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FRACTIONAL ADVECTION–DIFFUSION EQUATION WITH MEMORY AND ROBIN-TYPE BOUNDARY CONDITION

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Abstract. The one-dimensional fractional advection–diffusion equation with Robin-type boundary conditions is studied by using the Laplace and finite sine-cosine Fourier transforms. The mathematical model with memory is developed by employing the generalized Fick's law with time-fractional Caputo derivative. The influence of the fractional parameter (the non-local effects) on the solute concentration is studied. It is found that solute concentration can be minimized by decreasing the memory parameter. Also, it is found that, at small values of time the ordinary model leads to minimum concentration, while at large values of the time the fractional model is recommended.

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1. Introduction

Many real life problems are modeled by the advection–diffusion equation which can describe phenomena including nonlinearities and dispersion waves. For example, in aquatic systems or in atmosphere, the pollutants are distributed through diffusion and advection processes, and mathematical models are used to predict their transport. Pollution sources (diffuse or concentrated) are caused by agricultural, industrial, mining activities, etc. Mathematical modeling is an essential tool in determining of the spreading of pollutants in time and space, therefore in determining of the water/air quality.

There are many transport processes in complex systems which are controlled by non-exponential relaxation patterns or by the non-Brownian diffusion. These processes are well described by the fractional calculus. As an example, the fractional advection–diffusion equation can be used to model the transport of various quantities carried by the fluid flow in porous media. In last years, researchers have obtained some analytical or numerical solutions for the classical/fractional advection diffusion equation by considering various initial and boundary conditions.

Arshad *et al.* [1] have provided a numerical scheme to solve the time-space fractional advection–diffusion equation with spatial-fractional Riesz derivative and time-fractional Caputo derivative. The fractional advection–diffusion equation is transformed in an equivalent integral equation approximated by the trapezoidal formula. The stability and convergence of the proposed scheme have been discussed.

Keywords and phrases: Advection, diffusion, Caputo derivative, analytical solution.

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Povstenko and Kyrylychev [2] have obtained two fractional forms of the advection–diffusion equation by generalizing the classical constitutive equations of the probability current and Fick's law. For the obtained equations they have determined fundamental solutions for the Cauchy problems using Laplace and Fourier transforms.

Mohyud-Din *et al.* [3] have developed an implicit finite difference scheme to solve the time-fractional advection–diffusion equation with Caputo fractional derivatives. For spatial derivatives, authors have used the extended cubic B-splines functions.

Closed form approximate solutions to nonlinear heat/mass diffusion equation with power-law nonlinearity of the thermal/mass diffusivity have been studied by Hristov [4] using the integral-balance method and Kirchhoff transformation. Solutions to Dirichlet and Neumann boundary condition problems have been developed. Hristov [5] also has presented recent results on the approximate analytical integral-balance solutions of initial-boundary value problems to spatial-fractional diffusion equation with Riemann-Liouville derivatives.

Mojtabi and Deville [6] have studied one-dimensional linear advection–diffusion equation with Dirichlet homogeneous boundary conditions and obtained analytical and numerical solutions.

A new groundwater transport model with fractured aquifer and fractal nature exhibiting self-similarity has been developed by Allwright and Atangana [7]. Using fractal derivatives and fractal integrals, authors have studied the fractal advection–dispersion equation for the groundwater transport.

Avhale [8], by using the advection–diffusion equation has studied the concentration of nutrient entering in the aquatic root considered as a cylindrical surface. Using the separation of variables the advection–diffusion equation reduces at a Bessel equation.The non-linear boundary conditions by Michaelis-Menten type are used.

An explicit numerical algorithm based on improved spectral Galerkin method for solving the one-dimensional unsteady diffusion–convection–reaction equation was developed by Zhong *et al.* [9]. They obtained the explicit eigenvalues and eigenvectors based on the variables separation method and boundary conditions.

Povstenko and Klekot [10] have investigated the two-dimensional unsteady time-fractional advection–diffusion equation with Caputo time-fractional derivatives. They obtained fundamental solutions to the Cauhy problem using the integral transform methods. Singh *et al.* [11] formulated a numerical algorithm based upon operational matrices of integration for Jacobi polynomials and the collocation method for the approximate solution of the nonlinear Lane-Emden equations. Two interesting methods for finding approximate solutions for the fractional vibration equation have been proposed by Singh [12] and Singh *et al.* [13].These methods are based on the Jacobi polynomials respectively, on the operational matrices of the Legendre scaling function. In these methods the fractional vibration equation is converted into algebraic equation of Sylvester form which is numerically solved. A novel approximate method based on operational matrices of fractional integrations and differentiations for fractional Navier–Stokes equation in polar coordinate using Legendre scaling functions as a basis has been developed by Sing [14]. The convergence analysis, error analysis and numerical stability of the proposed method have been presented. Other interesting fractional mathematical models were studied in the references [15–18].

Frequently, the problems modeled by the partial differential equations are subjected at boundary conditions of Dirichlet or Neuman type.

The simulation of many diffusion phenomena require the solution of partial differential equations with domain subjected to Robin-type boundary conditions. As example, the solidification of multicomponent alloys, where Robin-type conditions describe the solute-rejection relations [19]. On the other hand, in some problems the evolution of the free boundary depends on the gradients of the solution, therefore, it is important to study the mathematical models in which the boundary conditions are of Robin type.

Generally, analytical solutions of such problems are difficult to obtain. Elegant numerical methods for boundary value problems with boundary conditions of Robin type have been developed by Papac *et al.* [20, 21].

A novel numerical technique for solving the time variable fractional order mobile-immobile advection–dispersion equation with the Coimbra variable time fractional derivative has been developed by Abdelkawy *et al.* [22]. The Coimbra variable time-fractional derivative is suitable for modeling of dynamical systems. Problems with Neumann and mixed boundary conditions have been investigated. Bhrawy and Baleanu [23] have applied an efficient Legendre–Gauss–Lobatto collocation (L–GL–C) method to solve the space-fractional advection diffusion equation with nonhomogeneous initial-boundary conditions. In their approach, the finding of solution for the space-fractional advection–diffusion equation is reducing to the solution of a system of ordinary differential equations. The proposed numerical method is efficient and gives accurate solutions.

The present paper aims to study the one-dimensional time-fractional advection–diffusion equation in the domain $(x,t) \in [0,1] \times [0,\infty)$ and Robin-type boundary conditions.

In order to develop the mathematical model with memory, the generalized Fick's law with time-fractional Caputo derivative is considered.

By using the Laplace transform with respect to the time variable t and the finite sine-cosine Fourier transform with the respect to the spatial variable x, the analytical solution to the fractional/ordinary advection–diffusion equation is determined.

The loading at $x = 0$ is considered in general form; therefore the obtained solution can be used for several types of external loadings.

To be able to numerical calculations for the concentration field, the positive roots of the transcendental equation $\tan x = \frac{(k_1+k_2)x}{k_1 k_2 x^2 - 1}$ needs to be known. In this paper we have obtained the positive roots of the above equation by using the subroutine "root $(\cdot)$" from the package MathCAD.

Finally, the influence of the fractional parameter α on the concentration field is analyzed in the case of the constant external loading in $x = 0$.

2. Statement of the problem

The conventional theory of a chemical species transfer is based on the local constitutive equation for the matter flux, given by [2],

$$\vec{j}(\mathbf{X},t) = -a_1 \nabla C(\mathbf{X},t) + \vec{v}(\mathbf{X},t) C(\mathbf{X},t), \tag{2.1}$$

in combination with the balanced equation of species [10],

$$\frac{\partial C(\mathbf{X},t)}{\partial t} = -\frac{\partial j_k}{\partial x_k}, \tag{2.2}$$

where $a_1[\mathrm{m}^2/2]$ is the constant diffusion coefficient, $\vec{v}$ is the fluid velocity, $C(\mathbf{X},t)$ is the concentration of the chemical species, $\vec{j}$ is the matter flux density vector and $\mathbf{X} = (x_1, x_2, x_3)$is a generic point in the space $\mathbb{R}^3$.

Using equation (2.1) into equation (2.2) we obtain the following equation

$$\frac{\partial C(\mathbf{X},t)}{\partial t} = a_1 \Delta C(\mathbf{X},t) - \vec{v} \cdot \nabla C(\mathbf{X},t), \tag{2.3}$$

which can be considered in terms of heat conduction or diffusion with additional velocity field [10].

In this paper we shall consider the one-dimensional case $j_1 = j,\ x_1 = x$ and constant drift velocity. Equations (2.1) and (2.3) become

$$j(x,t) = -a_1 \frac{\partial C(x,t)}{\partial x} + v_1 C(x,t), \tag{2.4}$$

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}, \tag{2.5}$$

$$\frac{\partial C(x,t)}{\partial t} = a_1 \frac{\partial^2 C(x,t)}{\partial x^2} - v_1 \frac{\partial C(x,t)}{\partial x}. \tag{2.6}$$

The above equations will be studied in the domain $(x,t) \in [0.L] \times [0,\infty),\ L > 0$.

In order to develop a mathematical model with memory, we consider the time non-local generalized constitutive flux equation (the generalized Fick's law) [2, 24, 25]

$$j(x,t) = {}^{C}D_t^{1-\alpha}\left(-a_\alpha \frac{\partial C(x,t)}{\partial x} + \mathrm{v}_\alpha C(x,t)\right), \tag{2.7}$$

where ${}^{C}D_t^{\alpha}(f(x,t))$ is the Caputo time-fractional derivative operator defined as [26]

$${}^{C}D_t^{\alpha}(f(x,t)) = \left(h_\alpha * \dot{f}\right)(t), \dot{f}(x,t) = \frac{\partial f(x,t)}{\partial t}, \tag{2.8}$$

where

$$h_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, 0 < \alpha < 1, \tag{2.9}$$

is the non-locality kernel.

It is observed that the Laplace transform of $h_\alpha(t)$ is

$$L\{h_\alpha(t)\} = \frac{1}{s^{1-\alpha}}, \tag{2.10}$$

therefore,

$$h_0(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1, h_1(t) = L^{-1}\{1\} = \delta(t), \tag{2.11}$$

where $\delta(t)$ is the Dirac's distribution.

Using the above properties of the non-locality kernel $h_\alpha(t)$, the definition (2.8) can be extended to the interval $\alpha \in [0,1]$, namely,

$${}^{C}D_t^{0}(f(x,t)) = (1 * \dot{f})(t) = f(x,t) - f(x,0), \tag{2.12}$$

$${}^{C}D_t^{1}(f(x,t)) = (\delta * \dot{f})(t) = \dot{f}(x,t). \tag{2.13}$$

The time-fractional Riemann-Liouville integral operator is defined as [24, 27]

$$I_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x,\tau)d\tau = (h_{1-\alpha} * f)(t), \quad 0 < \alpha < 1. \tag{2.14}$$

The following properties are easy to verify,

$${}^{C}D_t^{1-\alpha}(f(x,t)) = I_t^{\alpha}\dot{f}(x,t),\ {}^{C}D_t^{\alpha}(f(x,t)) = I_t^{1-\alpha}\dot{f}(x,t), \quad 0 \le \alpha \le 1 \tag{2.15}$$

$$\left({}^{C}D_t^{\alpha} \circ I_t^{\alpha}\right) f(x,t) = f(x,t) - f(x,0) \tag{2.16}$$

Remark. If $f(x,0)=0$, the Caputo derivative operator is invertible to the left so,

$$\left(I_t^{\alpha}\circ {}^{C}D_t^{\alpha}\right)f(x,t)=f(x,t), \tag{2.17}$$

Using equation (2.7) into the balance equation (2.5) we get the fractional equation of concentration,

$$\frac{\partial C(x,t)}{\partial t}=a_{\alpha}\,{}^{C}D_t^{1-\alpha}\frac{\partial^2 C(x,t)}{\partial x^2}-\mathrm{v}_{\alpha}\,{}^{C}D_t^{1-\alpha}\frac{\partial C(x,t)}{\partial x}, \tag{2.18}$$

where $a_{\alpha}[\frac{\mathrm{m}^2}{\mathrm{s}^{\alpha}}]$ is the generalized diffusion coefficient and $\mathrm{v}_{\alpha}[\frac{\mathrm{m}}{\mathrm{s}^{\alpha}}]$ is generalized drift velocity.

It is observed that for $\alpha=1$ the ordinary case is obtained.

Supposing that $C(x,0)=0$ and using the properties (2.15) and (2.17), equation (2.18) is written in equivalent forms

$${}^{C}I_t^{1-\alpha}\left(\frac{\partial C(x,t)}{\partial t}\right)=a_{\alpha}\frac{\partial^2 C(x,t)}{\partial x^2}-\mathrm{v}_{\alpha}\frac{\partial C(x,t)}{\partial x}, \tag{2.19}$$

$${}^{C}D_t^{\alpha}\left(C(x,t)\right)=a_{\alpha}\frac{\partial^2 C(x,t)}{\partial x^2}-\mathrm{v}_{\alpha}\frac{\partial C(x,t)}{\partial x}, \alpha\in(0,1], x\in[0,L], t\geq 0, \tag{2.20}$$

Along with the fractional differential equation (2.20) we consider the following initial condition and Robin-type boundary conditions:

$$C(x,0)=0, x\in[0,L], \tag{2.21}$$

$$C(0,t)-a\left.\frac{\partial C(x,t)}{\partial x}\right|_{x=0}=g_0 f(t), \tag{2.22}$$

$$C(L,t)+b\left.\frac{\partial C(x,t)}{\partial x}\right|_{x=L}=0. \tag{2.23}$$

Introducing the following dimensionless variables,

$$x^*=\frac{x}{L},\ \ t^*=\frac{t}{T},\ C^*=\frac{C}{C_{0,}},\ a_{\alpha}^*=\frac{a_{\alpha}T^{\alpha}}{L^2},\ \mathrm{v}_{\alpha}^*=\frac{T^{\alpha}\mathrm{v}_{\alpha}}{L},\ a^*=\frac{a}{L},\ b^*=\frac{b}{L},\ h^*(t^*)=\frac{g_0}{C_0}f(Tt^*), \tag{2.24}$$

with T the characteristic time and $C_0$the characteristic concentration and dropping the star notations, equations (2.20)–(2.23) are written as

$${}^{C}D_t^{\alpha}\left(C(x,t)\right)=a_{\alpha}\frac{\partial^2 C(x,t)}{\partial x^2}-\mathrm{v}_{\alpha}\frac{\partial C(x,t)}{\partial x},\ \alpha\in(0,1],\ x\in[0,1],\ t\geq 0, \tag{2.25}$$

$$C(x,0)=0, x\in[0,1], \tag{2.26}$$

$$C(0,t)-a\left.\frac{\partial C(x,t)}{\partial x}\right|_{x=0}=h(t), \tag{2.27}$$

$$C(1,t) + b\, \frac{\partial C(x,t)}{\partial x}\bigg|_{x=1} = 0. \tag{2.28}$$

3. Solution of the problem

To obtain the solution of the fractional differential equation (2.25) along with the initial conditions (2.26) and Robin-type boundary conditions (2.27) and (2.28), we shall use the Laplace transform with respect to time variable t, and the sine-cosine Fourier transform with respect to the spatial variable x.

Applying the Laplace transform to equations (2.25), (2.27), (2.28) and using the initial condition (2.26) we obtained the transformed problem as:

$$s^{\alpha}\bar{C}(x,s) = a_{\alpha}\frac{\partial^2 \bar{C}(x,s)}{\partial x^2} - \mathrm{v}_{\alpha}\frac{\partial \bar{C}(x,s)}{\partial x}, \tag{3.1}$$

$$\bar{C}(0,s) - a\, \frac{\partial \bar{C}(x,s)}{\partial x}\bigg|_{x=0} = \bar{h}(s), \tag{3.2}$$

$$\bar{C}(1,s) - a\, \frac{\partial \bar{C}(x,s)}{\partial x}\bigg|_{x=1} = 0. \tag{3.3}$$

Making the change of the unknown function by

$$\bar{C}(x,s) = \bar{\varphi}(x,s)\exp\left(\frac{\mathrm{v}_{\alpha}x}{2a_{\alpha}}\right), \tag{3.4}$$

we obtain the following problem for the function $\bar{\varphi}(x,s)$:

$$a_{\alpha}\frac{\partial^2 \bar{\varphi}(x,s)}{\partial x^2} = (s^{\alpha} + \gamma_{\alpha})\bar{\varphi}(x,s),\ \gamma_{\alpha} = \frac{\mathrm{v}_{\alpha}^2}{4a_{\alpha}},\ x \in [0,1], \tag{3.5}$$

$$\bar{\varphi}(0,s) - b_{\alpha}\, \frac{\partial \bar{\varphi}(x,s)}{\partial x}\bigg|_{x=0} = c_{\alpha}\bar{h}(s), \tag{3.6}$$

$$\bar{\varphi}(1,s) + d_{\alpha}\, \frac{\partial \bar{\varphi}(x,s)}{\partial x}\bigg|_{x=1} = 0, \tag{3.7}$$

where,

$$b_{\alpha} = \frac{2aa_{\alpha}}{2a_{\alpha} - a\mathrm{v}_{\alpha}}, \quad c_{\alpha} = \frac{2a_{\alpha}}{2a_{\alpha} - a\mathrm{v}_{\alpha}}, \quad d_{\alpha} = \frac{2ba_{\alpha}}{2a_{\alpha} + b\mathrm{v}_{\alpha}}. \tag{3.8}$$

To find the solution of equation (3.5), along with the Robin-type boundary conditions (3.6) and (3.7), we use the finite sine-cosine Fourier transform of the function $\bar{\varphi}(x,s)$, defined by

$$\tilde{\bar{\varphi}}(\xi_k,s) = \int_0^1 \bar{\varphi}(x,s)\left[\sin(\xi_k x) + b_{\alpha}\xi_k\cos(\xi_k x)\right]\mathrm{d}x, \tag{3.9}$$

where, $\xi_k, k = 1, 2, \dots,$ are the roots of an transcendental equation which will be later specified.

A straightforward calculus leads to,

$$I = \int_0^1 \frac{\partial^2 \bar{\varphi}(x,s)}{\partial x^2} \left[\sin(\xi_k x) + b_\alpha \xi_k \cos(\xi_k x)\right] \mathrm{d}x = -\xi_k^2 \tilde{\bar{\varphi}}(\xi_k, s) + \xi_k \left[\bar{\varphi}(0,s) - b_\alpha \left. \frac{\partial \bar{\varphi}(x,s)}{\partial x}\right|_{x=0}\right]$$

$$+\xi_k \cos\xi_k \left(b_\alpha \xi_k t_g \xi_k - 1\right) \left[\bar{\varphi}(1,s) + \frac{\sin\xi_k + b_\alpha \xi_k \cos\xi_k}{b_\alpha \xi_k^2 \sin\xi_k - \xi_k \cos\xi_k} \left.\frac{\partial \bar{\varphi}(x,s)}{\partial x}\right|_{x=1}\right].$$

Now we consider ξ_k being the root of the equation

$$\frac{\sin\xi_k + b_\alpha \xi_k \cos\xi_k}{b_\alpha \xi_k^2 \sin\xi_k - \xi_k \cos\xi_k} = d_\alpha, \tag{3.10}$$

or, equivalent with the equation

$$t_g \xi_k = \frac{(b_\alpha + d_\alpha)\xi_k}{b_\alpha d_\alpha \xi_k^2 - 1}. \tag{3.11}$$

The above integral I becomes

$$\begin{aligned} I &= \int_0^1 \frac{\partial^2 \bar{\varphi}(x,s)}{\partial x^2} \left[\sin(\xi_k x) + b_\alpha \xi_k \cos(\xi_k x)\right] \mathrm{d}x \\ &= -\xi_k^2 \tilde{\bar{\varphi}}(\xi_k, s) + \xi_k \left[\bar{\varphi}(0,s) - b_\alpha \left.\frac{\partial \bar{\varphi}(x,s)}{\partial x}\right|_{x=0}\right] + \xi_k \frac{b_\alpha^2 \xi_k^2 + 1}{b_\alpha d_\alpha \xi_k^2 - 1} \cos\xi_k \left[\bar{\varphi}(1,s) + d_\alpha \left.\frac{\partial \bar{\varphi}(x,s)}{\partial x}\right|_{x=1}\right] \end{aligned} \tag{3.12}$$

Now, by using the boundary conditions (3.6) and (3.7) we get

$$I = -\xi_k^2 \tilde{\bar{\varphi}}(\xi_k, s) + c_\alpha \bar{h}(s)\xi_k \tag{3.13}$$

and, the transformed form of equation (3.5) is

$$(s^\alpha + a_\alpha \xi_k^2 + \gamma_\alpha)\tilde{\bar{\varphi}}(\xi_k, s) = a_\alpha c_\alpha \bar{h}(s)\xi_k, \tag{3.14}$$

respectively,

$$\tilde{\bar{\varphi}}(\xi_k, s) = \frac{a_\alpha c_\alpha \xi_k}{s^\alpha + a_\alpha \xi_k^2 + \gamma_\alpha} \bar{h}(s). \tag{3.15}$$

It is easy to show that functions

$$\psi_k(x) = \sin(\xi_k x) + b_\alpha \xi_k \cos(\xi_k x), k = 1, 2, 3, \dots, \tag{3.16}$$

satisfy the following properties:

$$\int_0^1 \psi_k(x)\psi_n(x)\mathrm{d}x = 0, k \neq n, \tag{3.17}$$

$$I_{nn} = \int_0^1 \psi_n(x)\psi_n(x)\mathrm{d}x = \frac{b_\alpha^2\xi_n^2 - 1}{4\xi_n}\sin(2\xi_n) + \frac{1}{2}\left[1 + b_\alpha + b_\alpha^2\xi_n^2 - b_\alpha\cos(2\xi_n)\right]. \tag{3.18}$$

The inverse Fourier sine-cosine transform of the function $\tilde{\bar{\varphi}}(\xi_k, s)$ is given by

$$\bar{\varphi}(x, s) = \sum_{n=1}^{\infty} \frac{1}{I_{nn}} \tilde{\bar{\varphi}}(\xi_n, s)\psi_n(x). \tag{3.19}$$

Let us consider the function

$$f(x, t) = c_\alpha(1 - b_\alpha)h(t) - c_\alpha h(t)x. \tag{3.20}$$

A straightforward calculation shows that the sine-cosine Fourier transform of $f(x, t)$is given by

$$\tilde{f}(\xi_k, t) = B_k h(t), \tag{3.21}$$

where,

$$B_k = c_\alpha(1 - b_\alpha)(1 - \cos\xi_k + b_\alpha\xi_k\sin\xi_k) - \frac{c_\alpha}{\xi_k^2}\left[(b_\alpha\xi_k^2 + 1)\sin\xi_k + (b_\alpha - 1)\xi_k\cos\xi_k - b_\alpha\xi_k\right]. \tag{3.22}$$

Obviously, the inverse transform of $\tilde{f}(\xi_k, t)$is the function $f(x, t)$.
Now, we write the transform $\tilde{\bar{\varphi}}(\xi_k, s)$ given by equation (3.15) in the equivalent form

$$\begin{aligned} \tilde{\bar{\varphi}}(\xi_k, s) &= \tilde{\bar{f}}(\xi_k, s) + \frac{a_\alpha c_\alpha \xi_k}{s^\alpha + a_\alpha\xi_k^2 + \gamma_\alpha}\bar{h}(s) - \tilde{\bar{f}}(\xi_k, s) = \tilde{\bar{f}}(\xi_k, s) + \left\{\frac{a_\alpha c_\alpha \xi_k}{s^\alpha + a_\alpha\xi_k^2 + \gamma_\alpha} - B_k\right\}\bar{h}(s) \\ &= B_k\bar{h}(s) - \frac{B_k s^\alpha - a_\alpha c_\alpha\xi_k + B_k\left(a_\alpha\xi_k^2 + \gamma_\alpha\right)}{s^\alpha + a_\alpha\xi_k^2 + \gamma_\alpha}\bar{h}(s) = B_k\bar{h}(s) - \frac{B_k s^{\alpha-1}}{s^\alpha + D_k}s\bar{h}(s) + \frac{C_k s^{-1}}{s^\alpha + D_k}s\bar{h}(s) \end{aligned} \tag{3.23}$$

with,

$$C_k = a_\alpha c_\alpha\xi_k - B_k\left(a_\alpha\xi_k^2 + \gamma_\alpha\right), D_k = a_\alpha\xi_k^2 + \gamma_\alpha. \tag{3.24}$$

Using equation (3.23) into equation (3.19) and applying the inverse Laplace transform we get the solution

$$\begin{aligned} \varphi(x, t) = (1 - b_\alpha - x)\,c_\alpha h(t) &- \sum_{n=1}^{\infty}\frac{\psi_n(x)}{I_{nn}}B_n\int_0^t \dot{h}(t-\tau)E_{\alpha,1}\left(-D_n\tau^\alpha\right)\mathrm{d}\tau \\ &+ \sum_{n=1}^{\infty}\frac{\psi_n(x)}{I_{nn}}C_n\int_0^t \dot{h}(t-\tau)\tau^\alpha E_{\alpha,\alpha+1}\left(-D_n\tau^\alpha\right)\mathrm{d}\tau. \end{aligned} \tag{3.25}$$

where, $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$, $\alpha > 0,\ \beta > 0$, are the two-parameters Mittag-Leffler functions [28], with the property $L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha - d}\right\} = t^{\beta-1}E_{\alpha,\beta}\left(\mathrm{d}t^\alpha\right)$.

It is observed that, using equations (3.16) and (3.20), the function $\varphi(x, t)$satisfies the imposed boundary conditions (3.6) and (3.7).

Finally, the concentration of the chemical solute $C(x,t)$ is given by

$$C(x,t)=\varphi(x,t)\exp\left(\frac{\mathrm{v}_{\alpha}x}{2a_{\alpha}}\right). \tag{3.26}$$

4. Particular cases

In this section we will discuss some particular cases of the above studied problem.

4.1. The ordinary advection diffusion equation ($\alpha=1$)

Making $\alpha=1$ into the equation (3.25) and using the relationships

$$E_{1,1}\left(-D_n\tau\right)=\exp\left(-D_n\tau\right),\quad E_{1,2}\left(-D_n\tau\right)=\frac{1}{D_n}\left(1-e^{-D_n\tau}\right), \tag{4.1}$$

we find the simpler form of the function $\varphi(x,t)$, namely

$$\begin{aligned}\varphi(x,t)=\left(1-b_1-x\right)c_1h(t)-\sum_{n=1}^{\infty}\frac{\psi_n(x)}{I_{nn}}B_n\int_0^t\dot{h}(t-\tau)\exp\left(-D_n\tau\right)d\tau\\+\sum_{n=1}^{\infty}\frac{\psi_n(x)}{I_{nn}}\frac{C_n}{D_n}\int_0^t\dot{h}(t-\tau)\tau\left[1-\exp(-D_n\tau)\right]d\tau.\end{aligned} \tag{4.2}$$

4.2. The constant loading in $x=0$ ($h(t)$−the Heaviside's function)

In this case we consider $h(t)=H(t)=\frac{1}{2}sign(t)\left(1+sign(t)\right)$ – the unit step Heaviside's function.

Using the property, $\dot{h}(t)=\delta(t)$, where $\delta(t)$ is the Dirac's distribution, equations (3.25) and (4.2) become

$$\varphi(x,t)=\left(1-b_{\alpha}-x\right)c_{\alpha}H(t)-\sum_{n=1}^{\infty}\frac{B_n\psi_n(x)}{I_{nn}}E_{\alpha,1}\left(-D_nt^{\alpha}\right)+\sum_{n=1}^{\infty}\frac{C_n\psi_n(x)}{I_{nn}}t^{\alpha}E_{\alpha,\alpha+1}\left(-D_nt^{\alpha}\right),0<\alpha<1, \tag{4.3}$$

respectively,

$$\varphi(x,t)=\left(1-b_1-x\right)c_1H(t)-\sum_{n=1}^{\infty}\frac{B_n\psi_n(x)}{I_{nn}}\exp\left(-D_nt\right)+\sum_{n=1}^{\infty}\frac{\psi_n(x)}{I_{nn}}\frac{C_n}{D_n}t\left[1-\exp\left(-D_nt\right)\right],\alpha=1 \tag{4.4}$$

4.3. Concentration shock at the initial moment ($h(t)=\delta(t)$ – the Dirac's distribution)

Using $\bar{h}(s)=L\left\{\delta(t)\right\}=1$, equation (3.15) becomes

$$\tilde{\bar{\varphi}}(\xi_k,s)=\frac{a_{\alpha}c_{\alpha}\xi_k}{s^{\alpha}+D_k}, \tag{4.5}$$

with the inverse transforms

$$\phi(x,t)=\sum_{n=1}^{\infty}\frac{a_{\alpha}c_{\alpha}\xi_n}{I_{nn}}t^{\alpha-1}E_{\alpha,\alpha}(-D_nt^{\alpha}),\ 0<\alpha<1. \tag{4.6}$$

For $\alpha = 1$, equation (4.6) becomes

$$\phi(x,t) = \sum_{n=1}^{\infty} \frac{a_1 c_1 \xi_n}{I_{nn}} \exp(-D_n t). \tag{4.7}$$

5. Results and discussions

In this paper, the one-dimensional time-fractional advection–diffusion equation with Robin-type boundary conditions has been studied. In order to develop a mathematical model suitable to describe the memory effects, we considered a generalization of the Fick's law for the mass flux. The time-fractional advection–diffusion equation is studied in the domain $(x,t) \in 0,1] \times [0,\infty)$and the analytical solution for the concentration was obtained by employing the Laplace transform with respect to the time variable t and the finite sine-cosine Fourier transform with the respect to the spatial variable x. Based on the properties of the Caputo time-fractional derivative operator, the solution for the classical advection–diffusion equation has been obtained as a limiting case of the general solution corresponding to the fractional advection–diffusion equation.

To carrying out the numerical calculations for the concentration field it is necessary to know the positive roots of the transcendental equation

$$\tan x = \frac{(k_1 + k_2)x}{k_1 k_2 x^2 - 1}. \tag{5.1}$$

It is easy to observe that the above equation has a uniques root in each interval $((k-1)\pi, k\pi),\ k = 1, 2, 3, \ldots$ In order to find the roots of the transcendental equation (5.1) we have used the subroutine $\mathrm{root}(f(x), x, a, b)$from the package MathCAD.

To analyze the influence of the fractional parameter α on the concentration field and to compare the solution from the fractional case with the solution corresponding to classical advection–diffusion process ($\alpha = 1$), numerical calculations were carried out and results were plotted in Figures 1–3. At the boundary $x = 0$, we have considered the constant loading $h(t) = 2H(t)$.

In Figure 1 are plotted profiles of the non-locality kernel $h_{1-\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, with respect to the fractional parameter $\alpha \in (0,1]$ for three values of the time t, $t \in \{0.15,\ 0.50,\ 0.75\}$. It is observed form Figure 1 that values of the non-locality kernel decrease with respect to the time t, therefore, the damping of the mass flux will changes with the fractional parameter α and with the time t. As a consequence of this behavior of the non-locality kernel results the changes of the solute concentration with the fractional parameter α and the time t.

The influence of the fractional parameter on the solute concentration is analyzed by graphs from Figures 2 and 3.

In Figure 2 are sketched diagrams of the concentration $C(x,t)$, versus variable t, for three values of the spatial coordinate x. The concentration $C(x,t)$ has different behaviours for small values of the time t, respectively for large values of the time t. At small values of the time t the values of concentration decrease with the fractional parameter α. This behaviour is due to the weight function $h_{1-\alpha}(t)$ which increases if the fractional parameter decreases, $\alpha < 0.5$ (see Fig. 1).

For large values of the time t the damping effect of the non-locality kernel decreases, therefore the solute concentration increases with fractional parameter. For large values of the time t, the concentration tends to an asymptotic values $\phi_0(x)$ defined below. This behaviour is in accordance with the following property of function $\phi(x,t)$:

$$\lim_{t\to\infty} \phi(x,t) = \lim_{s\to 0} s \sum_{n=1}^{\infty} \frac{\psi_n(x)}{I_{nn}} \tilde{\bar{\varphi}}(\xi_n, s). \tag{5.2}$$

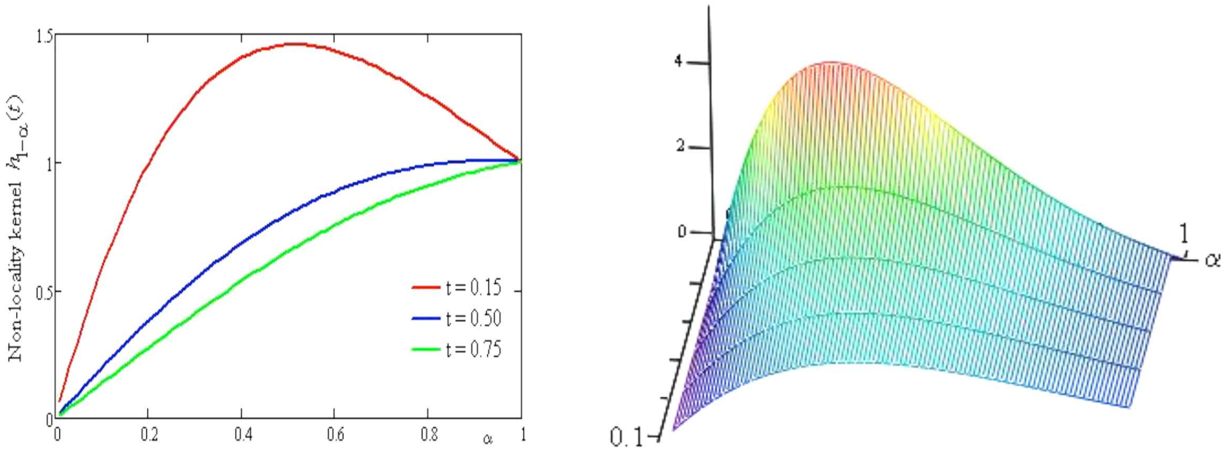

FIGURE 1. Variation of the non-locality kernel with the fractional parameter α.

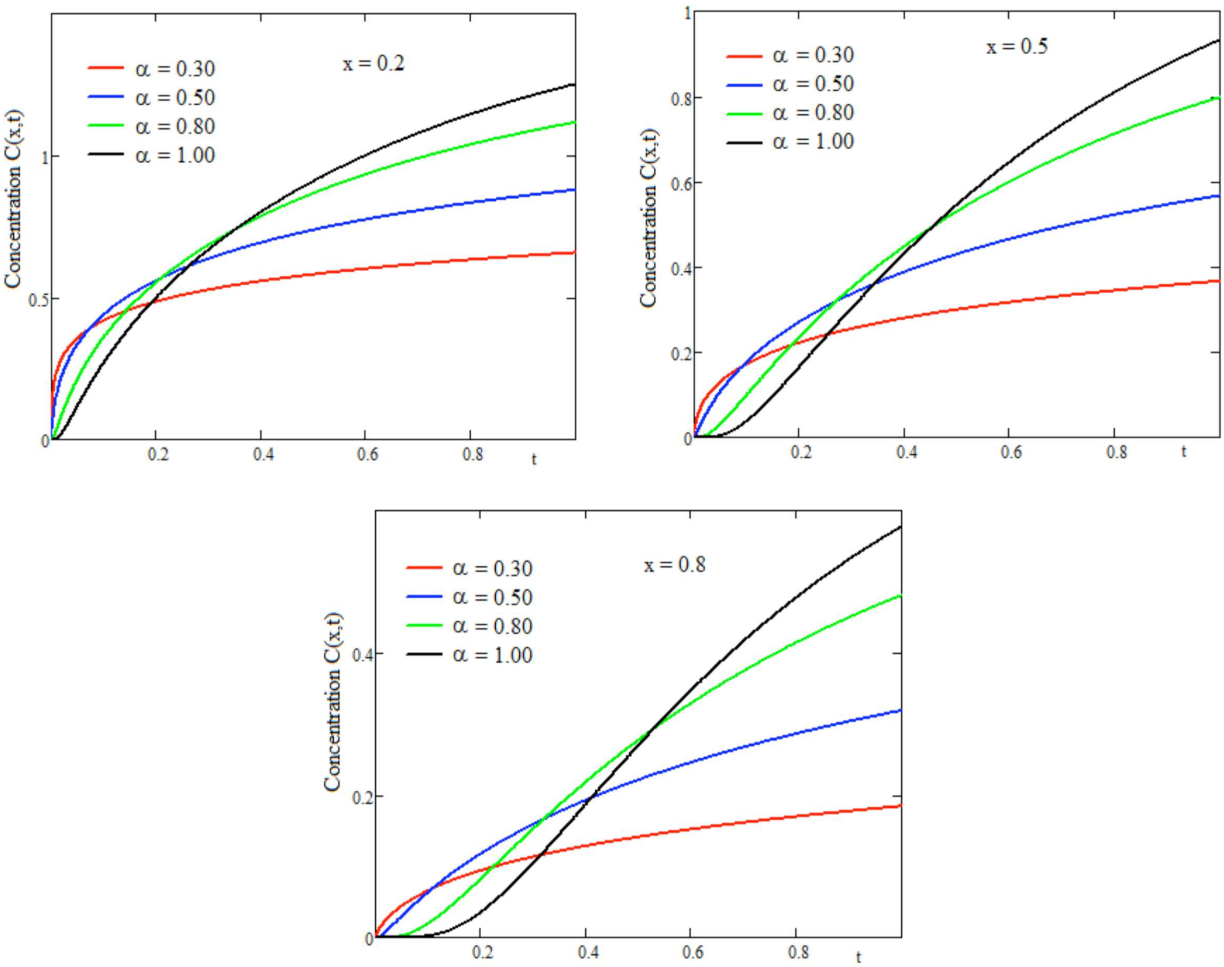

FIGURE 2. Diagrams of concentration $C(x,t)$ for different values of the fractional parameter α.

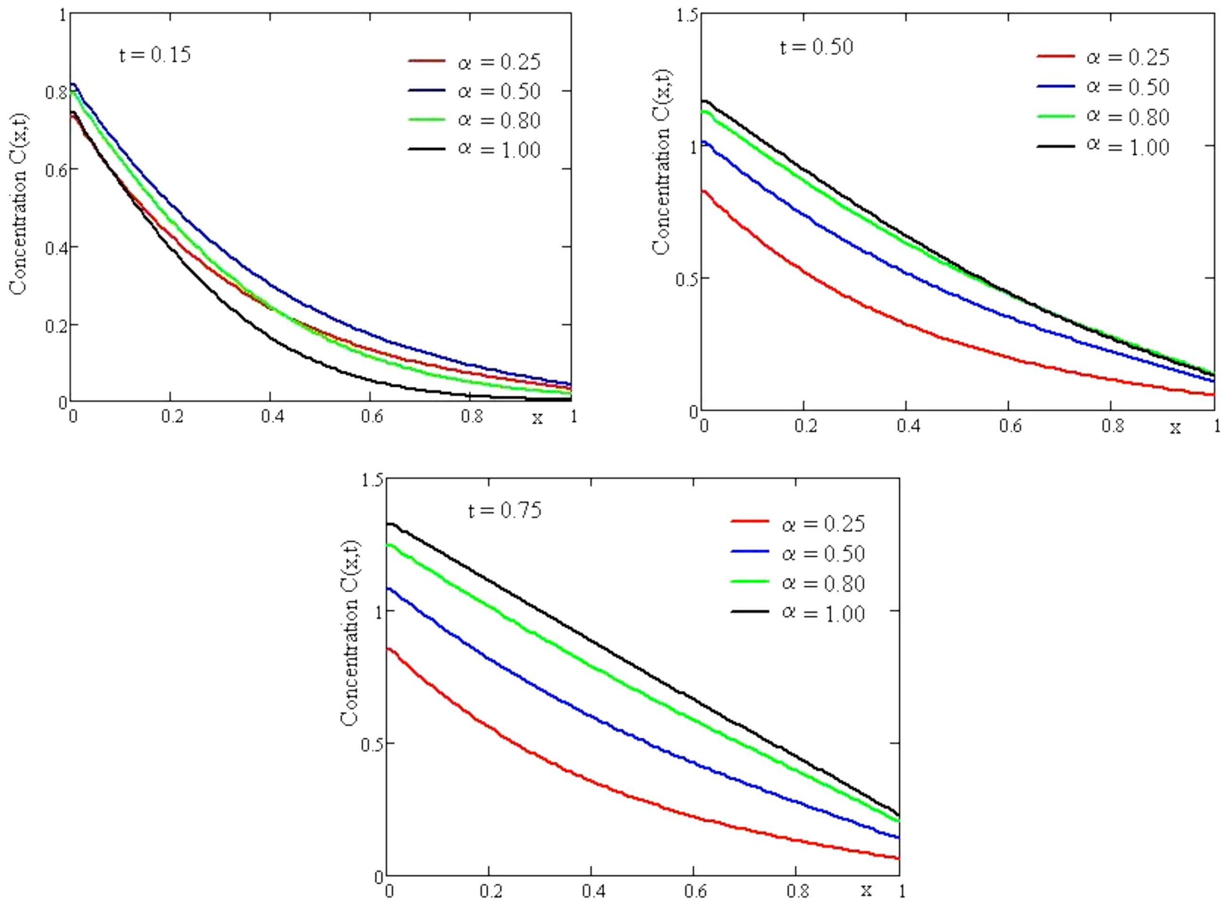

FIGURE 3. Spatial variation of the concentration $C(x,t)$ for different values of the fractional parameter α.

Using equation (3.15) we have

$$\lim_{t\to\infty}\phi(x,t)=\sum_{n=1}^{\infty}\frac{1}{I_{nn}}\lim_{s\to 0}\frac{a_\alpha c_\alpha \xi_n \psi_n(x)s}{s^\alpha+a_\alpha\xi_n^2+\lambda_\alpha}\frac{2}{s}=\sum_{n=1}^{\infty}\frac{2}{I_{nn}}\frac{a_\alpha c_\alpha\xi_n}{a_\alpha\xi_n^2+\lambda_\alpha}\psi_n(x)=\phi_0(x). \tag{5.3}$$

Variation of the concentration $C(x,t)$ in the spatial domain $x\in[0,1]$is illustrated in Figure 3. It is observed that the concentration $C(x,t)$ attains a maximum value close to the boundary $x=0$ and decreases with respect to the x variable.

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