Math. Model. Nat. Phenom. 13 (2018) 39
https://doi.org/10.1051/mmnp/2018042

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

ON GREEN FUNCTIONS FOR DIRICHELET SUB-LAPLACIANS ON A QUATERNION HEISENBERG GROUP

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Abstract. In the present paper, Green functions of Dirichlet boundary value problems are constructed for sub-Laplacians on certain unbounded domains of the quaternion Heisenberg group. Also, the explicit solutions of the Dirichlet problem are presented for the sub-Laplacian with non-zero boundary datum in wedge and strip domains. We also present Hardy and Rellich inequalities for the sub-Laplacians in terms of their fundamental solutions.

Mathematics Subject Classification. 22E30, 43A80

Accepted March 15, 2018.

1. Preliminaries

Let $\mathbb{H}$ be the set of all quaternions $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $1, i_1, i_2, i_3$ are the basis elements of $\mathbb{H}$ with following rules of multiplication

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1, \qquad i_1 i_2 = -i_2 i_1 = i_3, \qquad i_2 i_3 = -i_3 i_2 = i_1, \qquad i_3 i_1 = -i_1 i_3 = i_2.$$

Let $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 \in \mathbb{H}$. Then the real part of x is the real number x_0 and the imaginary part of x is the point $(x_1, x_2, x_3) \in \mathbb{R}^3$. Also, the real and imaginary parts of x are denoted by $\Re x$ and $\Im x$, respectively. It will be useful further to denote the imaginary parts such as

$$\Im_1 x = x_1, \quad \Im_2 x = x_2, \quad \Im_3 x = x_3.$$

The conjugate of x is denoted by

$$\overline{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3,$$

Keywords and phrases: Green function, Dirichlet sub-Laplacian, quaternion Heisenberg group.

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and the modulus $|x|$ is defined by

$$|x|^2 = x\overline{x} = \sum_{j=0}^{3} x_j^2.$$

The Grassmanian product (or the quaternion product) of x and y is defined by

$$xy = (x_0 y_0 - \Im x \cdot \Im y) + (x_0 \Im y + y_0 \Im x + \Im x \times \Im y),$$

where

$$\Im x \times \Im y = \det \begin{pmatrix} i_1 & i_2 & i_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Let $\mathbb{H}_q = \mathbb{H} \times \mathbb{R}^3$. Then $\mathbb{H}_q$ becomes a non-commutative (Lie) group with the group law

$$(x, t_1, t_2, t_3) \circ (y, \tau_1, \tau_2, \tau_3) = (x + y, t_1 + \tau_1 - 2\Im_1(\overline{y}x), t_2 + y_2 - 2\Im_2(\overline{y}x), t_3 + \tau_3 - 2\Im_3(\overline{y}x)),$$

for all $(x,t),(y,\tau) \in \mathbb{H}_q$. We note that $e = (0,0,0,0)$ is the identity element of $\mathbb{H}_q$ and the inverse of an element $(x, t_1, t_2, t_3) \in \mathbb{H}_q$ is $(-x, -t_1, -t_2, -t_3)$. The Haar measure on $\mathbb{H}_q$ coincides with the Lebesgue measure on $\mathbb{H} \times \mathbb{R}^3$ which is denoted by $\mathrm{d}x\mathrm{d}t$. Let $\mathfrak{h}_q$ be the Lie algebra of left invariant vector fields on $\mathbb{H}_q$. A basis of $\mathfrak{h}_q$ is given by $\{X_0, X_1, X_2, X_3\}$ and $\{T_1, T_2, T_3\}$, where

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial x_0} - 2x_1 \frac{\partial}{\partial t_1} - 2x_2 \frac{\partial}{\partial t_2} - 2x_3 \frac{\partial}{\partial t_3}, \\
X_1 &= \frac{\partial}{\partial x_1} + 2x_0 \frac{\partial}{\partial t_1} - 2x_3 \frac{\partial}{\partial t_2} + 2x_2 \frac{\partial}{\partial t_3}, \\
X_2 &= \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial t_1} + 2x_0 \frac{\partial}{\partial t_2} - 2x_1 \frac{\partial}{\partial t_3}, \\
X_3 &= \frac{\partial}{\partial x_3} - 2x_2 \frac{\partial}{\partial t_1} + 2x_1 \frac{\partial}{\partial t_2} + 2x_0 \frac{\partial}{\partial t_3},
\end{aligned}$$

and

$$T_k = \frac{\partial}{\partial t_k}, \quad k = 1, 2, 3.$$

The Lie brackets of these vector fields are given by

$$\begin{aligned}
[X_0, X_1] &= [X_3, X_2] = 4T_1, \\
[X_0, X_2] &= [X_1, X_3] = 4T_2, \\
[X_0, X_3] &= [X_2, X_1] = 4T_3.
\end{aligned}$$

Thus, the sub-Laplacian on $\mathbb{H}_q$ is given by

$$\mathcal{L} = \sum_{j=0}^{3} X_j^2 = \Delta_x - 4|x|^2 \Delta_t - 4 \sum_{k=1}^{3} (i_k x \cdot \nabla_x) \frac{\partial}{\partial t_k}, \tag{1.1}$$

where

$$\Delta_x = \sum_{k=0}^{3} \frac{\partial^2}{\partial x_k^2}, \quad \text{and} \quad \Delta_t = \sum_{k=1}^{3} \frac{\partial^2}{\partial t_k^2}.$$

Note that the fundamental solution of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_q$ was found by Tie and Wong in [18]. We restate their results in the following theorem.

Theorem 1.1. *The fundamental solution $\varGamma(\xi)$ of the sub-Laplacian $\mathcal{L}$ on the quaternion Heisenberg group $\mathbb{H}_q$ is given by*

$$\varGamma(\xi) := \varGamma(|x|, t) = \frac{2}{(2\pi)^{7/2}|x|^2} \int_{S^2} \frac{1}{(|x|^2 - i(t \cdot n))^3} \mathrm{d}\sigma, \tag{1.2}$$

where $\xi = (x, t) \in \mathbb{H}_q$, $n = (n_1, n_2, n_3)$ is a point on the unit sphere S^2 in $\mathbb{R}^3$ with center at the origin, and $\mathrm{d}\sigma$ is the surface measure on S^2. That is,

$$\mathcal{L}\varGamma_\zeta = -\Delta_\zeta, \tag{1.3}$$

where $\varGamma_\zeta(\xi) = \varGamma(\zeta^{-1} \circ \xi)$ and Δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{H}_q$.

The quaternion Heisenberg group is a special case of the model step two nilpotent Lie group (see [18]). For more general results of model step two nilpotent Lie group we refer to [1]. Moreover, it is a homogeneous Lie group with respect to the dilation

$$\Delta_\lambda : \mathbb{R}^7 \to \mathbb{R}^7, \quad \Delta_\lambda = (\lambda x, \lambda^2 t).$$

Thus,

$$d(\xi) = \frac{1}{\varGamma^{1/8}(\xi)}, \quad \xi = (x, t) \in \mathbb{H}_q, \tag{1.4}$$

is a homogeneous quasi-norm on $\mathbb{H}_q$ with respect to the dilation Δ_λ (see, *e.g.* [3]).

For $0 < \alpha < 1$, Folland and Stein (see [9, 10]) defined the anisotropic Hölder spaces $\digamma_\alpha(D)$, $D \subset \mathbb{H}_q$, by

$$\digamma_\alpha(D) = \left\{ f : D \to \mathbb{C} : \sup_{\substack{\xi,\zeta \in D \\ \xi \neq \zeta}} \frac{|f(\xi) - f(\zeta)|}{[d(\zeta^{-1} \circ \xi)]^\alpha} < \infty \right\},$$

where d is defined by the formula (1.4) in our case. For $k \in \mathbb{N}$ and $0 < \alpha < 1$, one defines $\digamma_{k+\alpha}(D)$ as the space of all $f : D \to \mathbb{C}$ such that all X_j derivatives of f of order k belong to $\digamma_\alpha(D)$. A bounded function f is called α-Hölder continuous in $D \subset \mathbb{H}_q$ if $f \in \digamma_\alpha(D)$.

The Green function for the Dirichlet sub-Laplacian in D is defined by the formula

$$G_D(\xi, \zeta) = \varGamma(\zeta^{-1} \circ \xi) - h_\zeta(\xi), \tag{1.5}$$

with

$$G_D(\xi, \zeta) = 0, \quad \xi \in \partial D, \tag{1.6}$$

where ∂D is the boundary of D. Here, $h_\zeta(\xi)$ is a harmonic function, that is,

$$\mathcal{L}h_\zeta(\xi) = 0 \quad \text{in } D, \tag{1.7}$$

having as boundary values (in the Perron–Wiener–Brelot sense) the fundamental solution with pole at $\zeta \in D$.

Let ∂D be the boundary of a smooth domain D in $\mathbb{H}_q$, $\mathrm{d}\nu$ is the volume element on $\mathbb{H}_q$, and $\langle X_j, \mathrm{d}\nu\rangle$ is the natural pairing between vector fields and differential forms. The following version of Green's second formula will be useful for our analysis, (given in [17], Prop. 3.10).

Proposition 1.2 (Green's second formula)**.** *Let $\mathcal{L}$ be the sub-Laplacian on $\mathbb{H}_q$, we have*

$$\int_D (u\mathcal{L}v - v\mathcal{L}u)\mathrm{d}\nu = \int_{\partial D} (u\langle\widetilde{\nabla}v, \mathrm{d}\nu\rangle - v\langle\widetilde{\nabla}u, \mathrm{d}\nu\rangle), \tag{1.8}$$

for any $u, v \in C^2(D)\bigcap C^1(\overline{D})$ and

$$\widetilde{\nabla}u = \sum_{k=0}^{3} (X_k u)\, X_k. \tag{1.9}$$

For further discussion on stratified groups we refer to [17] and [2], Section 7. The relation between the $(n-1)$ form under the integral in the right-hand side of (1.8) and the perimeter and surface measures on ∂D has been discussed in [17].

2. Green functions and representations of solutions on $\mathbb{H}_q$

In this section, Green functions of Dirichlet boundary value problems for sub-Laplacians $\mathcal{L}$ will be constructed on l-wedge like and l-strip like unbounded domains of the quaternion Heisenberg group by using the classical method of reflection. We also present solutions (in an explicit form) of the Dirichlet problem for the sub-Laplacian with non-zero boundary datum in those domains.

2.1. Green functions and representations of solutions in l-wedge like spaces.

Let us introduce the l-wedge like space in the following form

$$\mathbb{H}_q^{\ddagger} = \{\xi = (x_0, x_1, x_2, x_3, t_1, t_2, t_3)|\ x_0, \dots, x_l > 0\}, \quad l = 0, 1, 2, 3.$$

Let the point $\zeta = (y, \tau) = (y_0, y_1, y_2, y_3, \tau_1, \tau_2, \tau_3)$ lie in this l-wedge like space, i.e., $y_0 > 0, \dots, y_l > 0$.

$$\zeta_{x_k} := (y_0, \dots, -y_k, \dots, \tau_1, \tau_2, \tau_3),$$

is a symmetric point for the point ζ with respect to the hyperplane $x_k = 0$. Similarly, we have the symmetric point

$$\zeta_{x_k x_s} := (\dots, -y_k, -y_s, \dots, \tau_1, \tau_2, \tau_3),$$

for the point ζ_{x_k} with respect to the hyperplane $x_s = 0$ and so on. It is simple to show that the symmetry indices are invariant under permutations. We also use the notation for $\Gamma((\zeta_{(j,l)})^{-1} \circ \xi),\ j \le l$, over all possible

(j,l) combination symmetry arguments. The numbers of subindices can be reduced by writing $(\zeta_{(j,l)})^{-1}\circ\xi$ for $\zeta^{-1}_{x_{k_1}\dots x_{k_j}}\circ\xi$. For instance, in the case $l=2$ and taking $j=1$, then

$$\Gamma((\zeta_{(1,2)})^{-1}\circ\xi)=\Gamma((\zeta_{x_0x_1})^{-1}\circ\xi)+\Gamma((\zeta_{x_0x_2})^{-1}\circ\xi)+\Gamma((\zeta_{x_2x_1})^{-1}\circ\xi),$$

and if $l=2$, $j=2$, then

$$\Gamma((\zeta_{(2,2)})^{-1}\circ\xi)=\Gamma((\zeta_{x_0x_1x_2})^{-1}\circ\xi).$$

Proposition 2.1. *Let $G_{\mathbb{H}_q^\ddagger}$ be the Green function for the Dirichlet sub-Laplacian in $\mathbb{H}_q^\ddagger$, then we have*

$$G_{\mathbb{H}_q^\ddagger}(\xi,\zeta)=\Gamma(\zeta^{-1}\circ\xi)+\sum_{j=0}^{3}(-1)^j\Gamma((\zeta_{(j,4)})^{-1}\circ\xi). \tag{2.1}$$

Proof of Proposition 2.1. By definition $\zeta_{(j,l)}\notin\mathbb{H}_q^\ddagger$, $j=0,\dots,l$, it follows from (1.3) that

$$\mathcal{L}\Gamma((\zeta_{(j,l)})^{-1}\circ\xi)=-\Delta_{\zeta_{(j,l)}}=0 \text{ in } \mathbb{H}_q^\ddagger,$$

for every $\xi\in\mathbb{H}_q^\ddagger$ and $j=0,\dots,l$. Thus, the condition (1.7) holds for the function $\sum_{j=0}^{l}(-1)^j\Gamma((\zeta_{(j,l)})^{-1}\circ\xi)$, i.e., it is (sub)harmonic in $\mathbb{H}_q^\ddagger$. Now, it remains to check the boundary condition for $\mathbb{H}_q^\ddagger$, that is, the function $G_{\mathbb{H}_q^\ddagger}$ must become zero at $x_0=0$ and at infinity. It is simple to show that the d-distance from any point of the hyperplane $x_k=0$ to the points ζ and ζ_{x_k} is equal, that is, the Dirichlet condition holds for $G_{\mathbb{H}_q^\ddagger}$ at the hyperplanes $x_0=0,\dots,x_l=0$ and it is also clear (by the construction) that the function $G_{\mathbb{H}_q^\ddagger}$ is zero at infinity. It proves that

$$G_{\mathbb{H}_q^\ddagger}(\xi,\zeta)=0,\quad \xi\in\partial\mathbb{H}_q^\ddagger.$$

□

Let us consider a smooth open set $D\subset\mathbb{H}_q$ with boundary ∂D, and study the Dirichlet problem for the sub-Laplacian $\mathcal{L}$ in D.

Let us consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases}\mathcal{L}u=f & \text{in } \mathbb{H}_q^\ddagger,\\ u=\phi & \text{on } \partial\mathbb{H}_q^\ddagger,\end{cases} \tag{2.2}$$

where $f\in\digamma_\alpha(\mathbb{H}_q^\ddagger),\ 0<\alpha<1,\ \operatorname{supp} f\subset\mathbb{H}_q^\ddagger$, and $\phi\in C^\infty(\partial\mathbb{H}_q^\ddagger),\ \operatorname{supp}\phi\subset\{x_0=0\}\bigcup\dots\bigcup\{x_l=0\}$.

Theorem 2.2. *Let $u\in C^2(\mathbb{H}_q^\ddagger)\cap C^1(\overline{\mathbb{H}_q^\ddagger})$ be a unique solution of the boundary value problem for $f\in\digamma_\alpha(\mathbb{H}_q^\ddagger),\ 0<\alpha<1,\ \operatorname{supp} f\subset\mathbb{H}_q^\ddagger$, and $\phi\in C^\infty(\partial\mathbb{H}_q^\ddagger)$ and it can be represented by the formula*

$$u(\xi)=\int_{\mathbb{H}_q^\ddagger}G_{\mathbb{H}_q^\ddagger}(\xi,\zeta)f(\zeta)\mathrm{d}\nu(\zeta)-\int_{\partial\mathbb{H}_q^\ddagger}\phi(\zeta)\langle\widetilde{\nabla}G_{\mathbb{H}_q^\ddagger}(\xi,\zeta),\,\mathrm{d}\nu(\zeta)\rangle,\quad \xi\in\mathbb{H}_q^\ddagger, \tag{2.3}$$

where $\widetilde{\nabla}$ *is defined by* (1.9)*, or*

$$\widetilde{\nabla} G_{\mathbb{H}_q^{\ddagger}} = \sum_{k=0}^{3} \left(X_k G_{\mathbb{H}_q^{\ddagger}} \right) X_k.$$

Proof of Theorem 2.2. Assume that $u \in C^2(\mathbb{H}_q^{\ddagger}) \cap C^1(\overline{\mathbb{H}_q^{\ddagger}})$ tends to zero at infinity. Green's second formula (1.8) is in bounded domains, but it is still applicable for functions, with necessary decay rates at infinity, in unbounded domains. It can be shown by the standard argument using quasi-balls with radii $R \longrightarrow \infty$. Thus, if we apply Green's second formula (1.8) to the function u with $v(\zeta) = G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta)$, we shall obtain

$$u(\xi) = \int_{\mathbb{H}_q^{\ddagger}} G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta) f(\zeta) \mathrm{d}\nu(\zeta) - \int_{\partial \mathbb{H}_q^{\ddagger}} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta), \, \mathrm{d}\nu(\zeta) \rangle.$$

By using the properties of the Green function, we have

$$G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta) = 0, \quad \zeta \in \partial \mathbb{H}_q^{\ddagger},$$

and by construction the function $G_{\mathbb{H}_q^{\ddagger}}$ is symmetric, that is, $G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta) = G_{\mathbb{H}_q^{\ddagger}}(\zeta, \xi)$ in $\mathbb{H}_q^{\ddagger}$, so

$$\mathcal{L}_\zeta G_{\mathbb{H}_q^{\ddagger}}(\xi, \zeta) = -\Delta_\xi,$$

where Δ_ξ is the Dirac distribution at $\xi \in \mathbb{H}_q^{\ddagger}$. Now it is necessary to present that the function defined by (2.3) belongs to $C^2(\mathbb{H}_q^{\ddagger}) \cap C^1(\overline{\mathbb{H}_q^{\ddagger}})$. The volume potential (the first term of the right hand side in (2.3)) belongs to $C^2(\overline{\mathbb{H}_q^{\ddagger}})$ by Folland's theorem (see [9], Thm. 6.1, see also [10]) because of $f \in \digamma_\alpha(\mathbb{H}_q^{\ddagger})$, $\operatorname{supp} f \subset \mathbb{H}_q^{\ddagger}$. Hörmander's hypoellipticity theorem (see [12]) guarantees that every harmonic function is C^∞, hence the Dirichlet double layer potential (the second term of the right hand side in (2.3)) is in $C^2(\mathbb{H}_q^{\ddagger})$. On the other hand, since $\phi \in C^\infty(\partial \mathbb{H}_q^{\ddagger})$, $\operatorname{supp} \phi \subset \{x_0 = 0, \ldots, x_l = 0\}$ and the boundary hyperplanes $\{x_0 = 0\}, \ldots, \{x_l = 0\}$ have no characteristic points the Dirichlet double layer potential is continuous on the boundary by the Kohn–Nirenberg theorem (see [2], Thm. 3.12, which is a consequence of [13], Thm. 4 see also [5, 6]). □

Remark 2.3. Consider $(\mathbb{H}_q^{\ddagger})_a$, $a = (a_0, \ldots, a_l) \in \mathbb{R}^l$, l-wedge like space

$$\{\xi = (x_0, x_1, x_2, x_3, t_1, t_2, t_3) | \; x_0 > a_0, \ldots, x_l > a_l\},$$

but in this l-wedge like space the Green function $G_{(\mathbb{H}_q^{\ddagger})_a}$ has the same formula as the formula (2.1) in which the symmetry points are chosen, in this case, with respect to the hyperplanes $\{x_0 = a_0\}, \ldots, \{x_l = a_l\}$. Because of this, we will obtain an analogue of Theorem 2.2 with the same argument.

Now some simple cases of Theorem 2.2 and Proposition 2.1 with different (simpler) notations will be demonstrated. Firstly, a Green function for the Dirichlet sub-Laplacian in a half-space on $\mathbb{H}_q$ will be constructed as above. Let $\mathbb{H}_q^+$ be the half-space

$$\mathbb{H}_q^+ = \{\xi = (x_0, x_1, x_2, x_3, t_1, t_2, t_3) | \; x_0 > 0\}.$$

Let the point $\zeta = (y, \tau) = (y_0, y_1, y_2, y_3, \tau_1, \tau_2, \tau_3)$ lie in this half-space, $y_0 > 0$. The symmetric point ζ^* for the point ζ with respect to the hyperplane $x_0 = 0$ has the following form

$$\zeta^* = (y^*, \tau) := (-y_0, y_1, y_2, y_3, \tau_1, \tau_2, \tau_3).$$

Proposition 2.1 has the following direct consequence.

Corollary 2.4. *Let $G_{\mathbb{H}_q^+}$ be a Green function for a Dirichlet sub-Laplacian in $\mathbb{H}_q^+$. Then*

$$G_{\mathbb{H}_q^+}(\xi,\zeta)=\Gamma(\zeta^{-1}\circ\xi)-\Gamma((\zeta^*)^{-1}\circ\xi). \tag{2.4}$$

Let us consider the Dirichlet problem for the sub-Laplacian

$$\begin{cases}\mathcal{L}u=f & \text{in } \mathbb{H}_q^+,\\ u=\phi & \text{on } \partial\mathbb{H}_q^+,\end{cases} \tag{2.5}$$

for $f\in \digamma_\alpha(\mathbb{H}_q^+),\ \operatorname{supp} f\subset\mathbb{H}_q^+$, and $\phi\in C^\infty(\partial\mathbb{H}_q^+),\ \operatorname{supp}\phi\subset\{x_0=0\}$.

In this case Theorem 2.2 can be restated in the following form.

Corollary 2.5. *Let $u\in C^2(\mathbb{H}_q^+)\cap C^1(\overline{\mathbb{H}_q^+})$ be a solution of the boundary value problem (2.5). Then*

$$u(\xi)=\int_{\mathbb{H}_q^+}G_{\mathbb{H}_q^+}(\xi,\zeta)f(\zeta)\mathrm{d}\nu(\zeta)-\int_{\partial\mathbb{H}_q^+}\phi(\zeta)\langle\widetilde{\nabla}G_{\mathbb{H}_q^+}(\xi,\zeta),\,\mathrm{d}\nu(\zeta)\rangle,\quad \xi\in\mathbb{H}_q^+, \tag{2.6}$$

where

$$G_{\mathbb{H}_q^+}(\xi,\zeta)=\Gamma(\zeta^{-1}\circ\xi)-\Gamma((\zeta^*)^{-1}\circ\xi).$$

Now a Green function for the Dirichlet sub-Laplacian in a quadrant-space on $\mathbb{H}_q$ can be constructed. Let us denote the quadrant-space $\mathbb{H}_q^\oplus$

$$\mathbb{H}_q^\oplus=\{\xi=(x_0,x_1,x_2,x_3,t_1,t_2,t_3)|\ x_0>0,\ x_1>0\}.$$

Let the point $\zeta=(y,\tau)=(y_0,y_1,y_2,y_3,\tau_1,\tau_2,\tau_3)$ lie in this quadrant-space, $y_0>0$, $y_1>0$. Denote by

$$\zeta^*=(y^*,\tau):=(-y_0,y_1,y_2,y_3,\tau_1,\tau_2,\tau_3),$$

and

$$\overline{\zeta}=(\overline{y},\tau):=(y_0,-y_1,y_2,y_3,\tau_1,\tau_2,\tau_3),$$

the symmetric points for ζ with respect to the hyperplanes $x_0=0$ and $x_1=0$, respectively. The symmetric point for ζ^* with respect to the hyperplane $x_1=0$ and the symmetric point for $\overline{\zeta}$ with respect to the hyperplane $x_0=0$ has the following form

$$\overline{\zeta}^*=(\overline{y}^*,\tau)=(-y_0,-y_1,y_2,y_3,\tau_1,\tau_2,\tau_3).$$

Proposition 2.1 has the following another direct consequence.

Corollary 2.6. *Let the function $G_{\mathbb{H}_q^\oplus}$ be a Green function for a Dirichlet sub-Laplacian in $\mathbb{H}_q^\oplus$. Then*

$$G_{\mathbb{H}_q^\oplus}(\xi,\zeta)=\Gamma(\zeta^{-1}\circ\xi)+\Gamma((\overline{\zeta}^*)^{-1}\circ\xi)-\Gamma((\zeta^*)^{-1}\circ\xi)-\Gamma((\overline{\zeta})^{-1}\circ\xi). \tag{2.7}$$

2.2. Green functions and representations of solutions in l-strip spaces.

Let us have the l-strip like space with the following form

$$\mathbb{H}_q^{\models} = \{\xi = (x_0, x_1, x_2, x_3, t_1, t_2, t_3) |\ a > x_l > 0\},$$

for some $0 \leq l \leq 3$. Let the point $\zeta = (y, \tau) = (y_0, y_1, y_2, y_3, \tau_1, \tau_2, \tau_3)$ lie in this l-strip space, $a > y_l > 0$. We will use the notations

$$\zeta_{+,j} := (y_0, \ldots, y_l - 2aj, \ldots, y_3, \tau_1, \tau_2, \tau_3),$$

and

$$\zeta_{-,j} := (y_0, \ldots, -y_l + 2aj, \ldots, y_3, \tau_1, \tau_2, \tau_3),$$

for all $j = 0, 1, 2, 3$.

Proposition 2.7. *Let a function $G_{\mathbb{H}_q^{\models}}$ be a Green function for a Dirichlet sub-Laplacian in $\mathbb{H}_q^{\models}$. Then*

$$G_{\mathbb{H}_q^{\models}}(\xi, \zeta) = \sum_{j=-\infty}^{\infty} \left(\Gamma(\zeta_{+,j}^{-1} \circ \xi) - \Gamma(\zeta_{-,j}^{-1} \circ \xi)\right). \tag{2.8}$$

Proof of Proposition 2.7. It is easy to see that the term $\Gamma(\zeta_{+,0}^{-1} \circ \xi)$ represents the fundamental solution which is the $j = 0$ term of (2.8) and all the other terms are subharmonic functions in $\mathbb{H}_q^{\models}$. Thus, we only have to check that traces of (2.8) vanish on hyperplanes $x_l = 0$ and $x_l = a$. If $x_l = 0$, then (2.8) gives

$$\begin{aligned} G_{\mathbb{H}_q^{\models}}(\xi, \zeta)|_{x_l=0} = \sum_{j=-\infty}^{\infty} &\left(\Gamma\left((x_0 - y_0)^2 + \ldots + (-y_l + 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right)\right. \\ &\left.-\Gamma\left((x_0 - y_0)^2 + \ldots + (y_l - 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right)\right) = 0. \end{aligned} \tag{2.9}$$

If $x_l = a$, then (2.8) gives

$$\begin{aligned} G_{\mathbb{H}_q^{\models}}(\xi, \zeta)|_{x_l=a} &= \sum_{j=-\infty}^{\infty} \left(\Gamma\left((x_0 - y_0)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right)\right. \\ &\quad \left.-\Gamma\left((x_0 - y_0)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right)\right) \\ &= \sum_{j=0}^{\infty} \Gamma\left((x_0 - y_0)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right) \\ &\quad - \sum_{j=1}^{\infty} \Gamma\left((x_0 - y_0)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right) \\ &\quad + \sum_{j=-1}^{-\infty} \Gamma\left((x_0 - y_0)^2 + \ldots + (a - y_l + 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right) \\ &\quad - \sum_{j=0}^{-\infty} \Gamma\left((x_0 - y_0)^2 + \ldots + (a + y_l - 2aj)^2 + \ldots + (x_3 - y_3)^2, (t - \tau)\right) = 0. \end{aligned} \tag{2.10}$$

This completes the proof of Proposition 2.7. ∎

Let us consider the Dirichlet problem for the sub-Laplacian for $f \in \digamma_\alpha(\mathbb{H}_q^{\models}),\ 0<\alpha<1,\ \operatorname{supp} f \subset \mathbb{H}_q^{\models}$, and $\phi \in C^\infty(\partial\mathbb{H}_q^{\models}),\ \operatorname{supp}\phi \subset \{x_l=0\}\bigcup\{x_l=a\}$

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathbb{H}_q^{\models}, \\ u=\phi & \text{on } \partial\mathbb{H}_q^{\models}. \end{cases} \tag{2.11}$$

Theorem 2.8. *For $f \in \digamma_\alpha(\mathbb{H}_q^{\models}),\ 0<\alpha<1,\ \operatorname{supp} f \subset \mathbb{H}_q^{\models}$, and $\phi \in C^\infty(\partial\mathbb{H}_q^{\models})$, we have a unique solution $u \in C^2(\mathbb{H}_q^{\models}) \cap C^1(\overline{\mathbb{H}_q^{\models}})$ of the boundary value problem (2.11) and it can be represented by the formula*

$$u(\xi) = \int_{\mathbb{H}_q^{\models}} G_{\mathbb{H}_q^{\models}}(\xi,\zeta) f(\zeta) \mathrm{d}\nu(\zeta) - \int_{\partial\mathbb{H}_q^{\models}} \phi(\zeta) \langle \widetilde{\nabla} G_{\mathbb{H}_q^{\models}}(\xi,\zeta), \mathrm{d}\nu(\zeta)\rangle, \quad \xi \in \mathbb{H}_q^{\models}, \tag{2.12}$$

where $\widetilde{\nabla}$ is defined by (1.9), in particular,

$$\widetilde{\nabla} G_{\mathbb{H}_q^{\models}} = \sum_{k=0}^{3} \left(X_k G_{\mathbb{H}_q^{\models}} \right) X_k.$$

Proof of Theorem 2.8. The proof is the same as the proof of Theorem 2.2. ∎

3. Hardy type inequalities and uncertainty principles on $\mathbb{H}_q$

In this section, we present a Hardy type inequality on the quaternion Heisenberg group. The proof of Theorem 3.1 relies on properties of the fundamental solution of the sub-Laplacian $\mathcal{L}$ on the quaternion Heisenberg group.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $\alpha > 2-\beta$, $\beta > 2$. Then the following version of the Hardy inequality is valid:*

$$\left\| \Gamma^{\frac{\alpha}{2(2-\beta)}} |\nabla u| \right\|_{L_2(\mathbb{H}_q)} \geq \frac{|\beta+\alpha-2|}{2} \left\| \Gamma^{\frac{\alpha-2}{2(2-\beta)}} |\nabla \Gamma^{\frac{1}{2-\beta}}| |u| \right\|_{L_2(\mathbb{H}_q)}, \tag{3.1}$$

for any $u \in C_0^\infty(\mathbb{H}_q)$, where $\nabla := (X_0, X_1, X_2, X_3)$.

Proof of Theorem 3.1. Let $(\widetilde{\nabla} f)g := \sum_{k=0}^3 X_k f X_k g$ be for any differentiable functions f and g. Setting $u = d^\gamma q$ for some (real-valued) functions $d>0$, q, and a constant $\gamma \neq 0$ to be chosen later, we have

$$\begin{aligned}
(\widetilde{\nabla} u)u = (\widetilde{\nabla} d^\gamma q) d^\gamma q &= \sum_{k=0}^{3} X_k(d^\gamma q) X_k(d^\gamma q) \\
&= \gamma^2 d^{2\gamma-2} \sum_{k=0}^{3} (X_k d)^2 q^2 + 2\gamma d^{2\gamma-1} q \sum_{k=0}^{3} X_k d X_k q + d^{2\gamma} \sum_{k=0}^{3} (X_k q)^2 \\
&= \gamma^2 d^{2\gamma-2} ((\widetilde{\nabla} d) d) q^2 + 2\gamma d^{2\gamma-1} q (\widetilde{\nabla} d) q + d^{2\gamma} (\widetilde{\nabla} q) q.
\end{aligned}$$

Integrating by parts, we observe that

$$2\gamma \int_{\mathbb{H}_q} d^{\alpha+2\gamma-1} q (\widetilde{\nabla} d) q \mathrm{d}x = \frac{\gamma}{\alpha+2\gamma} \int_{\mathbb{H}_q} (\widetilde{\nabla} d^{\alpha+2\gamma}) q^2 \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q}(\widetilde{\nabla}q^2)d^{\alpha+2\gamma}\mathrm{d}x \\
&= -\frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q}q^2\mathcal{L}d^{\alpha+2\gamma}\mathrm{d}x,
\end{aligned}
$$

where we note that later on we will choose γ so that $d^{\alpha+2\gamma}=\Gamma$. Consequently, we get

$$
\begin{aligned}
\int_{\mathbb{H}_q} d^{\alpha}(\widetilde{\nabla}u)u\mathrm{d}x =&\gamma^2\int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla}d)d)\,q^2\mathrm{d}x+\frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q}(\widetilde{\nabla}d^{\alpha+2\gamma})q^2\mathrm{d}x \\
&+\int_{\mathbb{H}_q} d^{\alpha+2\gamma}(\widetilde{\nabla}q)q\mathrm{d}x \\
=&\gamma^2\int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla}d)d)\,q^2\mathrm{d}x-\frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q}q^2\mathcal{L}d^{\alpha+2\gamma}\mathrm{d}x \\
&+\int_{\mathbb{H}_q} d^{\alpha+2\gamma}(\widetilde{\nabla}q)q\mathrm{d}x \\
\geq&\gamma^2\int_{\mathbb{H}_q} d^{\alpha+2\gamma-2}((\widetilde{\nabla}d)d)\,q^2\mathrm{d}x-\frac{\gamma}{\alpha+2\gamma}\int_{\mathbb{H}_q}q^2\mathcal{L}d^{\alpha+2\gamma}\mathrm{d}x, \qquad (3.2)
\end{aligned}
$$

since $d>0$ and $(\widetilde{\nabla}q)q=|\nabla q|^2\geq 0$. On the other hand, it can be readily checked that for a vector field X we have

$$
\begin{aligned}
\frac{\gamma}{\alpha+2\gamma}X^2(d^{\alpha+2\gamma}) &= \gamma X(d^{\alpha+2\gamma-1}Xd) \\
&= \frac{\gamma}{2-\beta}X(d^{\alpha+2\gamma+\beta-2}X(d^{2-\beta})) \\
&= \frac{\gamma}{2-\beta}(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma+\beta-3}(Xd)X(d^{2-\beta}) \\
&\quad+\frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta}) \\
&= \gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(Xd)^2 \\
&\quad+\frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}X^2(d^{2-\beta}).
\end{aligned}
$$

Consequently, we get the equality

$$
-\frac{\gamma}{\alpha+2\gamma}\mathcal{L}d^{\alpha+2\gamma}=-\gamma(\alpha+2\gamma+\beta-2)d^{\alpha+2\gamma-2}(\widetilde{\nabla}d)d-\frac{\gamma}{2-\beta}d^{\alpha+2\gamma+\beta-2}\mathcal{L}d^{2-\beta}. \qquad (3.3)
$$

Since $q^2=d^{-2\gamma}u^2$, substituting (3.3) into (3.2) we obtain

$$
\begin{aligned}
\int_{\mathbb{H}_q} d^{\alpha}(\widetilde{\nabla}u)u\mathrm{d}x \geq&(-\gamma^2-\gamma(\alpha+\beta-2))\int_{\mathbb{H}_q} d^{\alpha-2}((\widetilde{\nabla}d)d)u^2\mathrm{d}x \\
&-\frac{\gamma}{2-\beta}\int_{\mathbb{H}_q}(\mathcal{L}d^{2-\beta})d^{\alpha+\beta-2}u^2\mathrm{d}x.
\end{aligned}
$$

Taking $d=\Gamma^{\frac{1}{2-\beta}}$, $\beta>2$, concerning the second term we observe that

$$\int_{\mathbb{H}_q}(\mathcal{L}\Gamma)\Gamma^{\frac{\alpha+\beta-2}{2-\beta}}|u|^2\mathrm{d}x=\left(\frac{1}{\Gamma(e)}\right)^{\frac{\alpha+\beta-2}{\beta-2}}u^2(e)=0,\ \alpha>2-\beta,\ \beta>2, \tag{3.4}$$

since Γ is the fundamental solution of the sub-Laplacian $\mathcal{L}$. Here $e=(0,0,0,0)$ is the identity element of $\mathbb{H}_q$. Thus, with $d=\Gamma^{\frac{1}{2-\beta}}$, $\beta>2$, we obtain

$$\int_{\mathbb{H}_q}\Gamma^{\frac{\alpha}{2-\beta}}(\widetilde{\nabla}u)u\,\mathrm{d}x\geq(-\gamma^2-\gamma(\alpha+\beta-2))\int_{\mathbb{H}_q}\Gamma^{\frac{\alpha-2}{2-\beta}}(\widetilde{\nabla}\Gamma^{\frac{1}{2-\beta}})\Gamma^{\frac{1}{2-\beta}}|u|^2\,\mathrm{d}x. \tag{3.5}$$

Now taking $\gamma=\frac{2-\beta-\alpha}{2}$, we arrive at (3.1). ☐

Theorem 3.1 implies the following uncertainty principles:

Corollary 3.2 (Uncertainty principle on $\mathbb{H}_q$)**.** *Let $\beta>2$. Then for any $u\in C_0^\infty(\mathbb{H}_q)$, we have*

$$\int_{\mathbb{H}_q}\Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\mathrm{d}x\int_{\mathbb{H}_q}|\nabla u|^2\mathrm{d}x\geq\left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{H}_q}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\mathrm{d}x\right)^2, \tag{3.6}$$

and also

$$\int_{\mathbb{H}_q}\frac{\Gamma^{\frac{2}{2-\beta}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2}|u|^2\mathrm{d}x\int_{\mathbb{H}_q}|\nabla u|^2\mathrm{d}x\geq\left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{H}_q}|u|^2\mathrm{d}x\right)^2. \tag{3.7}$$

Proof of Corollary 3.2. By taking $\alpha=0$ in the inequality (3.1), we get

$$\begin{aligned}\int_{\mathbb{H}_q}\Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\mathrm{d}x\int_{\mathbb{H}_q}|\nabla u|^2\mathrm{d}x&\geq\left(\frac{\beta-2}{2}\right)^2\int_{\mathbb{H}_q}\Gamma^{\frac{2}{2-\beta}}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\mathrm{d}x\int_{\mathbb{H}_q}\frac{|\nabla\Gamma^{\frac{1}{2-\beta}}|^2}{\Gamma^{\frac{2}{2-\beta}}}|u|^2\,\mathrm{d}x\\&\geq\left(\frac{\beta-2}{2}\right)^2\left(\int_{\mathbb{H}_q}|\nabla\Gamma^{\frac{1}{2-\beta}}|^2|u|^2\mathrm{d}x\right)^2,\end{aligned}$$

where we have used the Hölder inequality in the last line. This shows (3.6). The proof of (3.7) is similar. ☐

4. Rellich type inequalities on $\mathbb{H}_q$

In this section, we present a version of the Rellich inequality on the quaternion Heisenberg group $\mathbb{H}_q$.

Theorem 4.1. *Let $\alpha\in\mathbb{R}$, $\beta>\alpha>4-\beta$ and $\beta>2$. Then the following version of the Rellich inequality is valid:*

$$\left\|\frac{\Gamma^{\frac{\alpha}{2(2-\beta)}}}{|\nabla\Gamma^{\frac{1}{2-\beta}}|}|\mathcal{L}u|\right\|_{L_2(\mathbb{H}_q)}\geq\frac{(\beta+\alpha-4)(\beta-\alpha)}{4}\left\|\Gamma^{\frac{\alpha-4}{2(2-\beta)}}|\nabla\Gamma^{\frac{1}{2-\beta}}|u\right\|_{L_2(\mathbb{H}_q)}, \tag{4.1}$$

for any $u\in C_0^\infty(\mathbb{H}_q)$, where $\nabla=(X_0,X_1,X_2,X_3)$ is the gradient and $\mathcal{L}$ is the sub-Laplacian on the quaternion Heisenberg group $\mathbb{H}_q$ as defined in the introduction.

Proof of Theorem 4.1. A direct calculation shows that

$$\begin{aligned}
\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} =& \sum_{k=0}^{3} X_k^2 \Gamma^{\frac{\alpha-2}{2-\beta}} = (\alpha-2)\sum_{k=0}^{3} X_k \left(\Gamma^{\frac{\alpha-3}{2-\beta}} X_k \Gamma^{\frac{1}{2-\beta}}\right) \\
=&(\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}} \sum_{k=0}^{3} \left|X_k \Gamma^{\frac{1}{2-\beta}}\right|^2 + (\alpha-2)\Gamma^{\frac{\alpha-3}{2-\beta}} \sum_{k=0}^{3} X_k \left(X_k \Gamma^{\frac{1}{2-\beta}}\right) \\
=&(\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}} \sum_{k=0}^{3} \left|X_k \Gamma^{\frac{1}{2-\beta}}\right|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}} \sum_{k=0}^{3} X_k \left(\Gamma^{\frac{\beta-1}{2-\beta}} X_k \Gamma\right) \\
=&(\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}} \sum_{k=0}^{3} \left|X_k \Gamma^{\frac{1}{2-\beta}}\right|^2 + \frac{(\alpha-2)(\beta-1)}{2-\beta}\Gamma^{\frac{\alpha-3}{2-\beta}}\Gamma^{-1} \sum_{k=0}^{3} (X_k \Gamma^{\frac{1}{2-\beta}})(X_k\Gamma) \\
&+ \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma = (\alpha-2)(\alpha-3)\Gamma^{\frac{\alpha-4}{2-\beta}} \sum_{k=0}^{3} \left|X_k \Gamma^{\frac{1}{2-\beta}}\right|^2 \\
&+ (\alpha-2)(\beta-1)\Gamma^{\frac{\alpha-4}{2-\beta}} \sum_{k=0}^{3} (X_k \Gamma^{\frac{1}{2-\beta}})(X_k \Gamma^{\frac{1}{2-\beta}}) + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma \\
=&(\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma,
\end{aligned}$$

that is,

$$\mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}} = (\beta+\alpha-4)(\alpha-2)\Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 + \frac{\alpha-2}{2-\beta}\Gamma^{\frac{\beta+\alpha-4}{2-\beta}}\mathcal{L}\Gamma. \tag{4.2}$$

As before we can assume that u is real-valued. Multiplying both sides of (4.2) by u^2 and integrating over $\mathbb{H}_q$, since Γ is the fundamental solution of $\mathcal{L}$ and $\beta+\alpha-4>0$, we get

$$\int_{\mathbb{H}_q} u^2 \mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\, \mathrm{d}x = (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 u^2\, \mathrm{d}x. \tag{4.3}$$

On the other hand, integrating by parts, we have

$$\int_{\mathbb{H}_q} u^2 \mathcal{L}\Gamma^{\frac{\alpha-2}{2-\beta}}\, \mathrm{d}x = \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} \mathcal{L}u^2\, \mathrm{d}x = \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} (2u\mathcal{L}u + 2|\nabla u|^2)\, \mathrm{d}x, \tag{4.4}$$

Combining (4.3) and (4.4) we obtain

$$-2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u \mathrm{d}x + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 u^2 \mathrm{d}x = 2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} |\nabla u|^2 \mathrm{d}x. \tag{4.5}$$

By using (3.1) we establish

$$\begin{aligned}
&-2\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u \mathrm{d}x + (\beta+\alpha-4)(\alpha-2)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \mathrm{d}x \\
&\geq 2\left(\frac{\beta+\alpha-4}{2}\right)^2 \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2\, \mathrm{d}x.
\end{aligned} \tag{4.6}$$

It follows that

$$-\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u \mathrm{d}x \geq \left(\frac{\beta+\alpha-4}{2}\right)\left(\frac{\beta-\alpha}{2}\right)\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, \mathrm{d}x. \tag{4.7}$$

On the other hand, for any $\epsilon > 0$ Hölder's and Young's inequalities give

$$\begin{aligned} -\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-2}{2-\beta}} u\mathcal{L}u \mathrm{d}x &\leq \left(\int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \epsilon \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \mathrm{d}x + \frac{1}{4\epsilon} \int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 \mathrm{d}x. \end{aligned} \tag{4.8}$$

Inequalities (4.8) and (4.7) imply that

$$\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 \mathrm{d}x \geq \left(-4\epsilon^2 + (\beta+\alpha-4)(\beta-\alpha)\epsilon\right) \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, \mathrm{d}x.$$

Taking $\epsilon = \frac{(\beta+\alpha-4)(\beta-\alpha)}{8}$, we arrive at

$$\int_{\mathbb{H}_q} \frac{\Gamma^{\frac{\alpha}{2-\beta}}}{|\nabla \Gamma^{\frac{1}{2-\beta}}|^2} |\mathcal{L}u|^2 \mathrm{d}x \geq \frac{(\beta+\alpha-4)^2(\beta-\alpha)^2}{16} \int_{\mathbb{H}_q} \Gamma^{\frac{\alpha-4}{2-\beta}} |\nabla \Gamma^{\frac{1}{2-\beta}}|^2 |u|^2 \, \mathrm{d}x.$$

□

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