Math. Model. Nat. Phenom. 13 (2018) 35
https://doi.org/10.1051/mmnp/2018029

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

MULTILINEAR STOCKWELL TRANSFORMS

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Abstract. The main aim of this paper is to introduce multilinear versions of the Stockwell transforms (also named S-transforms) by using the fact that S-transforms can be written as convolution products. Further on we extend the multilinear S-transforms from the Schwartz class of rapidly decreasing functions to the space of tempered distributions. In the sequel we give a relation between multilinear S-transforms and multilinear pseudo-differential operators.

We also state and prove some boundedness results regarding multilinear S-transforms on the Lebegue's spaces $L^p(\mathbf{R}^n)$ and also on the Hörmander's spaces $B_{p,k}(\mathbf{R}^n)$, where $p \geq 1$ and k is a temperate weight function. In the end, a weak uncertainty principle for multilinear S-transforms and for its adjoint is also given.

Mathematics Subject Classification. 42C40, 47G10

Received November 29, 2017. Accepted February 1, 2018.

1. Introduction

In 1996 Stockwell, Mansinha and Lowe have introduced in [17] a new joint time-frequency transform of a signal $\varphi \in L^2(\mathbb{R})$ with respect to a normalised Gaussian function $\omega(t,\xi) = \frac{|\xi|}{\sqrt{2\pi}} e^{-\frac{t^2\xi^2}{2}}$, $t, \xi \in \mathbb{R}$, by $S(\varphi)(\tau,\xi) = = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(\tau-t)^2\xi^2}{2}} \varphi(t) e^{-2\pi i\, t\xi} \mathrm{d}t$, for all τ and ξ in $\mathbb{R}$. Since its appearing this time-frequency transform has been named Stockwell transform, sometimes referred to as the S-transform in the mathematical literature. It has been of interest in the last years in geophysics, in medical imaging, in signal procesing and in mathematics in general. It is easy to see that the Stockwell transform can be re-written as a convolution product. Indeed,

$$S_\omega(\varphi)(\tau,\xi) = \int_{\mathbb{R}} \omega(\tau - t, \xi)\varphi(t) e^{-2\pi i\, t\xi} \mathrm{d}t = (\omega(\cdot,\xi) * \varphi(\cdot) e^{-2\pi i \cdot \xi})(\tau),$$

where $\omega(t,\xi) = \frac{|\xi|}{\sqrt{2\pi}} e^{-\frac{t^2\xi^2}{2}}$, $t, \tau, \xi \in \mathbb{R}$.

In signal analysis, the term $S_\omega(\varphi)(\tau,\xi)$ gives the time-frequency content of a signal φ at time τ and frequency ξ when placing the window ω at time τ. Let us emphasize that the analyzing window of the Stockwell transform

Keywords and phrases: Fourier transforms, Stockwell transforms, pseudo-differential operators, uncertainty principle.

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can be adapted to the frequency to be analyzed, that is it depends of that frequency. Although the Gaussian window is chosen for several reasons (see [16]) any desired window may be employed.

Many extensions of the Stockwell transform have been proposed in the last years (see for example [3, 7, 8, 9, 12, 13, 14]). For other aspects or some applications of the Stockwell transform we refer to [1, 2, 5].

Let us first recall the one-dimensional modified Stockwell transform studied in [7, 8]. In this case $\omega(t,\xi) = |\xi|^{1/s}\overline{\Omega(-t\xi)} = |\xi|^{1/s}\mathcal{I}\Omega(t\xi)$, where $\mathcal{I}$ denote the involution operator defined by $\mathcal{I}\Omega(t) = \overline{\Omega(-t)}$, $1 \le s < \infty$, $\Omega \in L^2(\mathbb{R})$ and for a signal $\varphi \in L^2(\mathbb{R})$ its Stockwell transform is given by

$$S_{s,\Omega}(\varphi)(\tau,\xi) = \int_{\mathbb{R}} e^{-2\pi\, it\xi}|\xi|^{1/s}\overline{\Omega((t-\tau)\xi)}\varphi(t)\mathrm{d}t, \quad \tau,\xi \in \mathbb{R}.$$

Let us remark that when $\Omega(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ is the normalised Gaussian window and $s = 1$, then we recover the standard one-dimensional Stockwell transform.

When we take $\omega(t,\xi) = |\det A_\xi|^{-1/s}\mathcal{I}\Omega(A_\xi^{-1}t)$, where $\Omega \in L^2(\mathbb{R}^n)$, $A : \mathbb{R}^n \to GL(n,\mathbb{R}^n)$ is a matrix-valued function, $A(\xi) = A_\xi$, $\xi \in \mathbb{R}^n$ then for a signal $\varphi \in L^2(\mathbb{R}^n)$ its multi-dimensional modified Stockwell transform is given by $S_{s,A,\Omega}(\varphi)(\tau,\xi) = |\det A_\xi|^{-1/s}\int_{\mathbb{R}^n} e^{-2\pi it\cdot\xi}\overline{\Omega(A_\xi^{-1}(t-\tau))}\varphi(t)\mathrm{d}t$, $\tau,\xi \in \mathbb{R}^n$ and $t\cdot\xi = \sum_{j=1}^{n} t_j\xi_j$.

We notice that when $n = 1$, $s = 1$, $\Omega(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ and $A : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $A(\xi) = 1/\xi$, $\xi \neq 0$ we recover the standard one-dimensional Stockwell transform. The aim of this paper is to introduce and study the multilinear Stockwell transform.

In the beginning let us introduce some notation that is convenient for the multilinear Fourier analysis throughout this paper.

Points in $\mathbb{R}^{nm}$ are denoted by $x = (x_1,\dots,x_m)$, where $x_1,\dots,x_m$ are points in $\mathbb{R}^n$. For all points $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ in $\mathbb{R}^n$, the inner product $x\cdot y$ of x and y is given by $x\cdot y = \sum_{j=1}^{n} x_j\cdot y_j$ and in general for all points $x = (x_1,\dots,x_m)$, $y = (y_1,\dots,y_m)$ in $\mathbb{R}^{nm}$ the inner product $x\cdot y$ of x and y is given by $x\cdot y = \sum_{k=1}^{m} x_k\cdot y_k$, where $x_k\cdot y_k$ is the ordinary Euclidean inner product of x_k and y_k in $\mathbb{R}^n$. The sum $|x|$ of x in $\mathbb{R}^{nm}$ is given by $|x| = \sum_{k=1}^{m} x_k$. For all points x in $\mathbb{R}^n$ we denote by $\tilde{x}^m = (x,\dots,x)$-$m$-times.

The tensor product $\bigotimes_{j=1}^{m} f_j$ of the measurable functions $f_1,\dots,f_m$ on $\mathbb{R}^n$ is the function on $\mathbb{R}^{nm}$ defined by $(\bigotimes_{j=1}^{m} f_j)(x) = f_1(x_1)f_2(x_2)\dots f_m(x_m)$, for all $x = (x_1,\dots,x_m)$ in $\mathbb{R}^{nm}$.

We denote by $\mathrm{d}x$ the Lebesgue measure $\mathrm{d}x_1\mathrm{d}x_2\dots dx_m$ in $\mathbb{R}^{nm}$ and we also denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$ of tempered distributions.

Let $\omega \in \mathcal{S}(\mathbb{R}^{2n})$. Then for all $\varphi = (\varphi_1,\dots,\varphi_m) \in \mathcal{S}(\mathbb{R}^n)^m$ we define $S_\omega(\varphi)$ by

$$\begin{aligned} S_\omega(\varphi)(\tau,\xi) &= \int_{\mathbb{R}^{nm}} e^{-2\pi i\,|t|\cdot\xi}\omega(|\tilde{\tau}^m - t|,\xi)(\bigotimes_{j=1}^{m}\varphi_j)(t)\mathrm{d}t, \\ &= (\omega(|\cdot|,\xi) * \bigotimes_{j=1}^{m}\varphi_j e^{-2\pi i|\cdot|\cdot\xi})(\tau), \quad \tau,\xi \in \mathbb{R}^n. \end{aligned}$$

Now, let us consider $\omega^{\sim} \in \mathcal{S}(\mathbb{R}^{(m+1)n})$. Then for all $\varphi = (\varphi_1, \ldots, \varphi_m) \in \mathcal{S}(\mathbb{R}^n)^m$ we define $\tilde{S}_{\omega^{\sim}}(\varphi)$ by

$$\begin{aligned}\tilde{S}_{\omega^{\sim}}(\varphi)(\tau,\xi) &= \int_{\mathbb{R}^{nm}} e^{-2\pi i\ |t|\cdot\xi}\omega^{\sim}(\tau - t,\xi)(\bigotimes_{j=1}^{m}\varphi_j)(t)\mathrm{d}t,\\ &= (\omega^{\sim}(\cdot,\xi) * \bigotimes_{j=1}^{m}\varphi_j e^{-2\pi i|\cdot|\cdot\xi})(\tau), \quad \tau\in\mathbb{R}^{mn},\ \xi\in\mathbb{R}^n.\end{aligned}$$

Let us notice that if we take $\omega^{\sim}(t,\xi) = \omega(|t|,\xi)$, $t \in \mathbb{R}^{nm}$, $\xi \in \mathbb{R}^n$ then we get $S_\omega(\varphi)(\tau,\xi) = \tilde{S}_{\omega^{\sim}}(\varphi)(\tilde{\tau}^m,\xi)$, for all τ,ξ in $\mathbb{R}^n$. So, from now on we suppose that $\omega \in \mathcal{S}(\mathbb{R}^{(m+1)n})$ and that for all $\varphi = (\varphi_1,\ldots,\varphi_m) \in \mathcal{S}(\mathbb{R}^n)^m$, $S_\omega(\varphi)$ is defined by

$$S_\omega(\varphi)(\tau,\xi) = \int_{\mathbb{R}^{nm}} e^{-2\pi i\ |t|\cdot\xi}\omega(\tau - t,\xi)(\bigotimes_{j=1}^{m}\varphi_j)(t)\mathrm{d}t,\ \tau\in\mathbb{R}^{nm},\ \xi\in\mathbb{R}^n.$$

We call $S_\omega(\varphi)$ the multilinear Stockwell transform of the signal φ with respect to the window ω. When we take $m = 1$ we recover the definition of the S-transform which has been used by Singh in [15]. Now, if φ is a function defined on $\mathbb{R}^{nm}$ then its Fourier transform is taken to be the one given by

$$\mathcal{F}^{mn}(\varphi)(\xi) = \int_{\mathbb{R}^{nm}} e^{-2\pi i\ t\cdot\xi}\varphi(t)\mathrm{d}t, \quad \xi\in\mathbb{R}^{nm}.$$

Moreover if $\varphi = \varphi(t,s)$ is a function defined on $\mathbb{R}^{m+n}$ where $t\in\mathbb{R}^m$ and $s\in\mathbb{R}^n$, then we also denote by $\mathcal{F}_1^m(\varphi)$ and by $\mathcal{F}_2^n(\varphi)$ the partial Fourier transform with respect to the first m coordinates respectively with respect to the last n coordinates. We also denote by $\mathcal{F}^{m+n}(\varphi)$ the Fourier transform with respect to all coordinates in $\mathbb{R}^{m+n}$ and by $(\mathcal{F}_1^{-1})^m$ and by $(\mathcal{F}_2^{-1})^n$ the inverse partial Fourier transform with respect to the first m coordinates respectively with respect to the last n coordinates.

So, because we can rewritten the multilinear Stockwell transform as a convolution product as

$$S_\omega(\varphi)(\tau,\xi) = \left(\omega(\cdot,\xi) * \bigotimes_{j=1}^{m}\varphi_j e^{-2\pi\ i|\cdot|\cdot\xi}\right)(\tau), \quad \tau\in\mathbb{R}^{nm}, \quad \xi\in\mathbb{R}^n,$$

then applying the convolution property for the Fourier transform, we obtain

$$\begin{aligned}S_\omega(\varphi)(\tau,\xi) &= (\mathcal{F}_1^{-1})^{mn}\left[\mathcal{F}^{mn}\left(\bigotimes_{j=1}^{m}\varphi_j\right)(\alpha+\tilde{\xi})^m\mathcal{F}_1^{mn}(\omega)(\alpha,\xi)\right](\tau)\\ &= (\mathcal{F}_1^{-1})^{mn}\left[\left(\bigotimes_{j=1}^{m}\mathcal{F}^n\varphi_j\right)(\alpha+\tilde{\xi})^m\mathcal{F}_1^{mn}(\omega)(\alpha,\xi)\right](\tau), \quad \tau,\alpha\in\mathbb{R}^{nm}, \quad \xi\in\mathbb{R}^n.\end{aligned}$$

Further on we state some results in order to extend the multilinear S-transform from the Schwartz class of rapidly decreasing functions to the space of tempered distributions. More precisely we shall allow either the signal $\varphi\in\mathcal{S}'(\mathbb{R}^n)^m$ and the window $\omega\in\mathcal{S}(\mathbb{R}^{(m+1)n})$ or $\varphi\in\mathcal{S}(\mathbb{R}^n)^m$ and $\omega\in\mathcal{S}'(\mathbb{R}^{(m+1)n})$.

Proposition 1.1. *Let $\omega\in\mathcal{S}(\mathbb{R}^{(m+1)n})$ be a window function and let $\varphi\in\mathcal{S}(\mathbb{R}^n)^m$ be a signal. Then the multilinear S-transform of φ with respect to ω is a function in $\mathcal{S}(\mathbb{R}^{(m+1)n})$ that is $S_\omega(\varphi)\in\mathcal{S}(\mathbb{R}^{(m+1)n})$. Thus, S_ω maps $\mathcal{S}(\mathbb{R}^n)^m$ into $\mathcal{S}(\mathbb{R}^{(m+1)n})$.*

The proof of Proposition 1.1 is an easy consequence of the above formula and of the well-known result that the Fourier transform and its inverse are continuous isomorphisms from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Proposition 1.2. *Let $\omega \in \mathcal{S}(\mathbb{R}^{(m+1)n})$ be a window function. Then we can extend the mapping $S_\omega : \mathcal{S}(\mathbb{R}^n)^m \to \mathcal{S}(\mathbb{R}^{(m+1)n})$ in Proposition 1.1 to a mapping $S_\omega : \mathcal{S}'(\mathbb{R}^n)^m \to \mathcal{S}'(\mathbb{R}^{(m+1)n})$.*

Proof. We set forth three pieces of notation that will be used consistently throughout this paper. First, we shall use the symbol $\mathcal{R}$ to denote the reflections of a function in origin, that is $\mathcal{R}\varphi(x) = \varphi(-x)$, for all $x \in \mathbb{R}^n$, if φ is a function defined on $\mathbb{R}^n$. Second, we denote by $\langle \cdot, \cdot \rangle$ the brackets which are well defined by some form of duality and for any two measurable functions φ and ψ act as $\langle \varphi, \psi \rangle = \int_{\mathbb{R}^n} \varphi(x)\bar{\psi}(x)\mathrm{d}x$, if the integral in the right hand side exists (for example if φ and ψ are in $L^2(\mathbb{R}^n)$. Third, we denote by T_τ the translation operator defined by $\underline{(T_\tau\varphi)(t)} = \varphi(t-\tau)$. We recall that by $\mathcal{I}$ we have denoted the involution operator which works such as $\mathcal{I}\varphi(x) = \overline{\varphi(-x)}$. Now, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)^m$ and $\psi \in \mathcal{S}(\mathbb{R}^{(m+1)n})$, we have

$$S_\omega(\varphi)(\tau,\xi) = \left(\bigotimes_{j=1}^m \varphi_j(\cdot) e^{-2\pi\, i|\cdot|\cdot\xi} * \omega(\cdot,\xi) \right)(\tau) = \langle \bigotimes_{j=1}^m \varphi_j(\cdot)e^{-2\pi\, i|\cdot|\cdot\xi}, \overline{T_\tau \mathcal{R}\omega(\cdot,\xi)} \rangle,$$

$$\begin{aligned}
\langle S_\omega(\varphi)(\tau,\xi), \psi \rangle &= \langle \langle \bigotimes_{j=1}^m \varphi_j(\cdot)e^{-2\pi\, i|\cdot|\cdot\xi}, \overline{T_\tau \mathcal{R}\omega(\cdot,\xi)} \rangle, \psi \rangle \\
&= \int_{\mathbb{R}^{(m+1)n}} \langle \bigotimes_{j=1}^m \varphi_j(\cdot)e^{-2\pi\, i|\cdot|\cdot\xi}, \overline{T_\tau \mathcal{R}\omega(\cdot,\xi)} \rangle \bar{\psi}(\tau,\xi)\mathrm{d}\tau\, \mathrm{d}\xi \\
&= \langle \bigotimes_{j=1}^m \varphi_j(\cdot), \int_{\mathbb{R}^{(m+1)n}} e^{+2\pi i|\cdot|\cdot\xi} \overline{T_\tau \mathcal{R}\omega(\cdot,\xi)} \psi(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi \rangle \\
&= \langle \bigotimes_{j=1}^m \varphi_j(\cdot), \int_{\mathbb{R}^n} e^{+2\pi\, i|\cdot|\cdot\xi} \left(\int_{\mathbb{R}^{mn}} \overline{T_\tau \mathcal{R}\omega(\cdot,\xi)} \psi(\tau,\xi)\mathrm{d}\tau \right) \mathrm{d}\xi \rangle \\
&= \langle \bigotimes_{j=1}^m \varphi_j(\cdot), \int_{\mathbb{R}^n} e^{+2\pi\, i|\cdot|\cdot\xi} (\psi * \mathcal{I}\omega)(\cdot,\xi)\mathrm{d}\xi \rangle = \langle \bigotimes_{j=1}^m \varphi_j(\cdot), f_{\psi,\omega}(\cdot) \rangle,
\end{aligned}$$

where

$$f_{\psi,\omega}(t) = \int_{\mathbb{R}^n} e^{2\pi\, i|t|\cdot\xi} (\psi * \mathcal{I}\omega)(t,\xi)\mathrm{d}\xi = (\mathcal{F}_2^{-1})^n (\psi * \mathcal{I}\omega)(t,|t|),\ t \in \mathbb{R}^{mn}.$$

We notice that $f_{\psi,\omega} \in \mathcal{S}(\mathbb{R}^{mn})$. So, for any $\varphi \in \mathcal{S}'(\mathbb{R}^n)^m$ and $\psi \in \mathcal{S}(\mathbb{R}^{(m+1)n})$ we can define the tempered distributions $S_\omega(\varphi)$ by $\langle S_\omega(\varphi), \psi \rangle = \langle \bigotimes_{j=1}^m \varphi_j, f_{\psi,\omega} \rangle$, where $f_{\psi,\omega}$ is the function defined above and we have denoted by $\bigotimes_{j=1}^m \varphi_j$ the direct product of tempered distributions φ_j, $j = 1, 2, \ldots, m$. Let us recall that if $f \in \mathcal{S}'(\mathbb{R}^m)$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ are two tempered distributions on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, then we can define a new tempered distribution denoted by $f \otimes g \in \mathcal{S}'(\mathbb{R}^{m+n})$ and given by $\langle f \otimes g, \varphi \rangle = \langle f, \langle g, \varphi \rangle \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^{m+n})$. We shall call $f \otimes g$ the direct product of distributions f and g. Thus, the proof of Proposition 1.2 is complete. ∎

Proposition 1.3. *If $\omega \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$, then S_ω maps $\mathcal{S}(\mathbb{R}^n)^m$ into $\mathcal{S}'(\mathbb{R}^{(m+1)n})$.*

Proof. If $\varphi \in \mathcal{S}(\mathbb{R}^n)^m$ and $\psi \in \mathcal{S}(\mathbb{R}^{(m+1)n})$ then we define a function in the Schwartz class by

$$\chi_{\varphi,\psi}(\alpha,\xi) = \overline{\mathcal{F}^{mn}\left(\bigotimes_{j=1}^{m}\varphi_j\right)}(\alpha+\tilde{\xi}^m)\psi(\alpha,\xi) \in \mathcal{S}(\mathbb{R}^{(m+1)n}).$$

Now, for any $\omega \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$, $\varphi \in \mathcal{S}(\mathbb{R}^n)^m$ and $\psi \in \mathcal{S}(\mathbb{R}^{(m+1)n})$ let $S_\omega(\varphi) \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$ be the tempered distribution defined by $\langle \mathcal{F}_1^{mn}(S_\omega(\varphi)), \psi\rangle = \langle \mathcal{F}_1^{mn}(\omega), \chi_{\varphi,\psi}\rangle$ or equivalently $\langle S_\omega(\varphi), \mathcal{F}_1^{mn}(\psi)\rangle = \langle \omega, \mathcal{F}_1^{mn}(\chi_{\varphi,\psi})\rangle$. Thus, S_ω maps $\mathcal{S}(\mathbb{R}^n)^m$ into $\mathcal{S}'(\mathbb{R}^{(m+1)n})$ and the proof is complete. □

Let us remark that by taking $m = 1$ in Propositions 1.1–1.3 we recover Theorems 13–15, respectively in [15].

In the end of this section let us recall some facts concerning the Rihaczek transform. For a signal f in $L^2(\mathbb{R})$ the Rihaczek distribution Rf gives the time-frequency spectrum in time and frequency and is given by

$$Rf(x,\xi) = e^{ix\xi}\hat{f}(\xi)\overline{f(x)}, \quad x,\xi \in \mathbb{R}.$$

For f and g in $L^2(\mathbb{R}^n)$ the Rihaczek transform $R(f,g)$ is given by

$$R(f,g)(x,\xi) = e^{ix\xi}\hat{f}(\xi)\overline{g(x)}, \quad x,\xi \in \mathbb{R}^n.$$

The following theorem gives the Moyal identity for the Rihaczek transform.

Theorem 1.4. *For all functions f_1, g_1, f_2 and g_2 in $L^2(\mathbb{R}^n)$,*

$$(R(f_1,f_2), R(g_1,g_2)) = (f_1,f_2)_{L^2(\mathbb{R}^n)}\overline{(g_1,g_2)}_{L^2(\mathbb{R}^n)}.$$

For the proof of the theorem and for more details concerning the Rihaczek transform see for instance [4] and the references therein.

2. Relation between multilinear S-transforms and multilinear pseudo-differential operators

In the sequel we give a relation between multilinear S-transform and multilinear pseudo-differential operators which will may be useful in the study of S-transforms on distributions spaces.

By making a change of coordinate variables we get

$$\begin{aligned}
S_\omega(\varphi)(\tau,\xi) &= \int_{\mathbb{R}^{mn}} e^{-2\pi\, i|t|\cdot\xi}\omega(\tau-t,\xi)\left(\bigotimes_{j=1}^{m}\varphi_j\right)(t)\mathrm{d}t \\
&= e^{-2\pi\, i|\tau|\cdot\xi}\int_{\mathbb{R}^{mn}} e^{2\pi\, i|x|\cdot\xi}\omega(x,\xi)\left(\bigotimes_{j=1}^{m}\varphi_j\right)(\tau-x)\mathrm{d}x \\
&= e^{-2\pi\, i|\tau|\cdot\xi}\int_{\mathbb{R}^{mn}} e^{2\pi\, i|x|\cdot\xi}\omega(x,\xi)T_{-\tau}\left(\bigotimes_{j=1}^{m}\varphi_j\right)(-x)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= e^{-2\pi\, i|\tau|\cdot\xi}\int_{\mathbb{R}^{mn}} e^{2\pi\, i|x|\cdot\xi}\omega(x,\xi)\mathcal{F}^{mn}\left[(\mathcal{F}^{-1})^{mn}\left(T_{-\tau}\mathcal{R}\left(\bigotimes_{j=1}^{m}\varphi_j\right)\right)\right](x)\mathrm{d}x \\
&= e^{-2\pi\, i|\tau|\cdot\xi}\int_{\mathbb{R}^{mn}} e^{2\pi\, i|x|\cdot\xi}\omega(x,\xi)\mathcal{F}^{mn}\left[\bigotimes_{j=1}^{m}\left((\mathcal{F}^{-1})^{n}T_{\tau_j}\mathcal{R}\varphi_j\right)\right](x)\mathrm{d}x \\
&= e^{-2\pi\, i|\tau|\cdot\xi}K_\sigma\left((\mathcal{F}^{-1})^{n}T_{\tau}\mathcal{R}\varphi\right)(\xi),\ \tau\in\mathbb{R}^{mn},\ \xi\in\mathbb{R}^{n},
\end{aligned}
$$

where $\sigma(\xi,x)=\omega(x,\xi),\quad \xi\in\mathbb{R}^n,\ x\in\mathbb{R}^{mn}$ and in general

$$
K_\sigma(\varphi)(\xi)=\int_{\mathbb{R}^{mn}} e^{2\pi\, i|x|\cdot\xi}\sigma(\xi,x)\left(\bigotimes_{j=1}^{m}\mathcal{F}^{n}\varphi_j\right)(x)\mathrm{d}x,
$$

for all $\varphi=(\varphi_1,\dots,\varphi_m)\in\mathcal{S}(\mathbb{R}^n)^m$ and $\sigma\in\mathcal{S}(\mathbb{R}^{(m+1)n})$. K_σ is the multilinear pseudo-differential operator corresponding to the symbol σ. The reason for which K_σ is called a pseudo-differential operator can be found in reference [4].

Let us remark that we can write $S_\omega(\varphi)(\tau,\cdot)=M_{-|\tau|}K_\sigma((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi)(\cdot)$, where we define the modulation operator by $M_\eta\psi(\xi)=e^{-2\pi\, i\eta\cdot\xi}\psi(\xi)$, $\eta,\xi\in\mathbb{R}^n$.

The advantage of introducing $\sigma(\xi,x)$ consists in the possibility of re-writing the multilinear Stockwell transform as a multilinear pseudo-differential operators (see, for instance [4]).

Let $\omega\in\mathcal{S}(\mathbb{R}^{(m+1)n})$, $\psi\in\mathcal{S}(\mathbb{R}^n)$ be functions in the Schwartz class and let $\tau\in\mathbb{R}^{mn}$. Then we get

$$
\begin{aligned}
\langle S_\omega(\varphi)(\tau,\cdot),\psi\rangle &= \langle e^{-2\pi\, i|\tau|\cdot\xi}K_\sigma((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi),\psi\rangle \\
&= \int_{\mathbb{R}^n} e^{-2\pi\, i|\tau|\cdot\xi}K_\sigma((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi)(\xi)\bar{\psi}(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{mn}} e^{-2\pi\, i|\tau|\cdot\xi}\omega(x,\xi)R((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi,\psi)(\xi,x)\mathrm{d}x\mathrm{d}\xi,
\end{aligned}
$$

where we denote in general by $R(\varphi,\psi)$ the Rihaczek transform of φ in $L^2(\mathbb{R}^n)^m$ and ψ in $L^2(\mathbb{R}^n)$, which is the function on $\mathbb{R}^{(m+1)n}$ given by

$$
R(\varphi,\psi)(\xi,x)=e^{2\pi\, i|x|\cdot\xi}\left(\bigotimes_{j=1}^{m}\mathcal{F}^{n}\varphi_j\right)(x)\bar{\psi}(\xi),\quad \xi\in\mathbb{R}^n,\quad x\in\mathbb{R}^{mn}.
$$

Now, we define the modulation operator with respect to the second variable by $(M^2_\eta\omega)(\tau,\xi)=e^{2\pi\, i\eta\cdot\xi}\omega(\tau,\xi),\quad \tau\in\mathbb{R}^{mn},\ \xi\in\mathbb{R}^n$. Then we get,

$$
\langle S_\omega(\varphi)(\tau,\xi),\psi\rangle=\int_{\mathbb{R}^{(m+1)n}}(M^2_{-|\tau|}\omega)(x,\xi)R((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi,\psi)(\xi,x)\mathrm{d}x\mathrm{d}\xi.
$$

If $\omega\in\mathcal{S}'(\mathbb{R}^{(m+1)n})$, $\varphi\in\mathcal{S}(\mathbb{R}^n)^m$ and $\tau\in\mathbb{R}^{mn}$, then the "local spectrum" $S_\omega(\varphi)(\tau,\cdot)$ corresponding to the window ω and to the (multi)signal $\varphi\in\mathcal{S}(\mathbb{R}^n)^m$ can be defined as a linear mapping on $S(\mathbb{R}^n)$ by

$$
\langle S_\omega(\varphi)(\tau,\cdot),\psi\rangle=\langle M^2_{-|\tau|}\omega,\overline{R((\mathcal{F}^{-1})^nT_\tau\mathcal{R}\varphi,\psi)}\rangle,
$$

for all $\psi\in\mathcal{S}(\mathbb{R}^n)$. Thus, $S_\omega(\varphi)(\tau,\cdot)$ is a tempered distribution on $\mathbb{R}^n$.

3. The Moyal identity and L^2-boundedness

The following theorem gives the Moyal identity for the Rihaczek transform which is nothing else than a Plancherel formula.

Theorem 3.1. *For all $\varphi = (\varphi_1, \ldots, \varphi_m)$ and $\chi = (\chi_1, \ldots, \chi_m)$ in $L^2(\mathbb{R}^n)^m$ and all ψ_1 and ψ_2 in $L^2(\mathbb{R}^n)$,*

$$\langle R(\varphi, \psi_1), R(\chi, \psi_2)\rangle = \prod_{j=1}^{m} \langle \varphi_j, \chi_j\rangle \langle \overline{\psi_1, \psi_2}\rangle.$$

For a proof of Theorem 3.1 see, for instance [4]. An immediate corollary of the above theorem is the L^2-boundedness of multilinear Stockwell transform with L^2-window function.

Theorem 3.2. *Let $\omega \in L^2(\mathbb{R}^{(m+1)n})$. Then $S_\omega(\cdot)(\tau, \cdot) : L^2(\mathbb{R}^n)^m \to L^2(\mathbb{R}^n)$ is a bounded multilinear operator and $||S_\omega(\cdot)(\tau, \cdot)||_{B(L^2(\mathbb{R}^n)^m, L^2(\mathbb{R}^n))} \leq ||\omega||_{L^2(\mathbb{R}^{(m+1)n})}$, for all $\tau \in \mathbb{R}^n$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)^m$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then by Schwartz' inequality, the Moyal identity for the Rihaczek transform, the Fubini Theorem and the Plancherel Theorem we get

$$\begin{aligned}
|\langle S_\omega(\varphi)(\tau, \cdot), \psi\rangle| &= |\langle M^2_{-|\tau|}\omega, \overline{R((\mathcal{F}^{-1})^n T_\tau \mathcal{R}\varphi, \psi)}\rangle| \leq \\
&\leq ||\omega||_{L^2(\mathbb{R}^{(m+1)n})} ||R((\mathcal{F}^{-1})^n T_\tau \mathcal{R}\varphi, \psi)||_{L^2(\mathbb{R}^{(m+1)n})} = \\
&= ||\omega||_{L^2(\mathbb{R}^{(m+1)n})} \prod_{j=1}^{m} ||(\mathcal{F}^{-1})^n T_{\tau_j} \mathcal{R}\varphi_j||_{L^2(\mathbb{R}^n)} ||\psi||_{L^2(\mathbb{R}^n)} = \\
&= ||\omega||_{L^2(\mathbb{R}^{(m+1)n})} \prod_{j=1}^{m} ||T_{\tau_j} \mathcal{R}\varphi_j||_{L^2(\mathbb{R}^n)} \cdot ||\psi||_{L^2(\mathbb{R}^n)} = \\
&= ||\omega||_{L^2(\mathbb{R}^{(m+1)n})} \prod_{j=1}^{m} ||\varphi_j||_{L^2(\mathbb{R}^n)} \cdot ||\psi||_{L^2(\mathbb{R}^n)}
\end{aligned}$$

for all $\tau \in \mathbb{R}^{mn}$.

By a duality argument and a density argument we get

$$||S_\omega(\varphi)(\tau, \cdot)||_{L^2(\mathbb{R}^n)} \leq ||\omega||_{L^2(\mathbb{R}^{(m+1)n})} \prod_{j=1}^{m} ||\varphi_j||_{L^2(\mathbb{R}^n)},$$

for all $\varphi \in L^2(\mathbb{R}^n)^m$. Thus the proof is complete. ∎

4. $L^p - L^{p'}$ - boundedness $1 \leq p < \infty$

We give in the following a result on $L^p - L^{p'}$-boundedness of multilinear Stockwell transform. For this we need Hölder's inequality and Fubini's theorem.

Theorem 4.1. *Let $\omega \in L^{p'}(\mathbb{R}^{(m+1)n})$, $1 \leq p < \infty$ and let p' be the conjugate index of p. Then $S_\omega(\cdot)(\tau, \cdot) : L^p(\mathbb{R}^n)^m \to L^{p'}(\mathbb{R}^n)$ is a bounded multilinear operator and $||S_\omega(\cdot)(\tau, \cdot)||_{B(L^p(\mathbb{R}^n)^m, L^{p'}(\mathbb{R}^n))} \leq ||\omega||_{L^{p'}(\mathbb{R}^{(m+1)n})}$, for all $\tau \in \mathbb{R}^{nm}$.*

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_m) \in S(\mathbb{R}^n)^m$, $\psi \in L^p(\mathbb{R}^n)$ and $\tau \in \mathbb{R}^{mn}$. Then, by using the expression of multilinear S-transform as a multilinear pseudo-differential operator modulo a modulation operator, Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned}|\langle S_\omega(\varphi)(\tau,\cdot),\psi\rangle| &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^{nm}} e^{-2\pi i|\tau|\cdot\xi}\omega(x,\xi)R((\mathcal{F}^{-1})^n T_\tau\mathcal{R}\varphi,\psi)(\xi,x)\mathrm{d}x\mathrm{d}\xi\right| \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^{nm}} |\omega(x,\xi)|\,|R((\mathcal{F}^{-1})^n T_\tau\mathcal{R}\varphi,\psi)(\xi,x)|\mathrm{d}x\mathrm{d}\xi \\ &\leq ||\omega||_{L^{p'}(\mathbb{R}^{(m+1)n})}||R((\mathcal{F}^{-1})^n T_\tau\mathcal{R}\varphi,\psi)||_{L^p(\mathbb{R}^{(m+1)n})} \\ &= ||\omega||_{L^{p'}(\mathbb{R}^{(m+1)n})}\left\|\bigotimes_{j=1}^m T_{\tau_j}\mathcal{R}\varphi_j\right\|_{L^p(\mathbb{R}^{nm})}||\psi||_{L^p(\mathbb{R}^n)} \\ &= ||\omega||_{L^{p'}(\mathbb{R}^{(m+1)n})}\prod_{j=1}^m ||\varphi_j||_{L^p(\mathbb{R}^n)}||\psi||_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Since the dual of $L^p(\mathbb{R}^n)$ is $L^{p'}(\mathbb{R}^n)$ and using a density argument the theorem is proved. Let us remark that when we take $p=2$ in Theorem 4.1 we recover Theorem 3.2. □

5. A general boundedness result

In the sequel we give a general boundedness result for the multilinear S-transform by using Young's inequality for convolution. In the proof of our result we need the following theorem (see [11])

Theorem 5.1. *Let $p,q,r\geq 1$ and $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}=2$. Let $k\in L^p(\mathbb{R}^n)$, $f\in L^q(\mathbb{R}^n)$, $g\in L^r(\mathbb{R}^n)$. Then*

$$\begin{aligned}||f*g||_k &= \left|\int_{\mathbb{R}^n} k(x)(f*g)(x)dx\right| \\ &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x)f(x-y)g(y)\mathrm{d}y\mathrm{d}x\right| \leq C_{p,q,r,n}||k||_p||f||_q||g||_r,\end{aligned}$$

where the sharp constant $C_{p,q,r,n}=(C_pC_qC_r)^n$. Here we have $C_p=\left(\dfrac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$ and similar expressions for C_q and C_r (p' is the conjugate index of p, that is $1/p+1/p'=1$).

If we remember that the multilinear S-transform can be written as a convolution product and by using Theorem 5.1 we obtain the following boundedness result.

Theorem 5.2. *Let $p,q,r\geq 1$, $1/p+1/q+1/r=2$ and $k\in L^p(\mathbb{R}^{nm})$. Let $\varphi=(\varphi_1,\ldots,\varphi_m)\in L^r(\mathbb{R}^n)^m$ and $\omega(\cdot,\xi)\in L^q(\mathbb{R}^{nm})$, $\xi\in\mathbb{R}^n$. Then $||S_\omega(\varphi)(\cdot,\xi)||_k\leq C_{p,q,r,n}||k||_p\prod_{j=1}^m||\varphi_j||_r||\omega(\cdot,\xi)||_q$, $\xi\in\mathbb{R}^n$, where $C_{p,q,r,n}=(C_pC_qC_r)^n$, $C_p=\left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$ and similar expressions for C_q, C_r, with $1/p+1/p'=1$.*

Proof. If $k\in L^p(\mathbb{R}^{nm})$, $\varphi=(\varphi_1,\ldots,\varphi_m)\in L^r(\mathbb{R}^n)^m$ and $\omega(\cdot,\xi)\in L^q(\mathbb{R}^{nm})$, $\xi\in\mathbb{R}^n$, then

$$\begin{aligned}||S_\omega(\varphi)(\cdot,\xi)||_k &= \left\|\omega(\cdot,\xi)*\bigotimes_{j=1}^m\varphi_j e^{-2\pi i\,|\cdot|\cdot\xi}\right\|_k \leq C_{p,q,r,n}||k||_p||\omega(\cdot,\xi)||_q\left\|\bigotimes_{j=1}^m\varphi_j\right\|_r \\ &= (C_pC_qC_r)^n||k||_p||\omega(\cdot,\xi)||_q\prod_{j=1}^m||\varphi_j||_r,\ \xi\in\mathbb{R}^n.\end{aligned}$$

Thus the proof of the theorem is complete. □

For a measurable function $k : \mathbb{R}^n \to \mathbb{C}$, $k \neq 0$ a.e. we define the normed vector space $\mathcal{F}_k(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C}$, measurable function such that $\left|\int_{\mathbb{R}^n} k(x)f(x)\mathrm{d}x\right| < \infty\}$. Then we can restate Theorem 5.2 as follows.

Theorem 5.2′. *Let $p, q, r \geq 1$, $1/p + 1/q + 1/r = 2$ and $k \in L^p(\mathbb{R}^{nm})$. Let $\omega(\cdot, \xi) \in L^q(\mathbb{R}^{nm})$, $\xi \in \mathbb{R}^n$. Then $S_\omega(\cdot)(\cdot, \xi) : L^r(\mathbb{R}^n)^m \to \mathcal{F}_k(\mathbb{R}^{nm})$ is a bounded multilinear operator and $||S_\omega(\cdot)(\cdot, \xi)||_{B(L^r(\mathbb{R}^n)^m, \mathcal{F}_k(\mathbb{R}^{nm}))} \leq (C_pC_qC_r)^n||k||_p||\omega(\cdot, \xi)||_q$, $\xi \in \mathbb{R}^n$, where $C_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$ with $1/p + 1/p' = 1$ and we have similar expressions for C_q, C_r.*

6. Boundedness of multilinear S-transform on Hörmander's spaces $B_{p,k}$

A positive function k defined on $\mathbb{R}^n$ will be called a temperate weight function if there exist positive constants C and N such that

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^n.$$

We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of all such functions defined on $\mathbb{R}^n$. If $k \in \mathcal{K}(\mathbb{R}^n)$ and $1 \leq p \leq \infty$, we shall denote by $B_{p,k}$ the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{F}u = \hat{u}$ is a function and

$$||u||_{p,k} = (2\pi)^{-n}\left(\int_{\mathbb{R}^n} |k(\xi)\hat{u}(\xi)|^p\mathrm{d}\xi\right)^{1/p} < \infty, \quad 1 \leq p < \infty;$$

$$||u||_{\infty,k} = esssup|k(\xi)\hat{u}(\xi)|.$$

In the proof of our boundedness result we need the following theorem (see [10]).

Theorem 6.1. (L. Hörmander). *If $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ and $u_2 \in B_{\infty,k_2}$, then $u_1 * u_2 \in B_{\infty,k_1k_2}$ and the following estimate hold*

$$||u_1 * u_2||_{p,k_1k_2} \leq ||u_1||_{p,k_1}||u_2||_{\infty,k_2}, \quad 1 \leq p < \infty.$$

Before we state our boundedness result let us remark that if $k_j \in \mathcal{K}(\mathbb{R}^n)$, $j = 1, 2, \ldots, m$ are weight functions, then $k = \bigotimes_{j=1}^m k_j \in \mathcal{K}(\mathbb{R}^{nm})$ is also a weight function.

Theorem 6.2. *Let $k_j \in \mathcal{K}(\mathbb{R}^n)$, $\tilde{k} \in \mathcal{K}(\mathbb{R}^{nm})$, $1 \leq p < \infty$, $j = 1, 2, \ldots, m$, $\varphi_j \in B_{p,k_j} \cap \mathcal{E}'$, $j = 1, 2, \ldots, m$ and $\omega(\cdot, \xi) \in B_{\infty,\tilde{k}}(\mathbb{R}^{nm})$, $\xi \in \mathbb{R}^n$. Then $\bigotimes_{j=1}^m \varphi_j e^{-2\pi\, i|\cdot|\xi} \in B_{p,k} \cap \mathcal{E}'(\mathbb{R}^{nm})$, $k = \bigotimes_{j=1}^m k_j \in \mathcal{K}(\mathbb{R}^{nm})$ and $S_\omega(\cdot)(\cdot, \xi) : \prod_{j=1}^m B_{p,k_j}(\mathbb{R}^n) \to B_{p,k\tilde{k}}(\mathbb{R}^{nm})$ is a bounded multilinear operator such that*

$$||S_\omega(\cdot)(\cdot, \xi)||_{B(\prod_{j=1}^m B_{p,k_j}(\mathbb{R}^n), B_{p,k\tilde{k}}(\mathbb{R}^{nm}))} \leq (1 + C|\tilde{\xi}^m|)^N||\omega(\cdot, \xi)||_{\infty,\tilde{k}}, \quad \xi \in \mathbb{R}^n,$$

where $\tilde{\xi}^m = (\xi, \xi, \ldots, \xi) \in \mathbb{R}^{nm}$.

Proof. Using the hypothesis of Theorem 6.2 we can prove by a direct calculation that

$$\left\|\bigotimes_{j=1}^{m}\varphi_j e^{-2\pi\, i|\cdot|\cdot\xi}\right\|_{p,k} \leq (1+C|\tilde{\xi}^m|)^N \prod_{j=1}^{m}||\varphi_j||_{p,k_j},$$

for all $\varphi_j \in B_{p,k_j}\cap\mathcal{E}'$, $j=1,2,\dots,m$ and $\xi\in\mathbb{R}^n$, where C and N are suitable positive constants and $\tilde{\xi}^m = (\xi,\dots,\xi)\in\mathbb{R}^{nm}$. Thus by Theorem 6.1 we have

$$\begin{aligned}||S_\omega(\varphi)(\cdot,\xi)||_{p,k\tilde{k}} &\leq ||\bigotimes_{j=1}^{m}\varphi_j e^{-2\pi\, i|\cdot|\cdot\xi}||_{p,k}||\omega(\cdot,\xi)||_{\infty,\tilde{k}}\\ &\leq (1+C|\tilde{\xi}^m|)^N \prod_{j=1}^{m}||\varphi_j||_{p,k_j}||\omega(\cdot,\xi)||_{\infty,\tilde{k}}.\end{aligned}$$

Hence the proof of this theorem is complete. □

7. Uncertainty principles for multilinear S-transform and for multi-dimensional modified S-transform

Let now $\omega\in L^2(\mathbb{R}^{(m+1)n})$ a window and let S be the corresponding multilinear S-transform. Then we define the transform S_ω^* by

$$\langle S_\omega\varphi,\psi\rangle_{L^2(\mathbb{R}^{(m+1)n})} = \langle\bigotimes_{j=1}^{m}\varphi_j, S_\omega^*\psi\rangle_{L^2(\mathbb{R}^{mn})},$$

for all $\varphi=(\varphi_1,\dots,\varphi_m)\in L^2(\mathbb{R}^n)^m$ and $\psi\in L^2(\mathbb{R}^{n(m+1)})$. We shall call S_ω^* the adjoint of S_ω and we deduce from the previous relation that

$$S_\omega^*\psi(t) = \int_{\mathbb{R}^{(m+1)n}}\overline{\omega(\tau-t,\xi)}\psi(\tau,\xi)e^{2\pi\, i|t|\cdot\xi}\mathrm{d}\tau\mathrm{d}\xi,\quad t\in\mathbb{R}^{nm}.$$

Theorem 7.1. *(weak uncertainty principle for the adjoint transform S^*.) Suppose that $||\omega||_{L^2(\mathbb{R}^{(m+1)n})} = ||\psi||_{L^2(\mathbb{R}^{(m+1)n})} = 1$ and that $U\subseteq\mathbb{R}^{mn}$ and $\varepsilon\geq 0$ such that $\int_U|S_\omega^*\psi(t)|^2\mathrm{d}t\geq 1-\varepsilon$, then $|U|\geq 1-\varepsilon$, where we denote by $|U|$ the Lebesgue measure of U.*

Proof. By using Cauchy-Schwartz inequality we can write

$$|S_\omega^*\psi(t)| = \left|\int_{\mathbb{R}^{(m+1)n}}\overline{\omega(\tau-t,\xi)}\psi(\tau,\xi)e^{2\pi\, i|t|\cdot\xi}\mathrm{d}\tau\mathrm{d}\xi\right| \leq ||\omega||_2||\psi||_2,\ t\in\mathbb{R}^{mn}.$$

Hence by hypothesis and the above inequality we have

$$1-\varepsilon\leq\int_U|S_\omega^*\psi(t)|^2\mathrm{d}t\leq||S_\omega^*\psi||_\infty^2|U|\leq||\omega||_2^2||\psi||_2^2|U| = |U|.$$

Thus the proof is complete. □

A similar result can be state and prove for multilinear S-transform.

Theorem 7.2. *Let $\omega(\cdot,\xi) \in L^2(\mathbb{R}^{mn})$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in L^2(\mathbb{R}^n)^m$ and suppose that $||\omega(\cdot,\xi)||_2 = \left\| \bigotimes_{j=1}^{m} \varphi_j \right\|_2 = 1$ for all $\xi \in \mathbb{R}^n$. If $U \subseteq \mathbb{R}^{(m+1)n}$ and $\varepsilon \geq 0$ are such that $\int_U |S_\omega(\varphi)(\tau,\xi)|^2 \mathrm{d}\tau \mathrm{d}\xi \geq 1-\varepsilon$, then $|U| \geq 1-\varepsilon$. For more details and different aspects of the uncertainty principle we refer to [6].*

Acknowledgements. The author is grateful to the referees for the useful comments and suggestions in improving the presentation of the paper.

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