the sharp coefficient in a similar inequality for the modulus of the gradient.

Proposition 2.1. *Let x be an arbitrary point in $\mathbb{R}^n_+$ and let $\boldsymbol{z} \in \mathbb{S}^{n-1}$. The sharp coefficient $\mathcal{C}_{\alpha,\beta,p}(x; \boldsymbol{z})$ in the inequality*

$$|\,(\nabla u(x), \boldsymbol{z})\,| \leq \mathcal{C}_{\alpha,\beta,p}(x; \boldsymbol{z})\big\|f\big\|_p \tag{2.3}$$

is given by

$$\mathcal{C}_{\alpha,\beta,p}(x; \boldsymbol{z}) = \frac{C_{\alpha,\beta,p}(\boldsymbol{z})}{x_n^{2-n+\beta(1-\alpha)+((n-1)/p)}}, \tag{2.4}$$

where

$$C_{\alpha,\beta,1}(\boldsymbol{z}) = |k|\beta \sup_{\sigma\in\mathbb{S}^{n-1}_+} \left|\big(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\right|\big(\boldsymbol{e}_\sigma,\boldsymbol{e}_n\big)^{\beta}, \tag{2.5}$$

$$C_{\alpha,\beta,p}(\boldsymbol{z}) = |k|\beta \left\{ \int_{\mathbb{S}^{n-1}_+} \left|\big(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma, \boldsymbol{z}\big)\right|^{\frac{p}{p-1}} \big(\boldsymbol{e}_\sigma,\boldsymbol{e}_n\big)^{\frac{(\beta-n)p+n}{p-1}} d\sigma \right\}^{\frac{p-1}{p}} \tag{2.6}$$

for $1 < p < \infty$*, and*

$$C_{\alpha,\beta,\infty}(\boldsymbol{z}) = |k|\beta \int_{\mathbb{S}^{n-1}_+} \left|\big(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\right|\big(\boldsymbol{e}_\sigma,\boldsymbol{e}_n\big)^{\beta-n} d\sigma. \tag{2.7}$$

In particular, the sharp coefficient $\mathcal{C}_{\alpha,\beta,p}(x)$ *in the inequality*

$$|\nabla u(x)| \le \mathcal{C}_{\alpha,\beta,p}(x)\big\|f\big\|_p \tag{2.8}$$

is given by

$$\mathcal{C}_{\alpha,\beta,p}(x) = \frac{C_{\alpha,\beta,p}}{x_n^{2-n+\beta(1-\alpha)+((n-1)/p)}}, \tag{2.9}$$

where

$$C_{\alpha,\beta,p} = \sup_{|\boldsymbol{z}|=1} C_{\alpha,\beta,p}(\boldsymbol{z}). \tag{2.10}$$

Proof. Let $x = (x', x_n)$ be a fixed point in $\mathbb{R}^n_+$. The representation (2.1) implies

$$\frac{\partial u}{\partial x_i} = k\int_{\mathbb{R}^{n-1}} \left[\frac{\alpha\beta\delta_{ni}x_n^{\alpha\beta-1}}{|y-x|^{\beta}} + \frac{\beta x_n^{\alpha\beta}(y_i-x_i)}{|y-x|^{\beta+2}}\right] f(y')\mathrm{d}y',$$

that is

$$\nabla u(x) = k\beta x_n^{\alpha\beta-1}\int_{\mathbb{R}^{n-1}} \left[\frac{\alpha\boldsymbol{e}_n}{|y-x|^{\beta}} + \frac{x_n(y-x)}{|y-x|^{\beta+2}}\right] f(y')\mathrm{d}y'$$
$$= k\beta x_n^{\alpha\beta-1}\int_{\mathbb{R}^{n-1}} \frac{\alpha\boldsymbol{e}_n - (\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy}}{|y-x|^{\beta}} f(y')\mathrm{d}y',$$

where $\boldsymbol{e}_{xy} = (y-x)|y-x|^{-1}$. For any $\boldsymbol{z} \in \mathbb{S}^{n-1}$,

$$(\nabla u(x), \boldsymbol{z}) = k\beta x_n^{\alpha\beta-1}\int_{\mathbb{R}^{n-1}} \frac{(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})}{|y-x|^{\beta}} f(y')\mathrm{d}y'.$$

Hence,

$$\mathcal{C}_{\alpha,\beta,1}(x;\boldsymbol{z}) = |k|\beta x_n^{\alpha\beta-1} \sup_{y\in\partial\mathbb{R}^n_+} \frac{|(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{|y-x|^{\beta}}\ , \tag{2.11}$$

and

$$\mathcal{C}_{\alpha,\beta,p}(x;\boldsymbol{z}) = |k|\beta x_n^{\alpha\beta-1}\left\{\int_{\mathbb{R}^{n-1}} \frac{\big|(\alpha\boldsymbol{e}_n - (\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\boldsymbol{z})\big|^q}{|y-x|^{\beta q}}\,\mathrm{d}y'\right\}^{1/q} \tag{2.12}$$

for $1 < p \le \infty$, where $p^{-1}+q^{-1}=1$.

Taking into account the equality

$$\frac{x_n}{|y-x|} = (\boldsymbol{e}_{xy}, -\boldsymbol{e}_n), \tag{2.13}$$

by (2.11) we obtain

$$\mathcal{C}_{\alpha,\beta,1}(x;\boldsymbol{z}) = |k|\beta x_n^{\alpha\beta-1}\sup_{y\in\partial\mathbb{R}^n_+}\frac{|(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_{xy},\boldsymbol{e}_n)\boldsymbol{e}_{xy},\ \boldsymbol{z})|}{x_n^\beta}\left(\frac{x_n}{|y-x|}\right)^\beta$$
$$= \frac{|k|\beta}{x_n^{1+\beta(1-\alpha)}}\sup_{\sigma\in\mathbb{S}^{n-1}_-}\big|(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^\beta.$$

Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.4) for $p=1$ with the sharp constant (2.5).

Let $1 < p \le \infty$. Using (2.13) and the equality

$$\frac{1}{|y-x|^{\beta q}} = \frac{1}{x_n^{\beta q-n+1}}\left(\frac{x_n}{|y-x|}\right)^{\beta q-n}\frac{x_n}{|y-x|^n}$$

and replacing q by $p/(p-1)$ in (2.12), we conclude that (2.4) holds with the sharp constant

$$C_{\alpha,\beta,p}(\boldsymbol{z}) = |k|\beta\left\{\int_{\mathbb{S}^{n-1}_-}\big|(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|^{\frac{p}{p-1}}(\boldsymbol{e}_\sigma,-\boldsymbol{e}_n)^{\frac{(\beta-n)p+n}{p-1}}\,\mathrm{d}\sigma\right\}^{\frac{p-1}{p}},$$

where $\mathbb{S}^{n-1}_- = \{\sigma\in\mathbb{S}^{n-1} : (\boldsymbol{e}_\sigma,\boldsymbol{e}_n) < 0\}$. Replacing here $\boldsymbol{e}_\sigma$ by $-\boldsymbol{e}_\sigma$, we arrive at (2.6) for $1 < p < \infty$ and at (2.7) for $p=\infty$.

Estimate (2.8) with the sharp coefficient (2.9), where the constant $C_{\alpha,\beta,p}$ is given by (2.10), is an immediate consequence of (2.3) and (2.4). □

Remark 2.2. Formula (2.6) for the sharp constant $C_{\alpha,\beta,p}(\boldsymbol{z})$ in (2.4), $1<p<\infty$, can be written with the integral over the whole sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$,

$$C_{\alpha,\beta,p}(\boldsymbol{z}) = \frac{|k|\beta}{2^{(p-1)/p}}\left\{\int_{\mathbb{S}^{n-1}}\big|(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\boldsymbol{z})\big|^{\frac{p}{p-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{\frac{(\beta-n)p+n}{p-1}}\mathrm{d}\sigma\right\}^{\frac{p-1}{p}}. \tag{2.14}$$

A similar remark relates (2.5):

$$C_{\alpha,\beta,1}(\boldsymbol{z}) = |k|\beta\sup_{\sigma\in\mathbb{S}^{n-1}}\big|(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^\beta, \tag{2.15}$$

as well as formula (2.7):

$$C_{\alpha,\beta,\infty}(\boldsymbol{z}) = \frac{|k|\beta}{2} \int_{\mathbb{S}^{n-1}} \big|\big(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\big| \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\beta-n} \, d\sigma.$$

3. The case $p = 1$

In the next assertion we obtain the explicit formula for the sharp constant $C_{\alpha,\beta,1}$.

Theorem 3.1. *Let $f \in L^1(\mathbb{R}^{n-1})$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,\beta,1}(x)$ in the inequality*

$$|\nabla u(x)| \leq \mathcal{C}_{\alpha,\beta,1}(x) \big\| f \big\|_1 \tag{3.1}$$

is given by

$$\mathcal{C}_{\alpha,\beta,1}(x) = \frac{C_{\alpha,\beta,1}}{x_n^{1+\beta(1-\alpha)}}, \tag{3.2}$$

where

$$C_{\alpha,\beta,1} = |k|\beta|1-\alpha| \tag{3.3}$$

if

$$0 \leq \alpha \leq \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}+1} \quad \text{or} \quad \alpha \geq \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1} \tag{3.4}$$

and

$$C_{\alpha,\beta,1} = |k|\beta \left(\frac{\beta}{2\alpha-1}\right)^{\beta/2} \left(\frac{\alpha^2}{1+\beta}\right)^{(\beta+2)/2} \tag{3.5}$$

if

$$\frac{\sqrt{1+\beta}}{\sqrt{1+\beta}+1} < \alpha < \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1}. \tag{3.6}$$

If α satisfies condition (3.4), then the coefficient $\mathcal{C}_{\alpha,\beta,1}(x)$ is sharp also in the weaker inequality obtained from (3.1) by replacing ∇u by $\partial u/\partial x_n$.

Proof. The equality (3.2) for the sharp coefficient $\mathcal{C}_{\alpha,\beta,1}(x)$ in (3.1) was proved in Proposition 2.1. Using (2.5), (2.10) and the permutability of two suprema, we find

$$\begin{aligned} C_{\alpha,\beta,1} &= |k|\beta \sup_{|\boldsymbol{z}|=1} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \big|\big(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{\beta} \\ &= |k|\beta \sup_{\sigma \in \mathbb{S}^{n-1}_+} \big|\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{\beta}. \end{aligned} \tag{3.7}$$

Taking into account the equality

$$\big|\alpha \boldsymbol{e}_n - n(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma\big| = \Big(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma\Big)^{1/2} = \Big(\alpha^2 + (1-2\alpha)(\boldsymbol{e}_\sigma,\ \boldsymbol{e}_n)^2\Big)^{1/2},$$

and using (3.7), we arrive at the representation

$$C_{\alpha,\beta,1} = |k|\beta \sup_{\sigma \in \mathbb{S}^{n-1}_+} \Big(\alpha^2 + (1-2\alpha)(\boldsymbol{e}_\sigma,\ \boldsymbol{e}_n)^2\Big)^{1/2} \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{\beta}. \tag{3.8}$$

We denote $t = \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)$. Let us introduce the function

$$f(t) = \big(\alpha^2 + (1-2\alpha)t^2\big)^{1/2} t^\beta, \tag{3.9}$$

where $t \in [0,1]$, $\alpha \geq 0$ and $\beta > 0$. By (3.8),

$$C_{\alpha,\beta,1} = |k|\beta \max_{0\leq t \leq 1} f(t). \tag{3.10}$$

Taking into account that $f(t) > 0$ for $t \in (0,1)$ and any $\alpha \geq 0$, $\beta > 0$, we can consider the function $F(t) = f^2(t)$ on the interval $t \in (0,1)$ instead of $f(t)$. We have

$$F'(t) = 2\Big(\alpha^2\beta + (1-2\alpha)(1+\beta)t^2\Big)t^{2\beta-1}. \tag{3.11}$$

If $0 \leq \alpha \leq 1/2$, then $F'(t) > 0$ for $t \in (0,1)$. If $\alpha > 1/2$, then the positive root of the equation $F'(t) = 0$ is

$$t_1 = \sqrt{\frac{\alpha^2\beta}{(2\alpha-1)(1+\beta)}}. \tag{3.12}$$

Herewith, if

$$\frac{\alpha^2\beta}{(2\alpha-1)(1+\beta)} \geq 1, \tag{3.13}$$

then $t_1 \notin (0,1)$. Solving inequality (3.13) with respect to α, we obtain intervals for which (3.13) holds:

$$\alpha \leq \alpha_1 = \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}+1} \quad \text{and} \quad \alpha \geq \alpha_2 = \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1}.$$

Hence, $F'(t) = (f^2(t))' > 0$ for $t \in (0,1)$ if $\alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$. This, by (3.9) and (3.10), proves the equality (3.3) for (3.4).

Furthermore, by (2.5),

$$C_{\alpha,\beta,1}(\boldsymbol{e}_n) = |k|\beta \sup_{\sigma \in \mathbb{S}^{n-1}_+} \big|\alpha - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)^2\big| \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{\beta} \geq |k|\beta|1-\alpha|.$$

Hence, by $C_{\alpha,\beta,1} \geq C_{\alpha,\beta,1}(\boldsymbol{e}_n)$ and by (3.3) we obtain $C_{\alpha,\beta,1} = C_{\alpha,\beta,1}(\boldsymbol{e}_n)$, which completes the proof for the case $\alpha \leq \alpha_1$ as well as in the case $\alpha \geq \alpha_2$.

Now, we consider the case $t_1 < 1$, that is

$$\frac{\alpha^2\beta}{(2\alpha-1)(1+\beta)} < 1.$$

The last inequality holds for $\alpha \in (\alpha_1, \alpha_2)$. Differentiating (3.11), we obtain

$$F''(t) = 2\Big(\alpha^2\beta + (1-2\alpha)(1+\beta)(1+2\beta)t^2\Big)t^{2(\beta-1)}.$$

After calculations, we have

$$F''(t_1) = -4\alpha^2\beta^2\left(\frac{\alpha^2\beta}{(2\alpha-1)(1+\beta)}\right)^{\beta-1}.$$

Since $\alpha > \alpha_1 > 1/2$ and $\beta > 0$, by the last equality we conclude that $F''(t_1) < 0$. Hence, the function $F(t)$ and, as a consequence, the function $f(t)$ attains its maximum on $[0, 1]$ at the point $t_1 \in (0, 1)$.

Substituting t_1 from (3.12) in (3.9) and using (3.10), we arrive at (3.5) for the case $\alpha_1 < \alpha < \alpha_2$. □

4. The case $\alpha = 0$

In this section we consider integral (2.1) with $\alpha = 0$ that is

$$u(x) = k\int_{\mathbb{R}^{n-1}} \frac{f(y')}{|y-x|^\beta}\, dy',$$

where $x \in \mathbb{R}^n_+$, β satisfies inequality (2.2) and $f \in L^p(\mathbb{R}^{n-1})$. Here we solve extremal problem (2.10) with $\alpha = 0$ and obtain the explicit value for $C_{0,\beta,p}$. Namely, we prove

Theorem 4.1. *Let $\alpha = 0$ in (2.1) and let any of the following conditions holds:*
(i) $\beta \geq n-1$ *and* $p \in [1, \infty)$,
(ii) $\beta > n-1$ *and* $p = \infty$,
(iii) $\beta < n-1$ *and* $p \in [1, (n-1)/(n-1-\beta))$.
Then for any $x \in \mathbb{R}^n_+$ the sharp constant $C_{0,\beta,p}$ in the inequality

$$|\nabla u(x)| \leq \frac{C_{0,\beta,p}}{x_n^{2-n+\beta+((n-1)/p)}}\, \|f\|_p \tag{4.1}$$

is given by $C_{0,\beta,1} = |k|\beta$, and

$$C_{0,\beta,p} = |k|\beta \left\{ \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{(\beta-n+3)p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(\beta+2)p}{2(p-1)}\right)} \right\}^{\frac{p}{p-1}} \tag{4.2}$$

for $p > 1$.

The constant $C_{0,\beta,p}$ is sharp under conditions of the Theorem also in the weaker inequality obtained from (4.1) by replacing ∇u by $\partial u/\partial x_n$.

Proof. Let $\alpha = 0$ in (2.1) and $p = 1$. By (2.10) and (2.15),

$$C_{0,\beta,1} = |k|\beta \sup_{|\boldsymbol{z}|=1} \sup_{\sigma\in\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma,\ \boldsymbol{z})\big|\big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\beta+1} \le |k|\beta. \tag{4.3}$$

On the other hand,

$$C_{0,\beta,1} \ge |k|\beta \sup_{\sigma\in\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma,\ \boldsymbol{e}_n)\big|\big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\beta+1} = |k|\beta,$$

which, together with (4.3), implies $C_1 = |k|\beta$. We note that by (2.5),

$$C_{0,\beta,1}(\boldsymbol{e}_n) = |k|\beta \sup_{\sigma\in\mathbb{S}^{n-1}_+} \big(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\big)^{\beta+2} = |k|\beta,$$

that is $C_{0,\beta,1} = C_{0,\beta,1}(\boldsymbol{e}_n)$.

Let now $\alpha = 0$ in (2.1) and $p > 1$. By (2.10) and (2.14) we have

$$C_{0,\beta,p} = \frac{|k|\beta}{2^{(p-1)/p}} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\frac{p}{p-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\frac{(\beta-n+1)p+n}{p-1}} \mathrm{d}\sigma \right\}^{\frac{p-1}{p}}. \tag{4.4}$$

Let us denote by $\mu = p/(p-1)$ and $\lambda = ((\beta - n + 1)p + n)/(p-1)$ the powers in (4.4). Obviously, $\lambda > 0$ for $\beta \ge n-1$ and any $p > 1$. That is, $\lambda > 0$ if condition (i) is satisfied.

If $\beta > n-1$ and $p = \infty$, then $\lambda = \beta - n + 1 > 0$. Therefore, $\lambda > 0$ if condition (ii) holds.

If $\beta < n-1$, then $\lambda > 0$ for $p < n/(n-\beta-1)$. For $\beta - n + 1 < 0$, by inequality (2.2), we have $p < (n-1)/(n-\beta-1)$. So, $\lambda > 0$ if condition (iii) is satisfied.

By Hölder's inequality, we obtain

$$\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\mu} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda} \mathrm{d}\sigma \le \left\{ \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\mu\frac{\lambda+\mu}{\mu}} \mathrm{d}\sigma \right\}^{\frac{\mu}{\lambda+\mu}} \left\{ \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda\frac{\lambda+\mu}{\lambda}} \mathrm{d}\sigma \right\}^{\frac{\lambda}{\lambda+\mu}}. \tag{4.5}$$

Obviously, the value of the first integral on the right-hand side of the last inequality is independent of $\boldsymbol{z}$. Therefore,

$$\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\lambda+\mu} \mathrm{d}\sigma = \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda+\mu} \mathrm{d}\sigma\ ,$$

which, in view of (4.5), implies

$$\int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\mu} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda} \mathrm{d}\sigma \le \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda+\mu} \mathrm{d}\sigma\ . \tag{4.6}$$

On the other hand,

$$\sup_{|z|=1} \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{z})\big|^{\mu} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda} \mathrm{d}\sigma \ge \int_{\mathbb{S}^{n-1}} \big|(\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\big|^{\lambda+\mu} \mathrm{d}\sigma,$$

which together with (4.6) leads to

$$\sup_{|z|=1}\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{z})\big|^{\mu}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{\lambda}\mathrm{d}\sigma=\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{\lambda+\mu}\mathrm{d}\sigma.$$

The last equality, in view of (4.4), implies

$$C_{0,\beta,p}=\frac{|k|\beta}{2^{(p-1)/p}}\left\{\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{\frac{(\beta-n+2)p+n}{p-1}}\mathrm{d}\sigma\right\}^{\frac{p-1}{p}}. \tag{4.7}$$

Comparing (2.14) with $\alpha=0$, $\boldsymbol{z}=\boldsymbol{e}_n$ and (4.7), we conclude that $C_{0,\beta,p}=C_{0,\beta,p}(\boldsymbol{e}_n)$. This proves that the constant $C_{0,\beta,p}$ is sharp also in the weaker inequality obtained from (4.1) by replacing ∇u by $\partial u/\partial x_n$.

Evaluating the integral in (4.7), we find

$$\int_{\mathbb{S}^{n-1}}\big|(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\big|^{\frac{(\beta-n+2)p+n}{p-1}}\mathrm{d}\sigma=2\omega_{n-1}\int_0^{\pi/2}\cos^{\frac{(\beta-n+2)p+n}{p-1}}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta$$
$$=\omega_{n-1}B\left(\frac{(\beta-n+3)p+n-1}{2(p-1)},\frac{n-1}{2}\right)=\frac{2\pi^{(n-1)/2}\Gamma\left(\frac{(\beta-n+3)p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(\beta+2)p}{2(p-1)}\right)},$$

which together with (2.8), (2.9), where $\alpha=0$, and (4.7) proves (4.1) and (4.2). □

5. Reduction of the extremal problem to finding of the supremum by parameter of a double integral

The next assertion is based on the representation for $\mathcal{C}_{\alpha,\beta,p}(x)$, obtained in Proposition 2.1.

Proposition 5.1. *Let $f\in L^p(\mathbb{R}^{n-1})$, $p>1$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,\beta,p}(x)$ in the inequality*

$$|\nabla u(x)|\leq\mathcal{C}_{\alpha,\beta,p}(x)\big\|f\big\|_p \tag{5.1}$$

is given by

$$\mathcal{C}_{\alpha,\beta,p}(x)=\frac{C_{\alpha,\beta,p}}{x_n^{2-n+\beta(1-\alpha)+((n-1)/p)}}, \tag{5.2}$$

where

$$C_{\alpha,\beta,p}=|k|\beta(\omega_{n-2})^{(p-1)/p}\sup_{\gamma\geq 0}\frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^{\pi}d\varphi\int_0^{\pi/2}\mathcal{F}_{n,p}(\varphi,\vartheta;\alpha,\beta,\gamma)\,d\vartheta\right\}^{\frac{p-1}{p}}. \tag{5.3}$$

Here

$$\mathcal{F}_{n,p}(\varphi,\vartheta;\alpha,\beta,\gamma)=\big|\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\big|^{p/(p-1)}\cos^{((\beta-n)p+n)/(p-1)}\vartheta\sin^{n-2}\vartheta\sin^{n-3}\varphi \tag{5.4}$$

with

$$\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)=\cos^2\vartheta-\alpha+\gamma\cos\vartheta\sin\vartheta\cos\varphi. \tag{5.5}$$

Proof. The equality (5.2) for the sharp coefficient $\mathcal{C}_{\alpha,\beta,p}(x)$ in (5.1) was proved in Proposition 2.1. Since the integrand in (2.6) does not change when $\boldsymbol{z}\in\mathbb{S}^{n-1}$ is replaced by $-\boldsymbol{z}$, we may assume that $z_n=(\boldsymbol{e}_n,\boldsymbol{z})>0$ in (2.10).

Let $\boldsymbol{z}'=\boldsymbol{z}-z_n\boldsymbol{e}_n$. Then $(\boldsymbol{z}',\boldsymbol{e}_n)=0$ and hence $z_n^2+|\boldsymbol{z}'|^2=1$. Analogously, with $\sigma=(\sigma_1,\dots,\sigma_{n-1},\sigma_n)\in\mathbb{S}^{n-1}_+$, we associate the vector $\boldsymbol{\sigma}'=\boldsymbol{e}_\sigma-\sigma_n\boldsymbol{e}_n$.

Using the equalities $(\boldsymbol{\sigma}',\boldsymbol{e}_n)=0$, $\sigma_n=\sqrt{1-|\boldsymbol{\sigma}'|^2}$ and $(\boldsymbol{z}',\boldsymbol{e}_n)=0$, we find an expression for $\big(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)$ as a function of $\boldsymbol{\sigma}'$:

$$\begin{aligned}\big(\alpha\boldsymbol{e}_n-(\boldsymbol{e}_\sigma,\boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\big)&=\alpha z_n-\sigma_n\big(\boldsymbol{e}_\sigma,\boldsymbol{z}\big)=\alpha z_n-\sigma_n\big(\boldsymbol{\sigma}'+\sigma_n\boldsymbol{e}_n,\ \boldsymbol{z}'+z_n\boldsymbol{e}_n\big)\\&=\alpha z_n-\sigma_n\big[\big(\boldsymbol{\sigma}',\boldsymbol{z}'\big)+z_n\sigma_n\big]=-\big[(1-|\boldsymbol{\sigma}'|^2)-\alpha\big]z_n-\sqrt{1-|\boldsymbol{\sigma}'|^2}\,\big(\boldsymbol{\sigma}',\boldsymbol{z}'\big).\end{aligned} \tag{5.6}$$

Let $\mathbb{B}^n=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n:|x|<1\}$. By (2.6) and (5.6), taking into account that $\mathrm{d}\sigma=\mathrm{d}\sigma'/\sqrt{1-|\boldsymbol{\sigma}'|^2}$, we may write (2.10) as

$$\begin{aligned}C_{\alpha,\beta,p}&=|k|\beta\sup_{\boldsymbol{z}\in\mathbb{S}^{n-1}_+}\left\{\int_{\mathbb{B}^{n-1}}\frac{\mathcal{H}_{\alpha,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\beta-n)p+n)/(2p-2)}}{\sqrt{1-|\boldsymbol{\sigma}'|^2}}\,\mathrm{d}\sigma'\right\}^{\frac{p-1}{p}}\\&=|k|\beta\sup_{\boldsymbol{z}\in\mathbb{S}^{n-1}_+}\left\{\int_{\mathbb{B}^{n-1}}\mathcal{H}_{\alpha,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\beta-n-1)p+n+1)/(2p-2)}\mathrm{d}\sigma'\right\}^{\frac{p-1}{p}},\end{aligned} \tag{5.7}$$

where

$$\mathcal{H}_{\alpha,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)=\Big|\big[(1-|\boldsymbol{\sigma}'|^2)-\alpha\big]z_n+\sqrt{1-|\boldsymbol{\sigma}'|^2}\,\big(\boldsymbol{\sigma}',\boldsymbol{z}'\big)\Big|^{p/(p-1)}. \tag{5.8}$$

Using the well known formula (*e.g.* Prudnikov, Brychkov and Marichev [6], 3.3.2(3)),

$$\int_{\mathbb{B}^n}g\big(|\boldsymbol{x}|,(\boldsymbol{a},\boldsymbol{x})\big)\mathrm{d}x=\omega_{n-1}\int_0^1 r^{n-1}\mathrm{d}r\int_0^\pi g\big(r,|\boldsymbol{a}|r\cos\varphi\big)\sin^{n-2}\varphi\,\mathrm{d}\varphi\,,$$

we obtain

$$\begin{aligned}&\int_{\mathbb{B}^{n-1}}\mathcal{H}_{\alpha,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\beta-n-1)p+n+1)/(2p-2)}\,\mathrm{d}\sigma'\\&=\omega_{n-2}\int_0^1 r^{n-2}\big(1-r^2\big)^{((\beta-n-1)p+n+1)/(2p-2)}\mathrm{d}r\int_0^\pi\mathcal{H}_{\alpha,p}\big(r,r|\boldsymbol{z}'|\cos\varphi\big)\sin^{n-3}\varphi\mathrm{d}\varphi\,.\end{aligned}$$

Making the change of variable $r=\sin\vartheta$ on the right-hand side of the last equality, we find

$$\begin{aligned}&\int_{\mathbb{B}^{n-1}}\mathcal{H}_{\alpha,p}\big(|\boldsymbol{\sigma}'|,(\boldsymbol{\sigma}',\boldsymbol{z}')\big)\big(1-|\boldsymbol{\sigma}'|^2\big)^{((\beta-n-1)p+n+1)/(2p-2)}\,\mathrm{d}\sigma'\\&=\omega_{n-2}\int_0^\pi\sin^{n-3}\varphi\mathrm{d}\varphi\int_0^{\pi/2}\mathcal{H}_{\alpha,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big)\sin^{n-2}\vartheta\cos^{\frac{(\beta-n)p+n}{p-1}}\vartheta\mathrm{d}\vartheta\,,\end{aligned} \tag{5.9}$$

where, by (5.8),

$$\mathcal{H}_{\alpha,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = \Big|\big(\cos^2\vartheta-\alpha\big)z_n+|\boldsymbol{z}'|\cos\vartheta\sin\vartheta\cos\varphi\Big|^{p/(p-1)}.$$

Introducing here the parameter $\gamma = |\boldsymbol{z}'|/z_n$ and using the equality $|\boldsymbol{z}'|^2+z_n^2=1$, we obtain

$$\mathcal{H}_{\alpha,p}\big(\sin\vartheta,\ |\boldsymbol{z}'|\sin\vartheta\cos\varphi\big) = (1+\gamma^2)^{-p/(2p-2)}\big|\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\big|^{p/(p-1)}, \tag{5.10}$$

where $\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)$ is given by (5.5).

By (5.7), taking into account (5.9) and (5.10), we arrive at (5.3). □

6. The case $p=2$

In the next assertion we obtain the explicit formula for $C_{\alpha,\beta,2}$.

Theorem 6.1. *Let $f\in L^2(\mathbb{R}^{n-1})$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,\beta,2}(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{C}_{\alpha,\beta,2}(x)\big\|f\big\|_2 \tag{6.1}$$

is given by

$$\mathcal{C}_{\alpha,\beta,2}(x) = \frac{C_{\alpha,\beta,2}}{x_n^{\beta(1-\alpha)+((3-n)/2)}}\ , \tag{6.2}$$

where

$$C_{\alpha,\beta,2} = |k|\beta\left\{\frac{\pi^{(n-1)/2}\Gamma\left(\frac{2\beta+3-n}{2}\right)}{\Gamma(\beta+2)}\left[\frac{2\alpha^2\beta(\beta+1)}{2\beta+1-n}-2\alpha(\beta+1)+\frac{2\beta+3-n}{2}\right]\right\}^{1/2} \tag{6.3}$$

for $(n-1)/2<\beta\le n-1$. The same formula for $C_{\alpha,\beta,2}$ holds for $\beta>n-1$ and

$$\alpha\le\alpha_1 = \frac{(1+\beta)(2\beta+1-n)-\sqrt{(1+\beta)(2\beta+1-n)(\beta+1-n)}}{2\beta(1+\beta)}\ ,$$

or

$$\alpha\ge\alpha_2 = \frac{(1+\beta)(2\beta+1-n)+\sqrt{(1+\beta)(2\beta+1-n)(\beta+1-n)}}{2\beta(1+\beta)}\ .$$

If $\beta>n-1$ and $\alpha_1<\alpha<\alpha_2$, then

$$C_{\alpha,\beta,2} = |k|\beta\left\{\frac{\pi^{(n-1)/2}\Gamma\left(\frac{2\beta+3-n}{2}\right)}{2\Gamma(\beta+2)}\right\}^{1/2}\ . \tag{6.4}$$

If (i) *$(n-1)/2<\beta\le n-1$ or* (ii) *$\beta>n-1$, $\alpha\le\alpha_1$ or $\alpha\ge\alpha_2$, then the coefficient $\mathcal{C}_{\alpha,\beta,2}(x)$ is sharp under conditions of the Theorem also in the weaker inequality obtained from* (6.1) *by replacing ∇u by $\partial u/\partial x_n$.*

Proof. The equality (6.2) for the sharp coefficient $\mathcal{C}_{\alpha,\beta,2}(x)$ in (6.1) was proved in Proposition 2.1. By (5.3), (5.4) and (5.5),

$$C_{\alpha,\beta,2} = |k|\beta\sqrt{\omega_{n-2}}\ \sup_{\gamma\geq 0}\ \frac{1}{\sqrt{1+\gamma^2}}\left\{\int_0^{\pi} \mathrm{d}\varphi \int_0^{\pi/2} \mathcal{F}_{n,2}(\varphi,\vartheta;\alpha,\beta,\gamma)\,\mathrm{d}\vartheta\right\}^{1/2}, \tag{6.5}$$

where

$$\mathcal{F}_{n,2}(\varphi,\vartheta;\alpha,\beta,\gamma)=\left(\cos^2\vartheta-\alpha+\gamma\cos\vartheta\sin\vartheta\cos\varphi\right)^2\cos^{2\beta-n}\vartheta\sin^{n-2}\vartheta\sin^{n-3}\varphi.$$

The last equality and (6.5) imply

$$C_{\alpha,\beta,2} = |k|\beta\sqrt{\omega_{n-2}}\ \sup_{\gamma\geq 0}\ \frac{1}{\sqrt{1+\gamma^2}}\left\{\mathcal{I}_1+\gamma^2\mathcal{I}_2\right\}^{1/2}, \tag{6.6}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_0^{\pi}\sin^{n-3}\varphi\,\mathrm{d}\varphi\int_0^{\pi/2}\left(\cos^2\vartheta-\alpha\right)^2\sin^{n-2}\vartheta\cos^{2\beta-n}\vartheta\,\mathrm{d}\vartheta\\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{2\beta+3-n}{2}\right)}{2\Gamma(\beta+2)}\left\{\frac{2\alpha^2\beta(\beta+1)}{2\beta+1-n}-2\alpha(\beta+1)+\frac{2\beta+3-n}{2}\right\}\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\mathcal{I}_2 &= \int_0^{\pi}\sin^{n-3}\varphi\cos^2\varphi\,\mathrm{d}\varphi\int_0^{\pi/2}\sin^{n}\vartheta\cos^{2(\beta+1)-n}\vartheta\,\mathrm{d}\vartheta\\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{2\beta+3-n}{2}\right)}{4\Gamma(\beta+2)}.\end{aligned} \tag{6.8}$$

By (6.6) we have

$$C_{\alpha,\beta,2} = |k|\beta\sqrt{\omega_{n-2}}\ \max\left\{\mathcal{I}_1^{1/2},\mathcal{I}_2^{1/2}\right\}. \tag{6.9}$$

Further, by (6.7) and (6.8),

$$\begin{aligned}\frac{\mathcal{I}_1}{\mathcal{I}_2}-1 &= 4\alpha^2\frac{\beta(\beta+1)}{2\beta+1-n}-4\alpha(\beta+1)+2(\beta+1)-n\\ &= \frac{4\alpha^2\beta(\beta+1)-4\alpha(\beta+1)(2\beta+1-n)+(2\beta+2-n)(2\beta+1-n)}{2\beta+1-n}.\end{aligned} \tag{6.10}$$

We note that, by (6.9) with $p=2$, $2\beta-n+1>0$. By

$$f(\alpha)=4\alpha^2\beta(\beta+1)-4\alpha(\beta+1)(2\beta+1-n)+(2\beta+2-n)(2\beta+1-n)$$

we denote the numerator of fraction (6.10). The roots of the equation $f(\alpha) = 0$ are

$$\alpha_{1,2} = \frac{(\beta+1)(2\beta+1-n) \pm \sqrt{(\beta+1)(2\beta+1-n)(\beta+1-n)}}{2\beta(\beta+1)} . \tag{6.11}$$

It follows from (6.10) and (6.11) that $\mathcal{I}_1 \geq \mathcal{I}_2$ for $\beta + 1 - n \leq 0$. Combining the last condition for β with inequality $\beta > (n-1)/2$ and taking into account (6.7), (6.9), we arrive at formula (6.3) for the case $(n-1)/2 < \beta \leq n-1$.

Now, let $\beta > n-1$. Then, by (6.10),

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} - 1 \geq 0$$

for $\alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$, and

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} - 1 < 0$$

for $\alpha_1 < \alpha < \alpha_2$. This, by (6.9), proves (6.3) for $\alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$ and (6.4) for $\alpha_1 < \alpha < \alpha_2$.

In conclusion, we note that supremum in (6.6) is attained for $\gamma = 0$ in two cases: (i) $(n-1)/2 < \beta \leq n-1$, (ii) $\beta > n-1$ and $\alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$. Taking into account that $\gamma = |\boldsymbol{z}'|/z_n$, we conclude that $C_{\alpha,\beta,2} = C_{\alpha,\beta,2}(\boldsymbol{e}_n)$ for these cases. This proves that the coefficient $\mathcal{C}_{\alpha,\beta,2}(x)$ is sharp under conditions of the Theorem also in the weaker inequality obtained from (6.1) by replacing ∇u by $\partial u/\partial x_n$. $\square$

7. The case $p = \infty$

This section is devoted to the case $p = \infty$ with some restrictions on α and β. In the assertion below we obtain the explicit formula for $C_{\alpha,\beta,\infty}$ with any fixed $\beta > n-1$ and sufficiently large $\alpha > 1$. We note that inequality $\beta > n-1$ follows from (2.2) with $p = \infty$.

Theorem 7.1. *Let $f \in L^\infty(\mathbb{R}^{n-1})$, and let x be an arbitrary point in $\mathbb{R}^n_+$. Let β be a fixed and let $\alpha_n(\beta)$ be the root from the interval $(1, +\infty)$ of the equation*

$$\frac{2\Gamma\left(\frac{\beta-n}{2}+1\right)}{\sqrt{\pi}\big(\beta(\alpha-1)+n-1\big)\Gamma\left(\frac{\beta-n+1}{2}\right)} = \frac{\alpha-1}{1+\sqrt{1+(\alpha-1)^2}} \tag{7.1}$$

with respect to α.

If $\alpha \geq \alpha_n(\beta)$, then the sharp coefficient $\mathcal{C}_{\alpha,\beta,\infty}(x)$ in the inequality

$$|\nabla u(x)| \leq \mathcal{C}_{\alpha,\beta,\infty}(x)\big\|f\big\|_\infty \tag{7.2}$$

is given by

$$\mathcal{C}_{\alpha,\beta,\infty}(x) = \frac{C_{\alpha,\beta,\infty}}{x_n^{2-n+\beta(1-\alpha)}}, \tag{7.3}$$

where

$$C_{\alpha,\beta,\infty} = |k| \frac{\pi^{(n-1)/2}\, \Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \big((\alpha-1)\beta+n-1\big). \tag{7.4}$$

Under conditions of the Theorem, absolute value of the derivative of u with respect to the normal to the boundary of the half-space at any $x \in \mathbb{R}^n_+$ has the same supremum as $|\nabla u(x)|$.

Proof. First of all, we show that equation (7.1) has only one α-root $\alpha_n(\beta)$ on the interval $(1,+\infty)$ for any fixed $\beta > n-1$. In fact, the function

$$f(\alpha) = \frac{2\Gamma\left(\frac{\beta-n}{2}+1\right)}{\sqrt{\pi}\big(\beta(\alpha-1)+n-1\big)\Gamma\left(\frac{\beta-n+1}{2}\right)} \tag{7.5}$$

decreases, and the function

$$g(\alpha) = \frac{\alpha-1}{1+\sqrt{1+(\alpha-1)^2}} \tag{7.6}$$

increases on the interval $[1,\infty)$. The functions f, g are continuous, $f(1)>0, g(1)=0$, and

$$\lim_{\alpha\to+\infty} f(\alpha) = 0, \qquad \lim_{\alpha\to+\infty} g(\alpha) = 1\ .$$

So, the existence and uniqueness of the α-root $\alpha_n(\beta)$ of equation (7.1) on the interval $(1,+\infty)$ are proven.

The equality (7.3) for the sharp coefficient $\mathcal{C}_{\alpha,\beta,\infty}(x)$ in (7.2) was proved in Proposition 2.1. We pass to the limit as $p\to\infty$ in (5.3) and (5.4). This results at

$$C_{\alpha,\beta,\infty} = |k|\beta \sup_{\gamma\geq 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big|\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\big| \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta, \tag{7.7}$$

where $\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)$ is defined by (5.5).

Suppose that $\beta > n-1$ and $\alpha \geq \alpha_n(\beta)$ are fixed. We introduce three integrals

$$J(\gamma) = |k|\beta\omega_{n-2} \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big|\alpha - \cos^2\vartheta - \gamma\cos\vartheta\sin\vartheta\cos\varphi\big| \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta, \tag{7.8}$$

$$J_1 = |k|\beta\omega_{n-2} \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big|\alpha - \cos^2\vartheta\big| \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta, \tag{7.9}$$

and

$$J_2 = |k|\beta\omega_{n-2} \int_0^{\pi} \sin^{n-3}\varphi |\cos\varphi| \mathrm{d}\varphi \int_0^{\pi/2} \cos^{\beta-n+1}\vartheta \sin^{n-1}\vartheta \mathrm{d}\vartheta.$$

We note that

$$\frac{J(\gamma)}{\sqrt{1+\gamma^2}} \le \frac{J_1 + \gamma J_2}{\sqrt{1+\gamma^2}}. \tag{7.10}$$

Calculating J_1 and J_2, we obtain

$$J_1 = |k| \frac{\pi^{(n-1)/2}\, \Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \big((\alpha-1)\beta + n - 1\big), \tag{7.11}$$

and

$$J_2 = |k| \frac{2\pi^{(n-2)/2}\, \Gamma\left(\frac{\beta-n}{2}+1\right)}{\Gamma\left(\frac{\beta}{2}\right)}. \tag{7.12}$$

It follows from (7.11) and (7.12) that

$$\frac{J_2}{J_1} = \frac{2\Gamma\left(\frac{\beta-n}{2}+1\right)}{\sqrt{\pi}\big((\alpha-1)\beta+n-1\big)\Gamma\left(\frac{\beta-n+1}{2}\right)}. \tag{7.13}$$

We note that the right-hand side of the last equality coincides with the function $f(\alpha)$, defined by (7.5). Since $f(\alpha) \le g(\alpha)$ for $\alpha \ge \alpha_n(\beta)$ and $g(\alpha) < 1$, by (7.13) we conclude that

$$\frac{J_2}{J_1} = f(\alpha) < 1 \tag{7.14}$$

for $\alpha \ge \alpha_n(\beta)$. We find the interval of γ for which the inequality

$$\frac{J_1 + \gamma J_2}{\sqrt{1+\gamma^2}} \le J_1 \tag{7.15}$$

holds. Solving inequality (7.15) with respect to γ, we obtain

$$\gamma \ge \frac{2J_1 J_2}{J_1^2 - J_2^2} = \frac{2(J_2/J_1)}{1-(J_2/J_1)^2}. \tag{7.16}$$

We denote

$$\gamma_0 = \frac{2(J_2/J_1)}{1-(J_2/J_1)^2}.$$

By (7.10), (7.15) and (7.16),

$$\frac{J(\gamma)}{\sqrt{1+\gamma^2}} \le J_1 \ \text{ for } \ \gamma \ge \gamma_0. \tag{7.17}$$

Now, we show that $\alpha - 1 - \gamma_0 \geq 0$ for $\alpha \geq \alpha_n(\beta)$. Taking into account that $f(\alpha) \leq g(\alpha)$ for $\alpha \geq \alpha_n(\beta)$, by (7.6) and (7.14) we arrive at inequality

$$\frac{J_2}{J_1} \leq \frac{\alpha - 1}{1 + \sqrt{1 + (\alpha - 1)^2}}. \tag{7.18}$$

Using (7.18), after calculations we obtain

$$\gamma_0 = \frac{2(J_2/J_1)}{1 - (J_2/J_1)^2} \leq \alpha - 1,$$

which proves the inequality $\alpha - 1 - \gamma_0 \geq 0$ for $\alpha \geq \alpha_n(\beta)$.

Let $0 \leq \gamma \leq \gamma_0$. Taking into account that $\alpha - 1 - \gamma \geq 0$, by (7.8) and (7.9) we have $J(\gamma) = J_1$. Hence,

$$\frac{J(\gamma)}{\sqrt{1+\gamma^2}} = \frac{J_1}{\sqrt{1+\gamma^2}} \leq J_1 \ \text{ for } \ 0 \leq \gamma \leq \gamma_0,$$

which together with (7.17) leads to inequality

$$\frac{J(\gamma)}{\sqrt{1+\gamma^2}} \leq J_1$$

for any $\gamma \geq 0$. Therefore, in view of (7.7)-(7.9), we obtain

$$C_{\alpha,\beta,\infty} = \sup_{\gamma \geq 0} \frac{J(\gamma)}{\sqrt{1+\gamma^2}} \leq J_1 = J(0), \tag{7.19}$$

which together with inequality

$$\sup_{\gamma \geq 0} \frac{J(\gamma)}{\sqrt{1+\gamma^2}} \geq J(0)$$

results at

$$C_{\alpha,\beta,\infty} = J(0) = J_1.$$

In view of (7.11), the last equality proves (7.4). Since $\gamma = |\boldsymbol{z}'|/z_n$ and the supremum with respect to γ in (7.19) is attained at $\gamma = 0$, we conclude that the coefficient $\mathcal{C}_{\alpha,\beta,\infty}(x) = C_{\alpha,\beta,\infty} x_n^{n-2+\beta(\alpha-1)}$ is sharp under conditions of the Theorem also in the weaker inequality, obtained from (7.2) by replacing ∇u by $\partial u/\partial x_n$. □

Remark 7.2. As an example, we give a number of values of $\alpha_n(\beta)$, obtained by numerical solution of equation (7.1):

$$\alpha_3(2.5) \approx 1.2865,\ \alpha_3(3) \approx 1.4101,\ \alpha_3(3.5) \approx 1.4788,\ \alpha_3(4) \approx 1.521,\ \alpha_3(4.5) \approx 1.5482,\ \alpha_3(5) \approx 1.5664,$$
$$\alpha_4(3.5) \approx 1.207,\ \alpha_4(4) \approx 1.3079,\ \alpha_4(4.5) \approx 1.3698,\ \alpha_4(5) \approx 1.4115,\ \alpha_4(5.5) \approx 1.4413,\ \ \alpha_4(6) \approx 1.4631,$$
$$\alpha_5(4.5) \approx 1.1623,\ \alpha_5(5) \approx 1.2469,\ \alpha_5(5.5) \approx 1.3016,\ \alpha_5(6) \approx 1.3403,\ \alpha_5(6.5) \approx 1.3693,\ \ \alpha_5(7) \approx 1.3917,$$
$$\alpha_6(5.5) \approx 1.1316,\ \alpha_6(6) \approx 1.2063,\ \alpha_6(6.5) \approx 1.2548,\ \alpha_6(7) \approx 1.2903,\ \alpha_6(7.5) \approx 1.3176,\ \alpha_6(8) \approx 1.3393.$$

The representation for $C_{\alpha,\beta,\infty}$ with $\alpha \in (0, 1]$, obtained in the following auxiliary assertion, will be used later.

Lemma 7.3. *Let $0 < \alpha \le 1$ and $\beta > n-1$. Let $f \in L^{\infty}(\mathbb{R}^{n-1})$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,\beta,\infty}(x)$ in the inequality*

$$|\nabla u(x)| \le \mathcal{C}_{\alpha,\beta,\infty}(x) \big\| f \big\|_{\infty} \tag{7.20}$$

is given by

$$\mathcal{C}_{\alpha,\beta,\infty}(x) = C_{\alpha,\beta,\infty}\; x_n^{n-2+\beta(\alpha-1)}, \tag{7.21}$$

where

$$C_{\alpha,\beta,\infty} = |k|\beta \sup_{\gamma \ge 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(\alpha) + 2\int_0^{\pi} P\big(h_\gamma(\varphi)\big) \sin^{n-3}\varphi \, d\varphi \right\}. \tag{7.22}$$

Here

$$P(z) = \frac{(2\alpha)^{\beta-n+1} z^{n-1}}{\beta(4\alpha^2+z^2)^{\beta/2}} + \int_0^{\arctan\frac{z}{2\alpha}} \left\{ \frac{\beta-n+1}{\beta} - \alpha + \gamma\cos\varphi\cos\vartheta\sin\vartheta \right\} \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \, d\vartheta, \tag{7.23}$$

$$c_{n,\beta}(\alpha) = \left(\frac{\beta-n+1}{\beta} - \alpha \right) \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{\beta-n+1}{2}\right)}{2\Gamma\left(\frac{\beta}{2}\right)}\;, \tag{7.24}$$

and

$$h_\gamma(\varphi) = \gamma\cos\varphi + \Big(\gamma^2\cos^2\varphi + 4\alpha(1-\alpha)\Big)^{1/2}. \tag{7.25}$$

Proof. The formula (7.21) for the sharp coefficient $\mathcal{C}_{\alpha,\beta,\infty}(x)$ in (7.20) was proved in Proposition 2.1. By (7.7) and (5.5) we have

$$C_{\alpha,\beta,\infty} = |k|\beta \sup_{\gamma\ge 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big|\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\big| \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta, \tag{7.26}$$

where

$$\mathcal{G}(\varphi,\vartheta;\alpha,\gamma) = \cos^2\vartheta - \alpha + \gamma\cos\vartheta\sin\vartheta\cos\varphi. \tag{7.27}$$

First, we calculate the integral

$$\begin{aligned} c_{n,\beta}(\alpha) &= \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma) \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta \\ &= \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big\{ \cos^2\vartheta - \alpha + \gamma\cos\vartheta\sin\vartheta\cos\varphi \big\} \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta \\ &= \int_0^{\pi} \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\pi/2} \big(\cos^2\vartheta - \alpha \big) \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta \end{aligned}$$

$$= \left(\frac{\beta-n+1}{\beta} - \alpha\right) \frac{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{\beta-n+1}{2}\right)}{2\Gamma\left(\frac{\beta}{2}\right)}. \tag{7.28}$$

Now, we are looking for a solution of the equation

$$\cos^2\vartheta - \alpha + \gamma\cos\vartheta\sin\vartheta\cos\varphi = 0 \tag{7.29}$$

as a function ϑ of φ. We can rewrite (7.29) as the second order equation in $\tan\vartheta$:

$$-\alpha\tan^2\vartheta + \gamma\cos\varphi\tan\vartheta + (1-\alpha) = 0.$$

Since $0 \le \vartheta \le \pi/2$, we find that the nonnegative root of this equation is

$$\vartheta_\gamma(\varphi) = \arctan\frac{h_\gamma(\varphi)}{2\alpha}, \tag{7.30}$$

where

$$h_\gamma(\varphi) = \gamma\cos\varphi + \left(\gamma^2\cos^2\varphi + 4\alpha(1-\alpha)\right)^{1/2}. \tag{7.31}$$

We calculate the integral

$$\begin{aligned}
H_n(\varphi,\psi;\alpha,\gamma) &= \int_0^\psi \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta \\
&= \int_0^\psi \left(\cos^2\vartheta - \alpha + \gamma\cos\vartheta\sin\vartheta\cos\varphi\right)\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta \\
&= \int_0^\psi \left(\cos^2\vartheta - \alpha\right)\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta + \gamma\cos\varphi\int_0^\psi \cos^{\beta-n+1}\vartheta\sin^{n-1}\vartheta\mathrm{d}\vartheta \\
&= \frac{\sin^{n-1}\psi\cos^{\beta-n+1}\psi}{\beta} + \int_0^\psi\left\{\left(\frac{\beta-n+1}{\beta} - \alpha\right) + \gamma\cos\varphi\cos\vartheta\sin\vartheta\right\}\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta.
\end{aligned} \tag{7.32}$$

Obviously, $\mathcal{G}(\varphi,\vartheta;\alpha,\gamma) \ge 0$ for $0 \le \vartheta \le \vartheta_\gamma(\varphi)$ and $\mathcal{G}(\varphi,\vartheta;\alpha,\gamma) < 0$ for $\vartheta_\gamma(\varphi) < \vartheta \le \pi/2$. Hence,

$$\begin{aligned}
&\int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\pi/2} \left|\mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\right| \cos^{\beta-n}\vartheta\sin^{n-2}\vartheta d\vartheta \\
&= \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\vartheta_\gamma(\varphi)} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta d\vartheta \\
&\quad - \int_0^\pi \sin^{n-3}\varphi d\varphi \int_{\vartheta_\gamma(\varphi)}^{\pi/2} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta d\vartheta\ .
\end{aligned} \tag{7.33}$$

On the other hand, by (7.28),

$$c_{n,\beta}(\alpha) = \int_0^\pi \sin^{n-3}\varphi \mathrm{d}\varphi \int_0^{\vartheta_\gamma(\varphi)} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta + \int_0^\pi \sin^{n-3}\varphi \mathrm{d}\varphi \int_{\vartheta_\gamma(\varphi)}^{\pi/2} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta. \tag{7.34}$$

Using the equalities (7.33) and (7.34), and taking into account (7.32), we rewrite (7.26) as

$$C_{\alpha,\beta,\infty} = |k|\beta \sup_{\gamma\geq 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(\alpha) + 2\int_0^\pi \sin^{n-3}\varphi \, \mathrm{d}\varphi \int_0^{\vartheta_\gamma(\varphi)} \mathcal{G}(\varphi,\vartheta;\alpha,\gamma)\cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta \right\}$$
$$= |k|\beta \sup_{\gamma\geq 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(\alpha) + 2\int_0^\pi H_n\big(\varphi,\vartheta_\gamma(\varphi);\alpha,\gamma\big)\sin^{n-3}\varphi \mathrm{d}\varphi \right\}. \tag{7.35}$$

By (7.30),

$$\sin\vartheta_\gamma(\varphi) = \frac{h_\gamma(\varphi)}{\sqrt{4\alpha^2 + h_\gamma^2(\varphi)}}, \tag{7.36}$$

$$\cos\vartheta_\gamma(\varphi) = \frac{2\alpha}{\sqrt{4\alpha^2 + h_\gamma^2(\varphi)}}, \tag{7.37}$$

where $h_\gamma(\varphi)$ is defined by (7.31).

Using (7.36) and (7.37), we find

$$\sin^{n-1}\vartheta_\gamma(\varphi)\cos^{\beta-n+1}\vartheta_\gamma(\varphi) = \frac{(2\alpha)^{\beta-n+1} h_\gamma^{n-1}(\varphi)}{(4\alpha^2 + h_\gamma^2(\varphi))^{\beta/2}}. \tag{7.38}$$

By (7.32) and (7.38) we can write $H_n\big(\varphi,\vartheta_\gamma(\varphi);\alpha,\gamma\big)$ as

$$H_n\big(\varphi,\vartheta_\gamma(\varphi);\alpha,\gamma\big) = \frac{(2\alpha)^{\beta-n+1} h_\gamma^{n-1}(\varphi)}{\beta(4\alpha^2 + h_\gamma^2(\varphi))^{\beta/2}} + \int_0^{\arctan\frac{h_\gamma(\varphi)}{2\alpha}} \left\{ \left(\frac{\beta-n+1}{\beta} - \alpha \right) + \gamma\cos\varphi\cos\vartheta\sin\vartheta \right\} \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta,$$

which together with (7.35) leads to

$$C_{\alpha,\beta,\infty} = |k|\beta \sup_{\gamma\geq 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(\alpha) + 2\int_0^\pi P\big(h_\gamma(\varphi)\big)\sin^{n-3}\varphi \mathrm{d}\varphi \right\}, \tag{7.39}$$

where

$$P(z) = \frac{(2\alpha)^{\beta-n+1} z^{n-1}}{\beta(4\alpha^2 + z^2)^{\beta/2}} + \int_0^{\arctan\frac{z}{2\alpha}} \left\{ \left(\frac{\beta-n+1}{\beta} - \alpha \right) + \gamma\cos\varphi\cos\vartheta\sin\vartheta \right\} \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \mathrm{d}\vartheta. \tag{7.40}$$

Equalities (7.39), (7.40) together with (7.28), (7.31) prove the Lemma. □

In the next assertion we consider a particular case of (7.22) for $\alpha = 1$, $\beta \in (n-1, n]$. To find the explicit formula for $C_{1,\beta,\infty}$ we solve an extremal problem with a scalar parameter in the integrand of a double integral.

Theorem 7.4. *Let $\alpha = 1$ and $\beta \in (n-1, n]$. Let $f \in L^\infty(\mathbb{R}^{n-1})$, and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{1,\beta,\infty}(x)$ in the inequality*

$$|\nabla u(x)| \leq \mathcal{C}_{1,\beta,\infty}(x) \big\| f \big\|_\infty \tag{7.41}$$

is given by

$$\mathcal{C}_{1,\beta,\infty}(x) = \frac{|k| \pi^{(n-1)/2} (n-1) \Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} x_n^{n-2}. \tag{7.42}$$

The absolute value of the derivative of u with respect to the normal to the boundary of the half-space at any $x \in \mathbb{R}^n_+$ has the same supremum as $|\nabla u(x)|$.

Proof. The inequality (7.41) follows from (7.20). By (7.25), in the case $\alpha = 1$ we have $h_\gamma(\varphi) = 2\gamma \cos\varphi$ for $\varphi \in [0, \pi/2]$ and $h_\gamma(\varphi) = 0$ for $\varphi \in (\pi/2, \pi]$. Therefore, by (7.22)-(7.24) we obtain

$$C_{1,\beta,\infty} = |k|\beta \sup_{\gamma \geq 0} \frac{\omega_{n-2}}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(1) + 2\int_0^{\pi/2} U(\gamma\cos\varphi) \sin^{n-3}\varphi \, d\varphi \right\}, \tag{7.43}$$

where

$$c_{n,\beta}(1) = \frac{\sqrt{\pi}(1-n)\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{\beta-n+1}{2}\right)}{2\beta\Gamma\left(\frac{\beta}{2}\right)} \tag{7.44}$$

and

$$U(z) = \frac{z^{n-1}}{\beta\big(1+z^2\big)^{\beta/2}} + \int_0^{\arctan z} \left(\frac{1-n}{\beta} + z\cos\vartheta \sin\vartheta \right) \cos^{\beta-n}\vartheta \sin^{n-2}\vartheta \, d\vartheta. \tag{7.45}$$

Denoting

$$F(\gamma) = \frac{1}{\sqrt{1+\gamma^2}} \left\{ -c_{n,\beta}(1) + 2\int_0^{\pi/2} U(\gamma\cos\varphi) \sin^{n-3}\varphi \, d\varphi \right\}, \tag{7.46}$$

where $U(z)$ is defined by (7.45), we can rewrite (7.43) in the form

$$C_{1,\beta,\infty} = |k|\beta\omega_{n-2} \sup_{\gamma \geq 0} F(\gamma). \tag{7.47}$$

It follows from (7.46) that

$$\frac{dF}{d\gamma} = \frac{1}{(1+\gamma^2)^{3/2}} \left\{ c_{n,\beta}(1)\gamma + 2\int_0^{\pi/2} \left(-\gamma U(\gamma\cos\varphi) + (1+\gamma^2) \frac{\partial U(\gamma\cos\varphi)}{\partial\gamma} \right) \sin^{n-3}\varphi \, d\varphi \right\}. \tag{7.48}$$

Differentiating $U(\gamma\cos\varphi)$ with respect to γ, we obtain

$$\frac{\partial U(\gamma\cos\varphi)}{\partial\gamma} = \cos\varphi \int_0^{\arctan(\gamma\cos\varphi)} \cos^{\beta-n+1}\vartheta \sin^{n-1}\vartheta \mathrm{d}\vartheta. \tag{7.49}$$

Substituting $U(\gamma\cos\varphi)$ from (7.45) and $\partial U(\gamma\cos\varphi)/\partial\gamma$ from (7.49) into (7.48), we arrive at equality

$$\frac{\mathrm{d}F}{\mathrm{d}\gamma} = \frac{1}{(1+\gamma^2)^{3/2}}\Big\{\Phi_1(\gamma)+\Phi_2(\gamma)\Big\}, \tag{7.50}$$

where

$$\Phi_1(\gamma) = \frac{\gamma(n-1)}{\beta}\left\{-\frac{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{\beta-n+1}{2}\right)}{2\Gamma\left(\frac{\beta}{2}\right)} + 2\int_0^{\pi/2}\left(\int_0^{\arctan(\gamma\cos\varphi)}\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta\right)\sin^{n-3}\varphi\mathrm{d}\varphi\right\}$$

and

$$\Phi_2(\gamma) = 2\left\{-\frac{\gamma}{\beta}\int_0^{\pi/2}\frac{(\gamma\cos\varphi)^{n-1}\sin^{n-3}\varphi}{\left(1+\gamma^2\cos^2\varphi\right)^{\beta/2}}\mathrm{d}\varphi + \int_0^{\pi/2}\left(\int_0^{\arctan(\gamma\cos\varphi)}\cos^{\beta-n+1}\vartheta\sin^{n-1}\vartheta\mathrm{d}\vartheta\right)\cos\varphi\sin^{n-3}\varphi\mathrm{d}\varphi\right\}.$$

Estimating $\Phi_1(\gamma)$, we obtain

$$\Phi_1(\gamma) \le \frac{\gamma(n-1)}{\beta}\left\{-\frac{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{\beta-n+1}{2}\right)}{2\Gamma\left(\frac{\beta}{2}\right)} + 2\int_0^{\pi/2}\left(\int_0^{\pi/2}\cos^{\beta-n}\vartheta\sin^{n-2}\vartheta\mathrm{d}\vartheta\right)\sin^{n-3}\varphi\mathrm{d}\varphi\right\} = 0. \tag{7.51}$$

By differentiating $\Phi_2(\gamma)$, we arrive at equality

$$\frac{d\Phi_2}{\mathrm{d}\gamma} = 2\int_0^{\pi/2}\frac{(\gamma\cos\varphi)^{n-1}}{\left(1+\gamma^2\cos^2\varphi\right)^{(\beta+2)/2}}\left\{-\frac{n}{\beta}+\left(1-\frac{n}{\beta}\right)\gamma^2\cos^2\varphi+\cos\varphi\right\}\sin^{n-3}\varphi\mathrm{d}\varphi.$$

Therefore, $\Phi_2'(\gamma) < 0$ for $\gamma > 0$ and any $\beta \in (n-1, n]$. This together with $\Phi_2(0) = 0$ proves inequality $\Phi_2(\gamma) < 0$ for $\gamma > 0$ and any $\beta \in (n-1, n]$. Hence, by (7.50) and (7.51), $F'(\gamma) < 0$ for $\gamma > 0$ and any $\beta \in (n-1, n]$. So, by (7.47),

$$C_{1,\beta,\infty} = |k|\beta\omega_{n-2}F(0),$$

which, in view of (7.44)–(7.46), leads to

$$C_{1,\beta,\infty} = -|k|\beta\omega_{n-2}c_{n,\beta}(1) = |k|\frac{\pi^{(n-1)/2}(n-1)\Gamma\left(\frac{\beta-n+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}.$$

Combining the last formula with (7.21) in the case $\alpha = 1$, we arrive at (7.42). Since $\gamma = |\boldsymbol{z}'|/z_n$ and the supremum in (7.47) is attained at $\gamma = 0$, we conclude that the coefficient $\mathcal{C}_{1,\beta,\infty}(x) = C_{1,\beta,\infty}x_n^{n-2}$ is sharp under conditions of the Theorem also in the weaker inequality, obtained from (7.41) by replacing ∇u by $\partial u/\partial x_n$. □

8. Sharp estimates for harmonic and biharmonic functions

By $h^p(\mathbb{R}^n_+)$ we denote the Hardy space of harmonic functions on $\mathbb{R}^n_+$ which can be represented as the Poisson integral

$$v(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{x_n}{|y-x|^n} v(y')\mathrm{d}y' \tag{8.1}$$

with boundary values in $L^p(\mathbb{R}^{n-1})$, $1 \le p \le \infty$. Multiplying (8.1) on $x_n^{n\alpha-1}$, $\alpha \ge 0$, we obtain

$$x_n^{n\alpha-1} v(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \left(\frac{x_n^\alpha}{|y-x|} \right)^n v(y')\mathrm{d}y'. \tag{8.2}$$

On the right-hand side of the last equality is located generalized Poisson integral (1.2) with $k = 2/\omega_n$ and $\beta = n$.

Thus, we can apply the results of previous sections to obtain sharp pointwise estimates for $\left|\nabla\left(x_n^{n\alpha-1}v(x)\right)\right|$ in terms of the norm $L^p(\mathbb{R}^{n-1}), 1 \le p \le \infty$.

As consequence of Proposition 2.1 and Theorem 4.1 with $\beta = n$, we obtain

Corollary 8.1. *Let $v \in h^p(\mathbb{R}^n_+)$ and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,n,p}(x)$ in the inequality*

$$\left|\nabla\left(x_n^{n\alpha-1}v(x)\right)\right| \le \mathcal{C}_{\alpha,n,p}(x) \left\| v \right\|_p$$

is given by

$$\mathcal{C}_{\alpha,n,p}(x) = C_{\alpha,n,p}\; x_n^{n\alpha-2-(n-1)/p},$$

where

$$C_{\alpha,n,1} = \frac{2n}{\omega_n} \sup_{|\boldsymbol{z}|=1} \sup_{\sigma \in \mathbb{S}^{n-1}_+} \left|\left(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\right)\right| \left(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\right)^n,$$

$$C_{\alpha,n,p} = \frac{2n}{\omega_n} \sup_{|\boldsymbol{z}|=1} \left\{ \int_{\mathbb{S}^{n-1}_+} \left|\left(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma, \boldsymbol{z}\right)\right|^{\frac{p}{p-1}} \left(\boldsymbol{e}_\sigma, \boldsymbol{e}_n\right)^{\frac{n}{p-1}} d\sigma \right\}^{\frac{p-1}{p}}$$

for $1 < p < \infty$, and

$$C_{\alpha,n,\infty} = \frac{2n}{\omega_n} \sup_{|\boldsymbol{z}|=1} \int_{\mathbb{S}^{n-1}_+} \left|\left(\alpha \boldsymbol{e}_n - (\boldsymbol{e}_\sigma, \boldsymbol{e}_n)\boldsymbol{e}_\sigma,\ \boldsymbol{z}\right)\right| d\sigma.$$

In particular, the sharp constant $C_{0,n,p}$ in the inequality

$$\left| \nabla \left\{ \frac{v(x)}{x_n} \right\} \right| \le C_{0,n,p} x_n^{-2-(n-1)/p} \left\| v \right\|_p \tag{8.3}$$

is given by $C_{0,n,1} = 2n/\omega_n$, $C_{0,n,\infty} = 1$ *and*

$$C_{0,n,p} = \frac{2n}{\omega_n}\left\{\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3p+n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(n+2)p}{2(p-1)}\right)}\right\}^{\frac{p}{p-1}}$$

for $1 < p < \infty$.

The constant $C_{0,n,p}$ *is sharp in conditions of the Corollary also in the weaker inequality obtained from* (8.3) *by replacing* $\nabla\big(x_n^{n\alpha-1}v\big)$ *by* $\partial\big(x_n^{n\alpha-1}v\big)/\partial x_n$.

Concretizing Theorem 3.1 for $\beta = n$, we arrive at

Corollary 8.2. *Let* $v \in h^1(\mathbb{R}^n_+)$ *and let* x *be an arbitrary point in* $\mathbb{R}^n_+$. *The sharp coefficient* $\mathcal{C}_{\alpha,n,1}(x)$ *in the inequality*

$$\left|\nabla\big(x_n^{n\alpha-1}v(x)\big)\right| \le \mathcal{C}_{\alpha,n,1}(x)\big\|v\big\|_1 \tag{8.4}$$

is given by

$$\mathcal{C}_{\alpha,n,1}(x) = C_{\alpha,n,1}\; x_n^{n(\alpha-1)-1},$$

where

$$C_{\alpha,n,1} = \frac{2n}{\omega_n}|1-\alpha|$$

if

$$0 \le \alpha \le \frac{\sqrt{n+1}}{\sqrt{n+1}+1} \quad \text{or} \quad \alpha \ge \frac{\sqrt{n+1}}{\sqrt{n+1}-1}, \tag{8.5}$$

and

$$C_{\alpha,n,1} = \frac{2n}{\omega_n}\left(\frac{n}{2\alpha-1}\right)^{n/2}\left(\frac{\alpha^2}{n+1}\right)^{(n+2)/2}$$

if

$$\frac{\sqrt{n+1}}{\sqrt{n+1}+1} < \alpha < \frac{\sqrt{n+1}}{\sqrt{n+1}-1}.$$

In particular,

$$C_{1,n,1} = \frac{2(n-2)}{n\omega_n}\left\{\frac{(n-1)^2}{(n-2)(n+1)}\right\}^{(n+2)/2}.$$

If α *satisfies condition* (8.5), *then the coefficient* $\mathcal{C}_{\alpha,n,1}(x)$ *is sharp in conditions of the Corollary also in the weaker inequality obtained from* (8.4) *by replacing* $\nabla\big(x_n^{n\alpha-1}v\big)$ *by* $\partial\big(x_n^{n\alpha-1}v\big)/\partial x_n$.

Theorem 6.1 in the case $\beta = n$ leads to the following

Corollary 8.3. *Let $v \in h^2(\mathbb{R}^n_+)$ and let x be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $\mathcal{C}_{\alpha,n,2}(x)$ in the inequality*

$$\left|\nabla\left(x_n^{n\alpha-1}v(x)\right)\right| \leq \mathcal{C}_{\alpha,n,2}(x)\left\|v\right\|_2 \tag{8.6}$$

is given by

$$\mathcal{C}_{\alpha,n,2}(x) = C_{\alpha,n,2}\; x_n^{n\alpha-(n+3)/2},$$

where

$$C_{\alpha,n,2} = \left\{\frac{n}{2^{n-2}\omega_n}\left(n\alpha^2-(n+1)\alpha+\frac{n+3}{4}\right)\right\}^{1/2}$$

if

$$0 \leq \alpha \leq \frac{1}{2} \quad \text{or} \quad \alpha \geq \frac{1}{2}+\frac{1}{n}, \tag{8.7}$$

and

$$C_{\alpha,n,2} = \left\{\frac{n}{2^n\omega_n}\right\}^{1/2}$$

if

$$\frac{1}{2} < \alpha < \frac{1}{2}+\frac{1}{n}.$$

In particular,

$$C_{1,n,2} = \left\{\frac{n(n-1)}{2^n\omega_n}\right\}^{1/2}.$$

If α satisfies condition (8.7), then the coefficient $\mathcal{C}_{\alpha,n,2}(x)$ is sharp in conditions of the corollary also in the weaker inequality obtained from (8.6) by replacing $\nabla\left(x_n^{n\alpha-1}v)\right)$ by $\partial\left(x_n^{n\alpha-1}v)\right)/\partial x_n$.

As consequence of Theorem 7.1 with $\beta = n$ we obtain

Corollary 8.4. *Let $v \in h^\infty(\mathbb{R}^n_+)$ and let x be an arbitrary point in $\mathbb{R}^n_+$. Let α_n be the root from the interval $(1,+\infty)$ of the equation*

$$\frac{2}{\pi(n\alpha-1)} = \frac{\alpha-1}{1+\sqrt{1+(\alpha-1)^2}}$$

with respect to α.

If $\alpha \geq \alpha_n$, then the sharp coefficient $\mathcal{C}_{\alpha,n,\infty}(x)$ in the inequality

$$\left|\nabla\left(x_n^{n\alpha-1}v(x)\right)\right| \leq \mathcal{C}_{\alpha,n,\infty}(x)\left\|v\right\|_\infty$$

is given by

$$\mathcal{C}_{\alpha,n,\infty}(x) = (n\alpha - 1)\, x_n^{n\alpha-2}.$$

In conditions of the Corollary, the absolute value of the derivative of $x_n^{n\alpha-1}v$ *with respect to the normal to the boundary of the half-space at any* $x \in \mathbb{R}^n_+$ *has the same supremum as* $|\nabla(x_n^{n\alpha-1}v)|$.

Theorem 7.4 with $\beta = n$ implies

Corollary 8.5. *Let* $v \in h^\infty(\mathbb{R}^n_+)$ *and let* x *be an arbitrary point in* $\mathbb{R}^n_+$. *The sharp coefficient* $\mathcal{C}_{1,n,\infty}(x)$ *in the inequality*

$$\left|\nabla\left(x_n^{n-1}v(x)\right)\right| \le \mathcal{C}_{1,n,\infty}(x)\left\|v\right\|_\infty$$

is given by

$$\mathcal{C}_{1,n,\infty}(x) = (n-1)x_n^{n-2}.$$

The absolute value of the derivative of $x_n^{n-1}v(x)$ *with respect to the normal to the boundary of the half-space at any* $x \in \mathbb{R}^n_+$ *has the same supremum as* $\left|\nabla\left(x_n^{n-1}v(x)\right)\right|$.

We conclude this section with some remark. The following representation is well known (*e.g.* Schot [7])

$$w(x) = \frac{2n}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{x_n^3}{|y-x|^{n+2}} f_0(y')\mathrm{d}y' + \frac{2}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{x_n^2}{|y-x|^n} f_1(y')\mathrm{d}y' \tag{8.8}$$

for solution in $\mathbb{R}^n_+$ of the first boundary value problem for the biharmonic equation

$$\varDelta^2 w = 0 \ \ in \ \ \mathbb{R}^n_+, \quad w\big|_{x_n=0} = f_0(x')_+, \quad frac\partial w\partial x_n\Big|_{x_n=0} = f_1(x'), \tag{8.9}$$

where $y = (y', 0)$.

By w_0 we denote a solution of the problem (8.9) with $f_0 = 0$. By (8.8), we have

$$x_n^{n\alpha-2}w_0(x) = \frac{2}{\omega_n}\int_{\mathbb{R}^{n-1}} \frac{x_n^{n\alpha}}{|y-x|^n} f_1(y')\mathrm{d}y'. \tag{8.10}$$

The right-hand side of (8.10) is the same as in (8.2). So, by Proposition 2.1 we arrive at

$$\left|\nabla\left(x_n^{n\alpha-2}w_0(x)\right)\right| \le C_{\alpha,n,p}x_n^{n\alpha-2-(n-1)/p}\left\|\frac{\partial w_0}{\partial x_n}\right\|_p, \tag{8.11}$$

where the sharp constant $C_{\alpha,n,p}$ in (8.11) is the same as in Corollaries 8.1–8.5.

For instance, in the case $\alpha = 0, p = \infty$ by Corollary 8.1 we have the following inequality with the sharp coefficient

$$\left|\nabla\left\{\frac{w_0(x)}{x_n^2}\right\}\right| \le \frac{1}{x_n^2}\,\left\|\frac{\partial w_0}{\partial x_n}\right\|_\infty.$$

In another interesting case $\alpha = 1$ and $p = \infty$, Corollary 8.5 leads to the following inequality

$$\left|\nabla\left(x_n^{n-2} w_0(x)\right)\right| \leq (n-1) x_n^{n-2} \left\| \frac{\partial w_0}{\partial x_n} \right\|_\infty .$$

Acknowledgements. This research was supported by the Ministry of Education and Science of the Russian Federation, Agreement No. 02.a03.21.0008.

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