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OPTIMAL HARVESTING OF A SPATIALLY DISTRIBUTED RENEWABLE RESOURCE WITH ENDOGENOUS PRICING

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Abstract. In this paper, we focus on the exploitation of a renewable resource in a spatial setting. Building upon the spatial harvesting model of [Behringer and Upmann, *J. Econ. Dyn. Control* **42** (2014) 105–120], we endogenize the price for the resource assuming that after harvesting the good is non-durable, *i.e.* the harvesting yield must be supplied on the market instantaneously. We find necessary optimality conditions and use them to derive an iterative algorithm to improve at each step the harvesting effort. We find that with endogenous prices the full exploitation result of [Behringer and Upmann, *J. Econ. Dyn. Control* **42** (2014) 105–120] may cease to hold.

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1. Introduction

While classical microeconomic theory deals with homogenous consumption goods in a stationary framework, we consider a dynamic setting. More precisely, we consider the harvesting and sale of a renewable natural resource (fish, timber, etc.) the stock of which obeys a given law of growth. We consider the resources to be spatially distributed, thereby following extended taking into consideration demands from the discipline and policy makers (see [11]).

Recently [7] investigated optimal harvesting of a renewable resource that is spatially distributed over a *continuous* domain. Since in their model the agent is required to move in space, an optimal policy consists of an optimal choice of both, harvesting and movement. This approach, which has been generalized in [17] for a fully independent control choice, contrasts with previous analyses of *discrete* spaces, *e.g.* [14], but is similar to [8] who also consider a continuous spatial setting (see [15, 16], or [10] for early economic analyses). Harvesting models have also been intensively studied in [1, 12], or [2]. For optimal control problems for some economics models see [4], and [13].

The dynamic optimization problem in the model of [7] consists of a simultaneous choice of the speed of movement $\{v(t)\}_{t\in\mathcal{T}}$ and the harvesting rate $\{h(t)\}_{t\in\mathcal{T}}$. More precisely, the harvesting agent moves on a unit

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circle on which the resource, with stock $f(\cdot)$, is growing according to growth function $g(\cdot)$. The agent's location s is therefore on $\mathcal{S} = [0, 2\pi]$. $\mathcal{T}$ denotes the harvesting period or season $[0, T]$ and harvesting comes at a cost $C(\cdot)$, which may depend on the speed of the agent and the harvesting rate.

As the agent cannot harvest more than the entire resource stock at any particular location, we have $h(t) \leq \min\{\bar{h}, f(t, s(t))\}$, where $\bar{h}$ denotes the harvesting capacity of the agent. Harvesting takes place only at the actual location of the agent $x = s(t)$ and implies a downward jump in the stock of the resource $f(\cdot, x)$ at the set of arrival times of the agent at that location $x : J(x) = \{t_1(x), t_2(x), \ldots\}$. Therefore, the law of motion for the stock is

$$\frac{\partial f}{\partial t}(t, x) = g(f(t, x)), \ \forall t \in \mathcal{T} \backslash J(x), x \in S \tag{1.1}$$

$$f(t-, x) - f(t+, x) = h(t), \ \forall t \in J(x), x \in S \tag{1.2}$$

with initial level $f(0, x) = f_0(x)$ for all $x \in \mathcal{S}$.

By discounting the future at a rate $\rho \geq 0$, the agent's problem is

$$\max_{\{v,h\}} \int_0^T e^{-\rho t} \left(h(t) - C(v(t), h(t))\right) \mathrm{d}t$$

such that

$$\begin{aligned}
&\dot{s}(t) = v(t), && \forall t \in \mathcal{T} \\
&\frac{\partial f}{\partial t}(t, x) = g(f(t, x)), && \forall t \in \mathcal{T} \backslash J(x), x \in \mathcal{S} \\
&f(t-, x) - f(t+, x) = h(t), && \forall t \in J(x), x \in \mathcal{S} \\
&h(t) \in H(t) && \forall t \in \mathcal{T} \\
&f(0, x) = f_0(x), && \forall x \in \mathcal{S} \\
&s(0) = 0,
\end{aligned}$$

where $H(t) = [0, \min\{\bar{h}, f(t, s(t))\}]$. The last line implies that without loss of generality we let the agent start at $x = 0$ on the unit circle.

Let $\mathcal{F}$ be the solution of the set of differential equation (1.1) between two consecutive impulses, with $\mathcal{F}(f, 0) = f$. Note that (1.1) is *autonomous* and does not depend on time t directly but only *via* $f(\cdot)$. Hence, if we integrate (1.1) over the time of two consecutive rounds $t_{i-1}(x)$ and $t_i(x)$ we get

$$f(t_i(x)-, x) = \mathcal{F}(f(t_{i-1}(x)+, x), t_i(x) - t_{i-1}(x)).$$

For any fixed location equation (1.2) gives a mapping

$$f(t_i(x)+, x) = \mathcal{F}(f(t_{i-1}(x)+, x), t_i(x) - t_{i-1}(x)) - h(t_i(x)).$$

We thus have a problem where time and space of impulses are related, that is, we have an impure *impulse control problem.*

Behringer and Upmann [7] find that with exponential growth and constant speed, the resource will be fully extinguished by the agent by the end of the planning horizon. As in the early literature on Walrasian economics, this work treats prices as exogenous, though. In order to fully explore the welfare economic consequences of trading renewable resource commodities, we endogenize prices in this paper.

2. Optimal harvesting policy

Consider a fixed location $x \in \mathcal{S}$. Instead of letting the agent control the harvest $h(t)$, we assume that the agent controls the harvesting share $\alpha(t)$ (*e.g.,* think of a fisher using a fishing net with a given mesh size) so that the harvest amounts to $h(t) = \alpha(t)f(t-)$. This is the common formulation in the resource literature. When fishing nets are used, fish is harvested as a share $\alpha(t)$ of the stock and so the yield from fishing is multiplicative in the stock.

We assume that the commodity, *i.e.* the harvested resource (fish caught), is *non-durable*, and so it cannot be stored but has to be consumed immediately after purchase. Therefore, the quantities supplied to the market do not accumulate over time. Let R denote the instantaneous profit, then the optimal control problem is:

$$\max_{\alpha \in \mathcal{A}} \int_0^T e^{-\rho t} R(\alpha(t), f^\alpha(t))\mathrm{d}t, \tag{2.1}$$

where $\mathcal{A} = \{\alpha \in L^\infty(0,T);\, 0 \leq \alpha(t) \leq 1 \text{ a.e.}\}$ is the set of admissible controls, and f^α is the function f in the previous section corresponding to $h(t) = \alpha(t)ft-)$. Here $f_0 \in L^\infty(0, 2\pi)$, $f_0(t) \geq 0$ a.e., $\rho \in (0, +\infty)$. As in [7] we assume *exponential growth* from here onwards as this simplifies the presence of an economic discounting factor ρ.

Assume from now on that the speed v is a fixed positive constant. Let us consider some arbitrary but fixed location $x = \mod (vt, 2\pi)$, where $\mod (vt, 2\pi) = vt - 2k\pi$, with $2k\pi \leq vt < 2(k+1)\pi$, $k \in \mathbb{N}$. We see that the location of the harvesting device is actually a function of t ($x = \mod (vt, 2\pi)$).

Then, we denote by $f^\alpha(t-)$ the level of the renewable resource at location $x = \mod (vt, 2\pi)$ *just before* harvesting. Likewise the level of the resource *immediately after* harvesting is denoted $f^\alpha(t+)$.

Assume there are k completed rounds until T, such that

$$k\frac{2\pi}{v} < T \leq (k+1)\frac{2\pi}{v}, \qquad k \in \mathbb{N}.$$

For convenience, we extend the time horizon beyond the end of the harvesting period as

$$\alpha(t) = 0 \quad \text{on} \quad \left[T, (k+1)\frac{2\pi}{v}\right]$$

to allow for k complete rounds of supply and a possibly incomplete round on the circle with the density after T being zero. This is equivalent to letting the stock collapse after time T. This convention notionally extends the time horizon but does not affect the optimization problem. It only relaxes the effect that the fixed time horizon has on the possibility to treat only integer rounds.[1]

We use the index variable $l = 0, 1, \ldots, k$ to refer to the rounds of harvesting. Then, $l = 0$ refers to the first round, $l = 1$ to the second round, etc. and eventually $l = k$ refers to the last, possibly incomplete, round $k+1$.

For any fixed location x, the travelling time for one complete round on the circle equals the duration between any two consecutive arrivals times. With constant speed v, the time necessary to circle around the periphery once equals $\theta = 2\pi/v$. Since this θ equals the time between two subsequent harvesting times, it also represents the growth time of the resource between two subsequent harvesting times. Hence, the stock (and more generally the density) is a function of the travelling time θ (or equivalently of speed v).

Then, using the above definitions we obtain

$$f^\alpha(t+) = (1 - \alpha(t))f^\alpha(t-)$$

[1] This slight divergency between the notation and the semantics, *viz.* the counting variable l and the wording, helps facilitate the analysis. In addition, the notional extension of the density function beyond the fixed time horizon makes our analysis much less cumbersome than it otherwise would be.

and because of exponential growth at r it also follows that

$$f^{\alpha}\left((t+\theta)-\right)=e^{r\theta}(1-\alpha(t))f^{\alpha}(t-). \tag{2.2}$$

Equation (2.2) thus states that the density at time $t+\theta$ just before harvesting equals the original density at t before harvesting, of which the harvesting share at t has been deducted and which has since grown according to the exponential growth rate.

Now for some round l on the circle that takes place at some time interval $t\in[l\theta,(l+1)\theta]$, we define

$$f_l^{\alpha}\left((t-\theta l)\right)\equiv f^{\alpha}(t-),\quad l\in\{0,1,\dots,k\},$$

the stock of the resource just before harvesting extended $l\in\{0,1,2,\dots,k\}$ periods into the past. We can then also define the stock of the resource l periods into the future (by adding time θl to the above) as

$$f_l^{\alpha}(t)\equiv f^{\alpha}\left((t+\theta l)-\right).$$

for any round $l\in\{0,1,2,\dots,k\}$.

Adding time θl to (2.2) we find

$$\begin{aligned}f^{\alpha}\left((t+\theta+\theta l)-\right)=f_{l+1}^{\alpha}(t)&=e^{r\theta}(1-\alpha(t+\theta l))f^{\alpha}((t+\theta l)-)\\&=e^{r\theta}(1-\alpha(t+\theta l))f_l^{\alpha}(t)\end{aligned}$$

which holds for all $l\in\{0,1,2,\dots,k\}$ as α does not impact f differently over rounds and we make use of the extended time horizon. We thus have for the time interval $t\in[0,\theta]$ that

$$f_{l+1}^{\alpha}(t)=e^{r\theta}(1-\alpha(t+\theta l))f_l^{\alpha}(t)$$

i.e. the density just before harvesting at any round $l+1$ is given by the original density in round l just before harvesting, of which the harvesting share in that round has been deducted and which since then has grown (for one round of time) according to the exponential growth rate. For the first period, where previous harvesting trivially cannot have a consequence for present harvest and hence α is not an argument to be considered, this reduces to

$$f_0^{\alpha}(t)=e^{\mathrm{rt}}f_0(tv),$$

where $x=$mod$(vt,2\pi)=vt$ if $t\in[0,\theta)$ gives the location in the first round. Thus, we find the relation between round $l\in\{0,1,2,\dots,k\}$ densities and the following round densities for $t\in[0,\theta]$ as:

$$\begin{cases}f_{l+1}^{\alpha}(t)=e^{r\theta}(1-\alpha(t+\theta l))f_l^{\alpha}(t)\\ f_0^{\alpha}(t)=e^{\mathrm{rt}}f_0(tv).\end{cases} \tag{2.3}$$

2.1. Market demand

As the harvesting yield is a non-durable good, it must be sold instantaneously on the market. We assume that market demand is characterized by a downward sloping inverse demand function of the form

$$P(h)=\frac{1}{1+C_0h},\quad C_0>0, \tag{2.4}$$

implying that the demand elasticity equals $\eta(h)\equiv hP'(h)/P(h)=-C_0h/(1+C_0h)$.

For the purpose of tractability, we assume that the harvesting cost only depends on the harvest but not on the stock. Moreover, assuming constant marginal cost of harvesting, the cost function reads as

$$C(v,h) = C_1 + C_2 h, \quad C_1, C_2 > 0. \tag{2.5}$$

Using equations (2.4) and (2.5), and taking into account that $h(t) = \alpha(t) f^{\alpha}(t-)$, the optimal control problem (3) is then given as maximizing the total discounted profit from harvesting:

$$\max_{\alpha\in\mathcal{A}} G(\alpha) = \max_{\alpha\in\mathcal{A}} \int_0^T e^{-\rho t}\left[p_0 \frac{\alpha(t) f^{\alpha}(t-)}{1 + C_0 \alpha(t) f^{\alpha}(t-)} - C_1 - C_2 \alpha(t) f^{\alpha}(t-)\right] \mathrm{d}t, \tag{2.6}$$

where $\mathcal{A} = \{\alpha \in L^{\infty}(0,T);\ 0 \le \alpha(t) \le 1 \text{ a.e.}\}$ is the set of admissible controls and p_0 is a positive constant representing, for example, taxes or subsidies on revenue. Since the agent acknowledges the effects of its supply on the market price, our analysis basically represents the case of a monopoly.

This objective can be rewritten as the sum of k completed and a possibly incomplete round on the circle as

$$\begin{aligned} G(\alpha) = {} & \sum_{l=0}^{k-1} \int_0^{\theta} e^{-\rho(t+\theta l)} \left[p_0 \frac{\alpha(t+\theta l) f_l^{\alpha}(t)}{1 + C_0 \alpha(t+\theta l) f_l^{\alpha}(t)} - C_2 \alpha(t+\theta l) f_l^{\alpha}(t)\right] \mathrm{d}t \\ & + \int_0^{T-\theta k} e^{-\rho(t+\theta k)} \left[p_0 \frac{\alpha(t+\theta k) f_k^{\alpha}(t)}{1 + C_0 \alpha(t+\theta k) f_k^{\alpha}(t)} - C_2 \alpha(t+\theta k) f_k^{\alpha}(t)\right] \mathrm{d}t \\ & - C_1 \frac{1}{\rho}(1 - e^{-\rho T}). \end{aligned}$$

The following Theorem gives the directional derivative of G.

Theorem 2.1. *For any* $\alpha \in L^{\infty}(0,T)$, $0 \le \alpha(t) \le 1$ a.e. *and* $w \in L^{\infty}(0,T)$ *such that* $0 \le \alpha(t) + \varepsilon w(t) \le 1$ a.e., *for sufficiently small* $\varepsilon > 0$, *we have that*

$$\begin{aligned} dG(\alpha)(w) = \sum_{l=0}^{k} \int_0^{\theta} e^{-\rho(t+\theta l)} & \left[p_0 \frac{w(t+\theta l) f_l^{\alpha}(t) + \alpha(t+\theta l) z_l(t)}{(1 + C_0 \alpha(t+\theta l) f_l^{\alpha}(t))^2}\right. \\ & \left. - C_2 \left(w(t+\theta l) f_l^{\alpha}(t) + \alpha(t+\theta l) z_l(t)\right)\right] \mathrm{d}t \end{aligned} \tag{2.7}$$

with

$$\begin{cases} z_{l+1}(t) = e^{r\theta}\left[-w\left(t+\theta l\right) f_l^{\alpha}(t) + \left(1 - \alpha\left(t+\theta l\right)\right) z_l(t)\right], \\ \qquad\qquad\qquad\qquad t \in [0,\theta), \quad l = 0, 1, \ldots, k-1, \\ z_0(t) = 0, \qquad t \in [0,\theta). \end{cases} \tag{2.8}$$

Here

$$\alpha(t) = 0,\ w(t) = 0,\ \text{a.e. } t \in [T, (k+1)\theta],$$

and

$$z_l = \lim_{\varepsilon\to 0} \frac{f_l^{\alpha+\varepsilon w} - f_l^{\alpha}}{\varepsilon} \quad in \quad L^{\infty}(0,T). \tag{2.9}$$

Proof. For any α, w satisfying the hypotheses we have

$$\begin{aligned} G(\alpha+\varepsilon w)-G(\alpha) = \sum_{l=0}^{k}\int_0^{\theta} e^{-\rho(t+\theta l)}\Bigg[&p_0\frac{(\alpha+\varepsilon w)(t+\theta l)f_l^{\alpha+\varepsilon w}(t)}{1+C_0(\alpha+\varepsilon w)(t+\theta l)f_l^{\alpha+\varepsilon w}(t)} \\ &-C_2(\alpha+\varepsilon w)(t+\theta l)f_l^{\alpha+\varepsilon w}(t)\Bigg]\mathrm{d}t \\ &-\sum_{l=0}^{k}\int_0^{\theta} e^{-\rho(t+\theta l)}\left[p_0\frac{\alpha(t+\theta l)f_l^{\alpha}(t)}{1+C_0\alpha(t+\theta l)f_l^{\alpha}(t)}-C_2\alpha(t+\theta l)f_l^{\alpha}(t)\right]\mathrm{d}t. \end{aligned}$$

Dividing by $\varepsilon > 0$, taking $\varepsilon \to 0$ and using (2.9) we get

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{G(\alpha+\varepsilon w)-G(\alpha)}{\varepsilon} = &\sum_{l=0}^{k}\int_0^{\theta} e^{-\rho(t+\theta l)}\Bigg[p_0\frac{w(t+\theta l)f_l^{\alpha}(t)+\alpha(t+\theta l)z_l(t)}{(1+C_0\alpha(t+\theta l)f_l^{\alpha}(t))^2} \\ &-C_2\left(w(t+\theta l)f_l^{\alpha}(t)+\alpha(t+\theta l)z_l(t)\right)\Bigg]\mathrm{d}t. \end{aligned}$$

□

Let us denote by q the adjoint state, *i.e.* q satisfies

$$\begin{cases} q_l(t) = e^{r\theta}\left(1-\alpha\left(t+\theta l\right)\right)q_{l+1}(t) \\ \qquad + e^{-\rho(t+\theta l)}\left[p_0\dfrac{\alpha(t+\theta l)}{(1+C_0\alpha(t+\theta l)f_l^{\alpha}(t))^2}-C_2\alpha(t+\theta l)\right], \\ \qquad t\in[0,\theta), \quad l=0,1,\dots,k-1, \\ q_k(t) = \begin{cases} e^{-\rho(t+\theta k)}\left[p_0\dfrac{\alpha(t+\theta k)}{(1+C_0\alpha^*(t+\theta k)f_k^{\alpha}(t))^2}-C_2\alpha(t+\theta k)\right], \\ \qquad t\in[0,T-\theta k), \\ 0, \quad t\in[T-\theta k,\theta]. \end{cases} \end{cases} \tag{2.10}$$

For the construction of the adjoint problems in optimal control theory we refer to [3, 5, 6].

Theorem 2.2. *For any* $\alpha \in L^\infty(0,T)$, $0 \le \alpha(t) \le 1$ a.e. *we have that*

$$\begin{aligned} dG(\alpha)(w) = &\sum_{l=0}^{k-1}\int_0^{\theta} w\left(t+\theta l\right)\left[e^{-\rho(t+\theta l)}\left(p_0\frac{1}{(1+C_0\alpha(t+\theta l)f_l^{\alpha}(t))^2}-C_2\right)-e^{r\theta}q_{l+1}(t)\right]f_l^{\alpha}(t)\mathrm{d}t \\ &+\int_0^{\theta} w\left(t+\theta k\right)e^{-\rho(t+\theta k)}\left(p_0\frac{1}{(1+C_0\alpha(t+\theta k)f_k^{\alpha}(t))^2}-C_2\right)f_k^{\alpha}(t)\mathrm{d}t, \end{aligned} \tag{2.11}$$

for any $w \in L^\infty(0,T)$ *such that* $0 \le \alpha+\varepsilon w \le 1$, a.e., *for sufficiently small* $\varepsilon > 0$ *(we extend* α *and* w *by the value 0 on* $[T,(k+1)\theta]$*).*

Proof. We multiply the first equation in (2.10) by $z_l(t)$, integrate on $[0,\theta)$ and add up over l to $k-1$. We get that

$$\sum_{l=0}^{k-1}\int_0^\theta q_l(t)z_l(t)\mathrm{d}t = \sum_{l=0}^{k-1}\int_0^\theta \Big[e^{r\theta}\left(1-\alpha\left(t+\theta l\right)\right)q_{l+1}(t)z_l(t)$$
$$+e^{-\rho(t+\theta l)}\left(p_0\frac{\alpha(t+\theta l)}{(1+C_0(t+\theta l)f_l^\alpha(t))^2}-C_2\alpha(t+\theta l)\right)z_l(t)\Big]\,\mathrm{d}t.$$

Rewritting (2.8) as

$$e^{r\theta}\left(1-\alpha\left(t+\theta l\right)\right)z_l(t)=z_{l+1}(t)+e^{r\theta}w\left(t+\theta l\right)f_l^\alpha(t),$$

we obtain that

$$\sum_{l=0}^{k-1}\int_0^\theta q_l(t)z_l(t)\mathrm{d}t = \sum_{l=0}^{k-1}\int_0^\theta q_{l+1}(t)\left[z_{l+1}(t)+e^{r\theta}w\left(t+\theta l\right)f_l^\alpha(t)\right]\mathrm{d}t$$
$$+\sum_{l=0}^{k-1}\int_0^\theta e^{-\rho(t+\theta l)}\left(p_0\frac{\alpha(t+\theta l)}{(1+C_0\alpha(t+\theta l)f_l^\alpha(t))^2}-C_2\alpha(t+\theta l)\right)z_l(t)\mathrm{d}t.$$

Since $z_0(t)=0$ and $q_k(t)$ satisfies the second equation in (2.10), we may conclude that

$$0=\sum_{l=0}^{k-1}\int_0^\theta e^{r\theta}w\left(t+\theta l\right)f_l^\alpha(t)q_{l+1}(t)\mathrm{d}t$$
$$+\sum_{l=0}^{k}\int_0^\theta e^{-\rho(t+\theta l)}\left[p_0\frac{\alpha(t+\theta l)z_l(t)}{(1+C_0\alpha(t+\theta l)f_l^\alpha(t))^2}-C_2\alpha(t+\theta l)z_l(t)\right]\mathrm{d}t.$$

Using (2.7) we obtain

$$\begin{aligned} dG(\alpha)(w) = &-\sum_{l=0}^{k-1}\int_0^\theta w\left(t+\theta l\right)e^{r\theta}f_l^\alpha(t)q_{l+1}(t)\mathrm{d}t \\ &+\sum_{l=0}^{k}\int_0^\theta e^{-\rho(t+\theta l)}\left[p_0\frac{w(t+\theta l)f_l^\alpha(t)}{(1+C_0\alpha(t+\theta l)f_l^\alpha(t))^2}-C_2w(t+\theta l)f_l^\alpha(t)\right]\mathrm{d}t. \end{aligned} \tag{2.12}$$

From (2.12) we get the conclusion. □

Existence of an optimal control can be proved as in [3, 6, 9]. Let α^* be such an *optimal* control. Then, for any $w\in L^\infty(0,T)$ such that only $0\le\alpha^*(t)+\varepsilon w(t)\le 1$ a.e., for sufficiently small $\varepsilon>0$ holds, we have that

$$G(\alpha^*)\ge G(\alpha^*+\varepsilon w).$$

Remark 2.3. Let α^* be an optimal control for the problem (2.6). We denote by

$$a_l(t)=\frac{p_0}{(1+C_0\alpha^*(t+\theta l)f_l^{\alpha^*}(t))^2}-C_2-e^{r\theta+\rho(t+\theta l)}q_{l+1}(t), \tag{2.13}$$

for $t \in [0,\theta)$ and $l = 0,1,\dots,k-1$. $a_l(t)$ is a strictly decreasing and continuous function of $\alpha^*(t+\theta l) \in [0,1]$. The set of all values of $a_l(t)$ is the closed interval

$$\left[\frac{p_0}{(1+C_0 f_l^{\alpha^*}(t))^2} - C_2 - e^{r\theta+\rho(t+\theta l)}q_{l+1}(t),\ p_0 - C_2 - e^{r\theta+\rho(t+\theta l)}q_{l+1}(t)\right].$$

If 0 belongs to this interval, then $\alpha^*(t+\theta l)$ is the unique value for which $a_l(t) = 0$. So, for this case

$$\alpha^*(t+\theta l) = \frac{1}{C_0 f_l^{\alpha^*}(t)}\left(\sqrt{\frac{p_0}{C_2 + e^{r\theta+\rho(t+\theta l)}q_{l+1}(t)}} - 1\right). \tag{2.14}$$

If 0 does not belong to this interval, then the optimal control α^* can be characterized as:

$$\alpha^*(t+\theta l) = \begin{cases} 0, & \text{if} \quad p_0 - C_2 - e^{r\theta+\rho(t+\theta l)}q_{l+1}(t) < 0 \\ 1, & \text{if} \quad \frac{p_0}{(1+C_0 f_l^{\alpha^*}(t))^2} - C_2 - e^{r\theta+\rho(t+\theta l)}q_{l+1}(t) > 0, \end{cases} \tag{2.15}$$

for $t \in [0,\theta)$ and $l = 0,1,\dots,k-1$.

For $t \in [0, T-\theta k)$, we obtain similar results denoting by

$$a_k(t) = \frac{p_0}{(1+C_0\alpha^*(t+\theta k) f_l^{\alpha^*}(t))^2} - C_2.$$

The theoretical results allow for numerical tests that extend the present framework to more realistic and heterogenous distributions of the resource.

3. Numerical tests

The previous results (Thms. 2.1 and 2.2, and Rem. 2.3) allow us to develop a conceptual iterative algorithm to improve at each step the control α, in order to obtain a higher value for the objective G (a gradient-type algorithm).

`Step 0.` `Set` $j := 0$ `and` $G^{(0)} := 0$
`Initialize` $\alpha^{(0)}(t)$.

`Step 1.` `Compute` $f^{(j+1)}$ `the solution of` (2.3) `corresponding to` $\alpha := \alpha^{(j)}$;
`Evaluate` $G^{(j+1)} := G$ `from` (2.6) `corresponding to` $f^\alpha := f^{(j+1)}$ `and` $\alpha := \alpha^{(j)}$.

`Step 2.` `if` $|G^{(j+1)} - G^{(j)}| < \varepsilon$ `or` $G^{(j+1)} \le G^{(j)}$ `then STOP;`
`else go to Step 3.`

`Step 3.` `Compute` $q^{(j+1)}$ `the solution of` (2.10) `corresponding to` $\alpha := \alpha^{(j)}$ `and` $f^\alpha := f^{(j+1)}$;
`Compute` $\alpha^{(j+1)}(t) := \min\{\max\{\alpha^{(j)}(t) + \tilde{\varepsilon} w(t),\ 0\},\ 1\}$, `where`

$$w_l(t) = f_l^{(j+1)}\left(\frac{p_0}{(1+C_0\alpha^{(j)}(t+\theta l) f_l^{\alpha^{(j)}}(t))^2} - C_2 - e^{r\theta+\rho(t+\theta l)}q_{l+1}^{(j+1)}\right),\quad l = 0,\dots,k-1,$$

$$w_k(t) = f_k^{(j+1)}\left(\frac{p_0}{(1+C_0\alpha^{(j)}(t+\theta k) f_k^{\alpha^{(j)}}(t))^2} - C_2\right),\quad t \in [0, T-\theta k),$$

`and` $\tilde{\varepsilon} > 0$ `is small.`

`Step 4.` $j := j+1$;
`go to Step 1.`

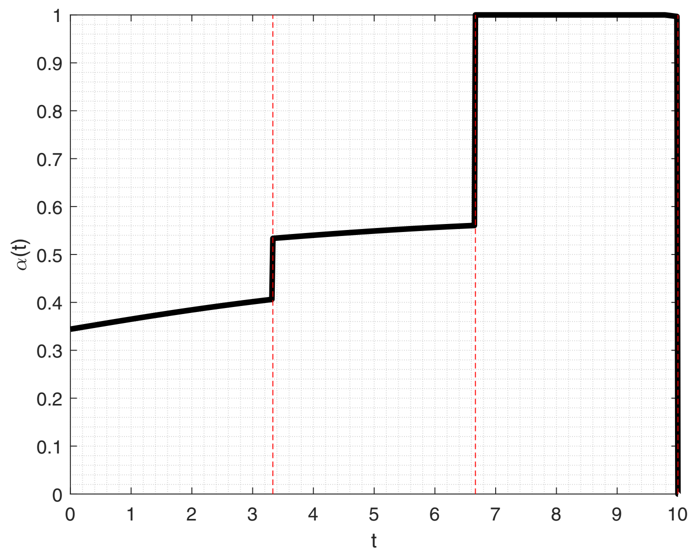

FIGURE 1. Test 1. The representation of final iteration of α.

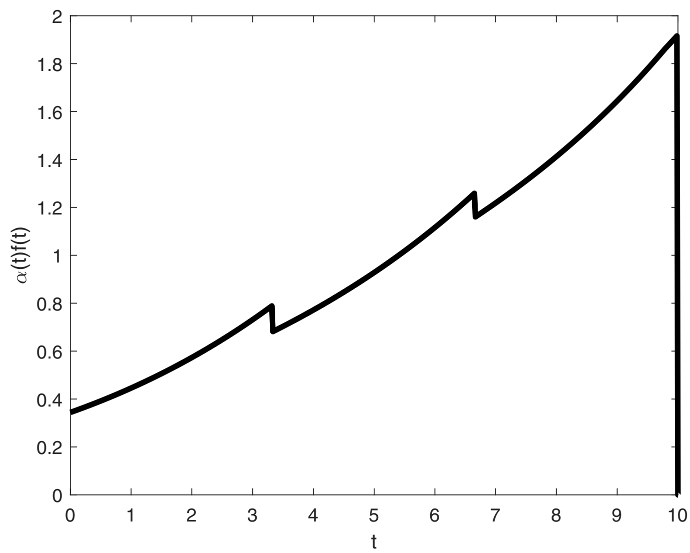

FIGURE 2. Test 1. The level of harvest corresponding to the last iteration.

Here $\varepsilon > 0$ (Step 2) is a prescribed convergence parameter. For more information about gradient methods, see [5].

3.1. Numerical examples

With fixed T and v we get $k = \lfloor T/\theta \rfloor$. The domain $[0, \theta)$ is discretized by m equidistant nodes, namely

$$t_i = (i-1)\Delta t, \; i = 1, 2, \ldots, m,$$

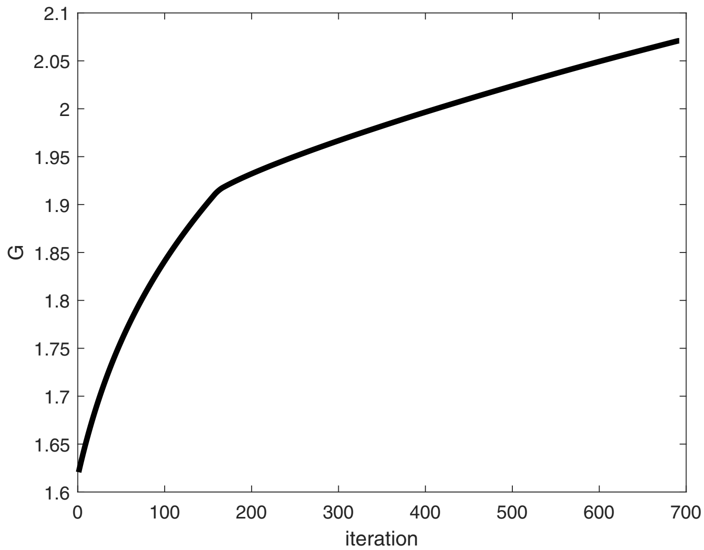

FIGURE 3. Test 2. The approximate values of G as a function of iterations.

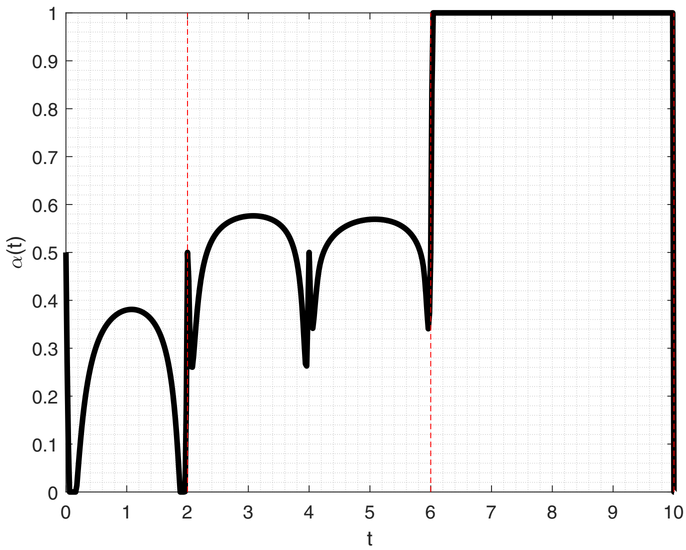

FIGURE 4. Test 2. The representation of final iteration of α.

with $\Delta t = \theta/m$. Thus, the interval $[l\theta, (l+1)\theta]$ is approximated by nodes

$$t_{lm+i} = t_i + \theta l,$$

for $i = 1, 2, \ldots, m$, and $l = 0, 1, 2, \ldots, k$, if $k < T/\theta$, and $l = 0, 1, 2, \ldots, k-1$, if $k = T/\theta$.

We set the positive constants: $C_0 = 1,\ C_1 = 0.1,\ C_2 = 0.01,\ p_0 = 1,\ \rho = 0.05$, and $r = 0.2$. We also take the tolerance $\varepsilon = 0.0001$.

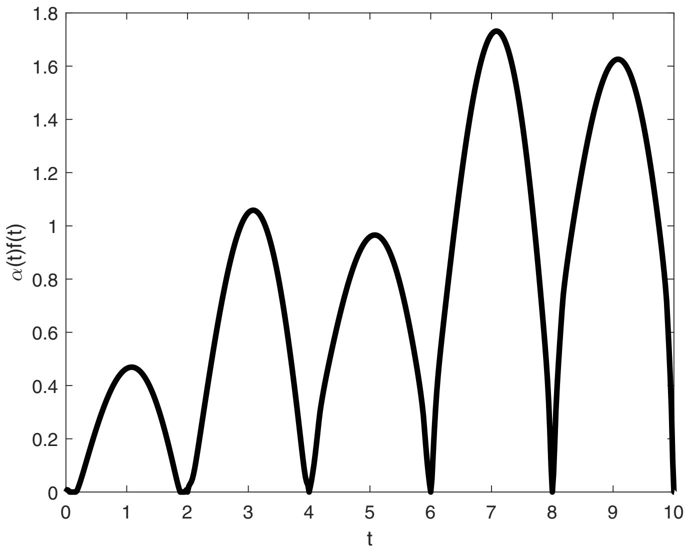

FIGURE 5. Test 2. The level of harvest corresponding to the last iteration.

TABLE 1. The approximate value of the cost functional G after a certain number of iterations.

Iteration	G
1	2.3204
10	2.3767
20	2.4285
30	2.4715
40	2.5079
50	2.5391
100	2.6492
120	2.6806
140	2.7078
168	2.7335

Test 1. For $v = 3\pi/5$ and $T = 10$, we obtain $k = 3$ complete rounds. The initial level of stock is assumed to be a constant, *e.g.* $f_0(x) = 1$. To start with, we take $\alpha^{(0)}(t) = 0.5$. For $m = 200$, the algorithm ends in 168 iterations, when the first condition in Step 2 is fulfilled.

In Table 1, it can be seen that the algorithm provides a higher value for the objective G at each step. The representation of the control α corresponding to the last iteration is in Figure 1, and the level of harvest after the last iteration of the algorithm is shown in Figure 2.

Let us notice that, for these numerical data, we have full exploitation of the resource in the last round of harvesting. This means that we have $\alpha(t) = 1$ for all $t \in [T - \theta, T]$, thus the final stock equals $f(T, x) = 0$, for all $x \in [0, 2\pi]$ (see Fig. 1, where by dashed vertical lines we indicate the three completed rounds).

Test 2. For $v = \pi/2$ and $T = 10$, we obtain $k = 2$ complete rounds and an incomplete one. We take $f_0(x) = |\sin x|$, and with the same initialization of $\alpha^{(0)}(t)$, the algorithm ends in 689 iterations, with the first condition in Step 2 fulfilled. In Figure 3 is the representation of the objective G *vs.* iterations, and in the Figure 4 is the control from the last iteration. The level of harvest resulting from the last iteration of the algorithm is shown in Figure 5.

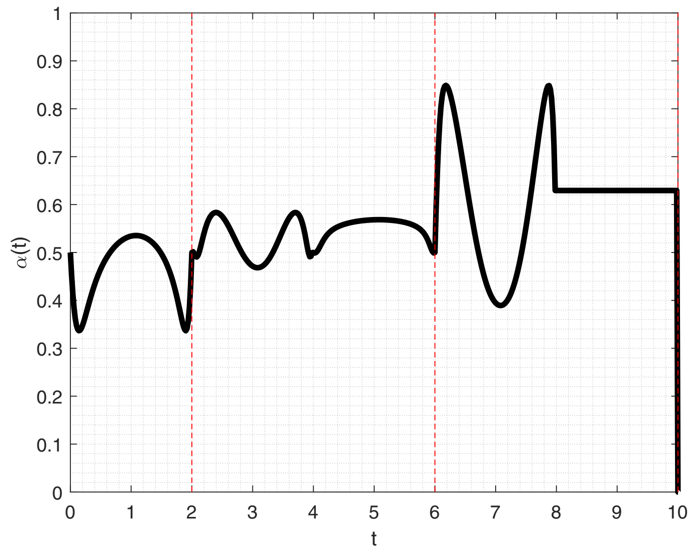

FIGURE 6. Test 3. The representation of final iteration of α, for $C_2 = 0.3$ (the 272th iteration).

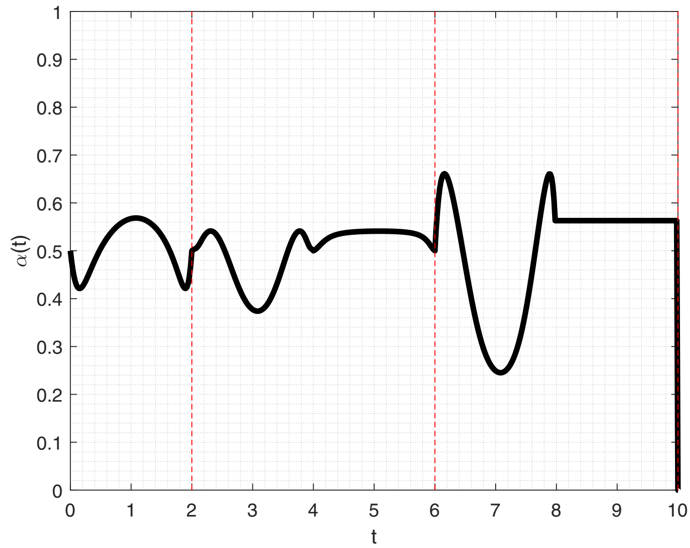

FIGURE 7. Test 3. The representation of final iteration of α, for $C_2 = 0.4$ (the 157th iteration).

We have full exploitation of the resource in the last round of harvesting, like in previous numerical test (see Fig. 4, where we indicate by dashed lines the completed rounds, backward in time).

Test 3. We take $v = \pi/2$ and $T = 10$. By increasing the value of the cost constant, *e.g.* $C_2 = 0.3$ or $C_2 = 0.4$, it can be observed that there is no more full exploitation (see Figs. 6 and 7). Since $\alpha(t) < 1$, for all $t \in [T - \theta, T]$, the final stock is positive everywhere. The distribution of the stock along the optimal path is represented in Figure 8, for $C_2 = 0.4$. For each point of the unit circle, we represent the stock of the resource at the end of the planning period; *i.e.* $f(x, T)$ for all $x \in [0, 2\pi]$. Thus, we represent the final stock in polar coordinates (r_f, ϕ), where $r_f = 1 + f(x, T)$ and $0 \leq \phi < 2\pi$.

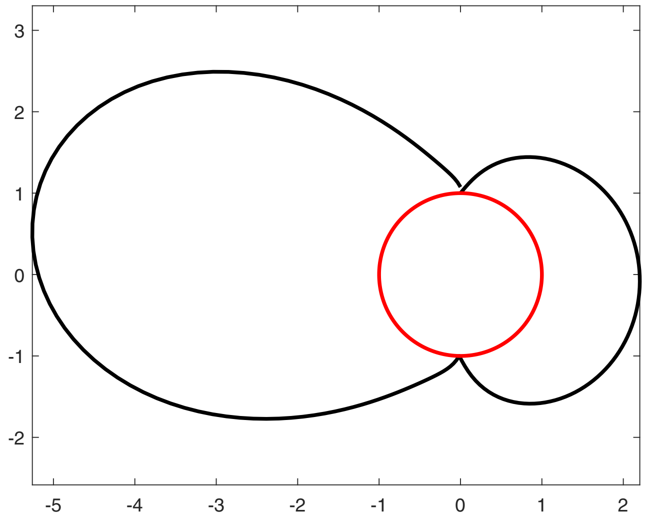

FIGURE 8. Test 3. The final resource distribution ($C_2 = 0.4$).

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REFERENCES

[1] S. Aniţa, Analysis and Control of Age-Dependent Population Dynamics. Kluwer Acad. Publ., Dordrecht (2000).
[2] L.-I. Aniţa, S. Aniţa and V. Arnăutu, Optimal harvesting for periodic age-dependent population dynamics with logistic term. *Appl. Math. Comput.* **215** (2009) 2701–2715.
[3] S. Aniţa, V. Arnăutu and V. Capasso, An Introduction to Optimal Control Problems in Life Sciences and Economics. From Mathematical Models to Numerical Simulation with Matlab. Birkhäuser, Boston (2011).
[4] S. Aniţa, V. Capasso, H. Kunze and D. La Torre, Dynamics and control of an integro-differential system of geographical economics. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **7** (2015) 8–26.
[5] V. Arnăutu and P. Neittaanmäki, Optimal Control from Theory to Computer Programs. Kluwer Acad. Publ., Dordrecht (2003).
[6] V. Barbu, Mathematical Methods in Optimization of Differential Systems. Kluwer Acad. Publ., Dordrecht (1994).
[7] S. Behringer and T. Upmann, Optimal harvesting of a spatial renewable resource. *J. Econ. Dyn. Control* **42** (2014) 105–120.
[8] A. Belyakov and V. Veliov, Constant versus periodic fishing: age structured optimal control approach. *MMNP* **9** (2014) 20–37.
[9] M. Brokate, Pontryagin's principle for control problems in age-dependent population dynamics. *J. Math. Biol.* **23** (1985) 75–101.
[10] C.W. Clark, F.H. Clarke and G.R. Munro, The optimal exploitation of renewable resource stocks: problems of irreversible investment. *Econometrica* **47** (1979) 25–47.
[11] R.T. Deacon, D.S. Brookshire, A.C. Fisher, A.V. Kneese, C.D. Kolstad, D. Scrogin, V.K. Smith, M. Ward and J. Wilen, Research trends and opportunities in environmental and natural resource economics. *Environ. Resource Econ.* **11** (1998) 383–397.
[12] N. Hritonenko and Y. Yatsenko, Optimization of harvesting age in integral age-dependent model of population dynamics. *Math. Biosci.* **195** (2005) 154–167.
[13] D. La Torre, T. Malik, O. Sharomi and R. Zaki, Dynamics and optimal control for a spatially-structured environmental-economic model. *Electron. J. Diff. Eqs.* **277** (2015) 1–15.
[14] J. Sanchirico and J. Wilen, Bioeconomics of spatial exploitation in a patchy environment. *J. Environ. Econ. Manag.* **42** (1999) 257–276.
[15] V.L. Smith, Economics of production from natural resources. *Am. Econ. Rev.* **58** (1968) 409–431.
[16] V.L. Smith, Control theory applied to natural and environmental resources, an exposition. *J. Environ. Econ. Manag.* **4** (1977) 1–24.
[17] M.I. Zelikin, L.V. Lokutsievskiy and S.V. Skopincev, On optimal harvesting of a resource on a circle. *Math. Notes* **102** (2017) 521–532.