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# CHARACTERIZATIONS OF TWO DIFFERENT FRACTIONAL OPERATORS WITHOUT SINGULAR KERNEL

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**Abstract.** In this paper, we analyze the behaviours of two different fractional derivative operators defined in the last decade. One of them is defined with the normalized sinc function (NSF) and the other one is defined with the Mittag-Leffler function (MLF). Both of them have a non-singular kernel. The fractional derivative operator defined with the MLF is developed by Atangana and Baleanu (ABO) in 2016 and the other operator defined with the normalized sinc function (NSFDO) is created by Yang *et al.* in 2017. These mentioned operators have some advantages to model the real life problems and to solve them. On the other hand, since the Laplace transform (LT) of the ABO can be calculated more easily, it can be preferred to solve linear/nonlinear problems. In this study, we use the perturbation method with coupled the LTs of these operators to analyze their performance in solving some fractional differential equations. Furthermore, by constructing the error analysis, we test the practicability and usefulness of the method.

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## 1. Introduction

Modelling real-life problems in the nature is very important to clarify and to reduce problems in our life. For this reason, many scientists have attached it to the importance of mathematical modelling. Because the problems in our life have some different variables, modelling them with partial differential equations (PDEs) is more important. Moreover, the multiplicity of problems which have memory effect in the nature has increased the importance of fractional calculus. Especially, in the last decade, some important fractional derivative operators have been developed. The spread of different types of problems has necessitated a multiplication of the fractional derivative operator. Because it is not possible for a derivative operator to solve all problems in the literature, new operators have been defined. In this context, we have mentioned two different types of fractional derivative operators in this study.

Fractional calculus has been used to model the complex real-life problems. It plays a very significant role in the area of engineering, chemistry, material science, physics, finance etc. In the branch of fractional calculus, fractional derivatives and fractional integrals are important aspects. Nowadays, many researchers and scientists study in this special branch [1–5].

*Keywords and phrases:* Atangana-Baleanu fractional derivative, normalized sinc function, Laplace perturbation method, Laplace transform, nonlinear equation.

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Atangana and Baleanu [6] used the MLF to define an operator (ABO) in 2016. The other definition, which is defined by Yang *et al.* [7] in 2017, is based on the normalized sinc function (NSFDO). Especially in recent years, many important theoretical results and applications have been obtained using an operator which has non-local and non-singular kernel such as the Caputo-Fabrizio, NSFDO and ABO [8–18]. Both of these operators have some illustrative advantages according to the operators defined previously such as Caputo [19] and conformable [20]. In the NSFDO and ABO, the integrals have been regarded as fractional. Since NSFDO has been developed later than the ABO, there are only a few studies on this operator. Nevertheless, after these definitions, many physical, mathematical, chemical, biological problems have been modelled and solved by using these suggested operators [21–26].

On the other hand, some special analytical and numerical approximation methods for fractional PDEs have been developed by using these mentioned operators. For instance, Laplace homotopy transform method (LHTM) [29], homotopy analysis transform method (HATM) [27, 28], an extended transport model [30], etc.

In this paper, the mentioned LHTM for numerical-approximate solutions of FPDEs is considered. In order to show the efficiency and accurateness of the method, it is applied to the Cauchy and Burgers' equations. When looking at the results, it can be seen clearly that the suggested method defined by NSFDO and ABO is very influential and infallible for solving FPDEs. However, computing the recurrence relations with the ABO is easier than the NSFDO.

## 2. Some basic definitions

**Definition 2.1.** The normalized sinc function (NSF) is defined as follows [31]:

$$sinc\left(t\right)=\frac{\sin\left(\pi t\right)}{\pi t},\qquad t\in R. \tag{2.1}$$

**Definition 2.2.** Let $\xi\left(t\right)$ be in $H^{1}\left(a,b\right),\ \ b>a$. A new fractional derivative operator which is defined with the normalized sinc function (NSFDO) of the function $\xi\left(t\right)$ of order $\alpha$ is defined by the following [7]:

$${}_{a}^{NSF}D_{t}^{\alpha}\xi\left(t\right)=\frac{\alpha\varpi\left(\alpha\right)}{1-\alpha}\int_{a}^{t}sinc\left(-\frac{\alpha\left(t-\varepsilon\right)}{1-\alpha}\right)\xi'\left(\varepsilon\right)\mathrm{d}\varepsilon,\ \ a\in\left(-\infty,t\right),\ \ 0<\alpha<1, \tag{2.2}$$

where $\varpi\left(\alpha\right)$ is a normalization function such that $\varpi\left(0\right)=\varpi\left(1\right)=1$.

**Definition 2.3.** The LT of the NSFDO is defined as [7]

$$\begin{aligned}\mathcal{L}\left\{{}_{0}^{NSF}D_{t}^{\alpha}\xi\left(t\right)\right\}&=\mathcal{L}\left\{\tfrac{\alpha\varpi(\alpha)}{1-\alpha}\int_{0}^{t}sinc\left(-\tfrac{\alpha(t-\varepsilon)}{1-\alpha}\right)\xi'\left(\varepsilon\right)\mathrm{d}\varepsilon\right\}\\&=\tfrac{\varpi(\alpha)}{\pi}\arctan\left(\tfrac{\alpha\pi}{s(1-\alpha)}\right)\left(s\xi\left(s\right)-\xi\left(0\right)\right),\end{aligned} \tag{2.3}$$

where $\mathcal{L}\left\{\xi\left(t\right)\right\}=\xi\left(s\right)$.

**Definition 2.4.** The Atangana-Baleanu (AB) operator in the sense of Caputo is given as follows [6]:

$${}_{0}^{ABC}D_{t}^{\alpha}\xi\left(t\right)=\frac{\varpi\left(\alpha\right)}{1-\alpha}\int_{0}^{t}\xi'\left(\tau\right)E_{\alpha}\left[-\frac{\alpha\left(t-\tau\right)^{\alpha}}{1-\alpha}\right]\mathrm{d}\tau, \tag{2.4}$$

where $\xi\in H^{1}\left(a,b\right),\ \ b>a,\ \ \alpha\in\left[0,1\right]$.

**Definition 2.5.** The LT of the ABC ${}_{0}^{ABC}D_{t}^{\alpha}\xi\left(t\right)$ is given by the following [6]

$$\mathcal{L}\left\{{}_{0}^{ABC}D_{t}^{\alpha}\xi\left(t\right)\right\}\left(s\right)=\frac{\varpi\left(\alpha\right)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\left\{\xi\left(t\right)\right\}\left(s\right)-s^{\alpha-1}\xi\left(0\right)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.5}$$

## 3. A recursive approximation with the new fractional operators

Let us take the following fractional nonlinear PDE [32, 33]:

$$
{}_{0}^{*}D_{t}^{(\alpha+\mu)}\xi\left(x,t\right)+\sigma\left[\xi\left(x,t\right)\right]+\rho\left[\xi\left(x,t\right)\right]=\theta\left(x,t\right),\quad \left(x,t\right)\in\left[0,1\right]\times\left[0,T\right],\quad \kappa-1<\alpha+\mu\leq\kappa, \tag{3.1}
$$

with initial conditions

$$
\frac{\partial^{z}\xi}{\partial t^{z}}\left(x,0\right)=f_{z}\left(x\right),\qquad z=0,1,...,\kappa-1, \tag{3.2}
$$

and the boundary conditions

$$
\xi\left(0,t\right)=\gamma_{0}\left(t\right),\qquad \xi\left(1,t\right)=\gamma_{1}\left(t\right),\qquad t\geq 0, \tag{3.3}
$$

where $f_z$, $\theta$, $\gamma_0$ and $\gamma_1$ are known functions and T $>0$ is a real number. We represent the linear part of the equation with $\sigma\left[.\right]$ and the nonlinear part with $\rho\left[.\right].$ In equation (3.1), ${}_{0}^{*}D_{t}^{(\alpha+\mu)}$ shows the NSFDO or the ABO fractional derivatives. We define the recursive approximations for solving problems (3.1)–(3.3). Using the LTs of the NSFDO (2.3) and ABO (2.5) we define the $\mathcal{L}\left\{\xi\left(x,t\right)\right\}\left(s\right)=\tilde{\psi}\left(x,s\right)$ for equation (3.1). Then we can write the homotopies for the NSFDO fractional derivative

$$
\begin{aligned}
\tilde{\psi}\left(x,s\right)=&-\left(\frac{\pi}{s\varpi\left(\alpha\right)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\xi\left(x,t\right)\right]+\rho\left[\xi\left(x,t\right)\right]\right\}\\
&+\frac{\pi}{s\varpi\left(\alpha\right)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\tilde{\theta}\left(x,s\right)+\frac{\xi\left(x,0\right)}{s}.
\end{aligned} \tag{3.4}
$$

In addition, we get the homotopies for the ABO

$$
\begin{aligned}
\tilde{\psi}\left(x,s\right)=&-\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{\sigma\left[\xi\left(x,t\right)\right]+\rho\left[\xi\left(x,t\right)\right]\right\}\\
&+\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\tilde{\theta}\left(x,s\right)+\tfrac{1}{s^{\alpha}}\left[s^{\alpha-1}\xi_{0}\left(x\right)\right],
\end{aligned} \tag{3.5}
$$

where $\tilde{\psi}\left(x,s\right)=\mathcal{L}\left\{\xi\left(x,t\right)\right\}$ and $\tilde{\theta}\left(x,s\right)=\mathcal{L}\left\{\theta\left(x,t\right)\right\}.$ Also, taking the LTs of the boundary conditions we get the following:

$$
\tilde{\psi}\left(0,s\right)=\mathcal{L}\left\{\gamma_{0}\left(t\right)\right\},\quad \tilde{\psi}\left(1,s\right)=\mathcal{L}\left\{\gamma_{1}\left(t\right)\right\},\quad s\geq 0. \tag{3.6}
$$

After that, applying the perturbation method we have the solution of equations (3.4) and (3.5):

$$
\tilde{\psi}\left(x,s\right)=\sum_{u=0}^{\infty}\lambda^{u}\tilde{\psi}_{u}\left(x,s\right),\qquad u=0,1,2,\ldots \tag{3.7}
$$

The nonlinear part in equation (3.1) can be evaluated from

$$
\rho\left[\xi\left(x,t\right)\right]=\sum_{u=0}^{\infty}\lambda^{u}\mathcal{N}_{u}\left(x,t\right), \tag{3.8}
$$

the polynomials $\mathcal{N}_u(x,t)$ are given in [34] as

$$\mathcal{N}_u(\xi_0,\xi_1,...,\xi_u)=\frac{1}{u!}\frac{\partial^u}{\partial\lambda^u}\left[\rho\left(\sum_{i=0}^{\infty}\lambda^i\xi_i\right)\right]_{\lambda=0},\quad u=0,1,2,... \tag{3.9}$$

Substituting equations (3.7) and (3.8) into equation (3.4), we get the recursive relation which gives the solution for the NSFDO:

$$\begin{aligned}\textstyle\sum_{u=0}^{\infty}\lambda^u\tilde{\psi}_u(x,s)=-\lambda&\left[\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\textstyle\sum_{u=0}^{\infty}\lambda^u\tilde{\psi}_u(x,t)\right]+\textstyle\sum_{u=0}^{\infty}\lambda^u\mathcal{N}_u(x,t)\right\}\right]\\&+\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\tilde{\theta}(x,s)+\frac{\xi(x,0)}{s},\end{aligned} \tag{3.10}$$

and substituting the equations (3.7) and (3.8) into equation (3.5), we get the recursive relation which gives the solution for the ABO:

$$\begin{aligned}\textstyle\sum_{u=0}^{\infty}\lambda^u\tilde{\psi}_u(x,s)=-\lambda&\left[\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\mathcal{L}\left\{\sigma\left[\textstyle\sum_{u=0}^{\infty}\lambda^u\tilde{\psi}_u(x,t)\right]+\textstyle\sum_{u=0}^{\infty}\lambda^u\mathcal{N}_u(x,t)\right\}\right]\\&+\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\tilde{\theta}(x,s)+\frac{1}{s^\alpha}\left[s^{\alpha-1}\xi_0(x)\right].\end{aligned} \tag{3.11}$$

Then, by solving equations (3.10) and (3.11) with respect to $\lambda$, we have the following NSFDO homotopies:

$$\lambda^0:\tilde{\psi}_0(x,s)=\frac{\xi(x,0)}{s}+\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\tilde{\theta}(x,s), \tag{3.12}$$

$$\lambda^1:\tilde{\psi}_1(x,s)=-\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\xi_0(x,t)\right]+\rho\left[\xi_0(x,t)\right]\right\},$$

$$\lambda^2:\tilde{\psi}_2(x,s)=-\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\xi_1(x,t)\right]+\rho\left[\xi_1(x,t)\right]\right\},$$

$$\vdots$$

$$\lambda^{n+1}:\tilde{\psi}_{n+1}(x,s)=-\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\xi_n(x,t)\right]+\rho\left[\xi_n(x,t)\right]\right\}.$$

Moreover, we define the following ABO homotopies:

$$\lambda^0:\tilde{\psi}_0(x,s)=\frac{1}{s^\alpha}\left[s^{\alpha-1}\xi_0(x)\right]+\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\tilde{\theta}(x,s), \tag{3.13}$$

$$\lambda^1:\tilde{\psi}_1(x,s)=-\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\mathcal{L}\left\{\sigma\left[\xi_0(x,t)\right]+\rho\left[\xi_0(x,t)\right]\right\},$$

$$\lambda^2 : \tilde{\psi}_2(x,s) = -\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\mathcal{L}\left\{\sigma\left[\xi_1(x,t)\right] + \rho\left[\xi_1(x,t)\right]\right\},$$

$$\vdots$$

$$\lambda^{n+1} : \tilde{\psi}_{n+1}(x,s) = -\left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\mathcal{L}\left\{\sigma\left[\xi_n(x,t)\right] + \rho\left[\xi_n(x,t)\right]\right\}.$$

When $\lambda \to 1$, we obtain that equations (3.12) and (3.13) show the approximate solution for problems (3.10), (3.11) and the solution is

$$\Re_n(x,s) = \sum_{p=0}^{n} \tilde{\psi}_p(x,s). \tag{3.14}$$

By applying the inverse LT of equation (3.14), we obtain the approximate solution of equation (3.1),

$$\xi_{approx}(x,t) \cong \xi_n(x,t) = \mathcal{L}^{-1}\left\{\Re_n(x,s)\right\}. \tag{3.15}$$

Furthermore, we test the stability of the recursive approximation we defined in Section 3 by applying it to some illustrative linear/nonlinear fractional problems. Considering the description in equation (3.14) we determine the measure of absolute inaccuracy ($MAI$) as

$$MAI = \left|\Re_n(x,t) - \xi_{exact}(x,t)\right|. \tag{3.16}$$

## 4. Applications of the mentioned recursive approximations

In this subsection of the paper, we show the efficiency of the recursive approximations defined in the previous section via the NSFDO and ABO fractional derivatives.

### 4.1. A new recursive approximation for fractional Cauchy problem

We consider the following homogeneous time-fractional Cauchy problem [35]:

$$\frac{\partial^\alpha \xi}{\partial t^\alpha} + x\frac{\partial \xi}{\partial x} = 0, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \tag{4.1}$$

with the initial condition

$$\xi(x,0) = x^2. \tag{4.2}$$

Firstly, we apply the solution method in the sense of fractional operator defined with the normalized sinc function to equation (4.1) with the initial condition (4.2). We get the following homotopies:

$$\lambda^0 : \tilde{\psi}_0(x,s) = \frac{\xi(x,0)}{s} + \frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\tilde{\theta}(x,s) = \frac{x^2}{s}, \tag{4.3}$$

$$\lambda^1 : \tilde{\psi}_1(x,s) = -\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\left(x\frac{\partial\tilde{\psi}_0(x,s)}{\partial x}\right) = \frac{-2x^2\pi}{s^2\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)},$$

$$\lambda^2 : \tilde{\psi}_2(x,s) = -\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\left(x\frac{\partial\tilde{\psi}_1(x,s)}{\partial x}\right) = \frac{x^2}{s}\left(\frac{-2\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)^2,$$

$$\vdots$$

$$\lambda^{n+1} : \tilde{\psi}_{n+1}(x,s) = -\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\left(x\frac{\partial\tilde{\psi}_n(x,s)}{\partial x}\right) = \frac{x^2}{s}\left(\frac{-2\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)^n,$$

From the last iterations, we have

$$\begin{aligned}\Re_n(x,s) &= \sum_{p=0}^{n}\tilde{\psi}_p(x,s) = \frac{x^2}{s} - \frac{2x^2\pi}{s^2\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)} + \cdots + \frac{x^2}{s}\left(\frac{-2\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)^n \\ &= \sum_{p=0}^{n}\frac{x^2}{s}\left(\frac{-2\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)^p.\end{aligned} \tag{4.4}$$

Taking the inverse LT of equation (4.4), the approximate solution of equation (4.1) is presented as follows:

$$\begin{aligned}\xi(x,t) &\cong \xi_n(x,t) = \mathcal{L}^{-1}\{\Re_n(x,s)\} \\ &= \frac{x^2}{2}e^{(\mathrm{A}+\mathrm{B})t}\left(1 - \frac{\pi\sqrt{\alpha\varpi(\alpha)}}{\sqrt{-32\alpha+\pi^2\alpha\varpi(\alpha)+32}}\right) + \frac{x^2}{2}e^{(\mathrm{A}-\mathrm{B})t}\left(1 + \frac{\pi\sqrt{\alpha\varpi(\alpha)}}{\sqrt{-32\alpha+\pi^2\alpha\varpi(\alpha)+32}}\right) + \cdots,\end{aligned} \tag{4.5}$$

where $\mathrm{A} = \frac{\pi^2\alpha}{4(1-\alpha)}$ and $\mathrm{B} = \frac{\pi\sqrt{\frac{\alpha}{\varpi(\alpha)}}\sqrt{-32\alpha+\pi^2\alpha\varpi(\alpha)+32}}{4(1-\alpha)}$.

For the special case $\alpha \to 1$, the exact solution of the Cauchy equation is presented by $\xi(x,t) = \lim_{n\to\infty}\Re_n(x,t) = x^2e^{-2t}$.

The numerical computation of equation (4.1) for the special cases $\alpha = 0.5$ and $\alpha = 0.75$ is shown in Figure 1.

Secondly, we solve the problem (4.1) and (4.2) by using the Laplace perturbation method in the ABO sense constructed by the Atangana-Baleanu operator. Taking into consideration that the equation is homogeneous, that is, $\tilde{\theta}(x,s) = \mathcal{L}\{\theta(x,t)\} = 0$, we evaluate the homotopies as follows:

$$\lambda^0 : \tilde{\psi}_0(x,s) = \frac{\xi(x,0)}{s} + \left(\frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha}\right)\tilde{\theta}(x,s) = \frac{x^2}{s}, \tag{4.6}$$

$$\lambda^1 : \tilde{\psi}_1(x,s) = -\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left(x\frac{\partial\tilde{\psi}_0(x,s)}{\partial x}\right) = -2x^2\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^{\alpha+1}}\right),$$

$$\vdots$$

$$\lambda^n : \tilde{\psi}_n(x,s) = -\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^\alpha}\right)\left(x\frac{\partial\tilde{\psi}_{n-1}(x,s)}{\partial x}\right) = x^2(-2)^n\frac{((1-\alpha)s^\alpha+\alpha)^n}{s^{n\alpha+1}}.$$

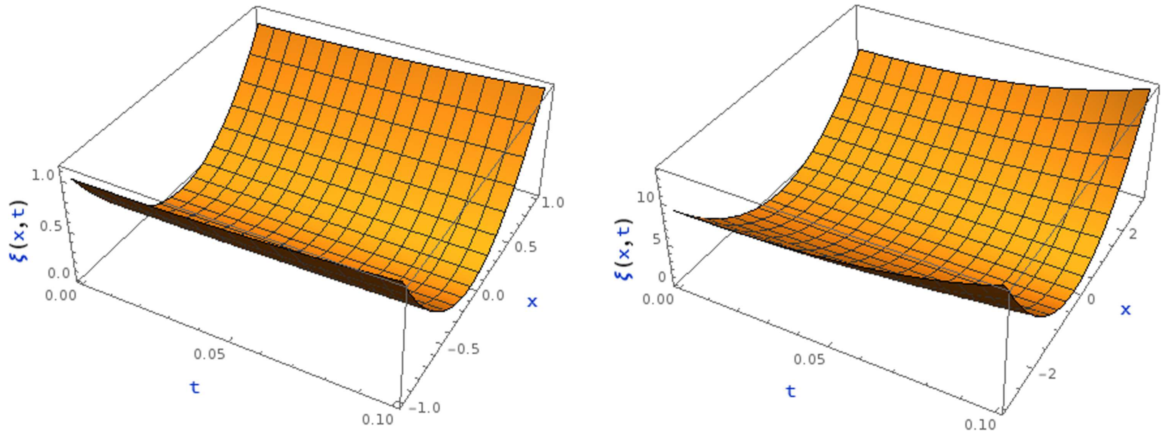


FIGURE 1. The solution function of (4.1) in the NSFDO sense for $\alpha = 0.5$ (*left*) and $\alpha = 0.75$ (*right*).

Considering equation (3.14) and by building the $n$th order approximate solution, we have

$$\begin{aligned} \Re_n(x,s) &= \textstyle\sum_{p=0}^{n} \tilde{\psi}_p(x,s) = \frac{x^2}{s} - 2x^2\left(\frac{(1-\alpha)s^\alpha+\alpha}{s^{\alpha+1}}\right) + \cdots + x^2(-2)^n \frac{((1-\alpha)s^\alpha+\alpha)^n}{s^{n\alpha+1}} \\ &= x^2 \textstyle\sum_{p=0}^{n} (-2)^p \frac{((1-\alpha)s^\alpha+\alpha)^p}{s^{p\alpha+1}}. \end{aligned} \tag{4.7}$$

Applying the inverse LT to equation (4.7) and taking into account equation (3.15), we get the approximate solution of (4.1) and (4.2) as follows:

$$\begin{aligned} \xi(x,t) &\cong \xi_n(x,t) = \mathcal{L}^{-1}\{\Re_n(x,s)\} \\ &= x^2\left(2\alpha - 1 - \tfrac{2\alpha t^\alpha}{\Gamma(\alpha+1)} + \tfrac{(2\alpha t^\alpha)^2}{\Gamma(2\alpha+1)} - \tfrac{(2\alpha t^\alpha)^3}{\Gamma(3\alpha+1)} + \tfrac{(2\alpha t^\alpha)^4}{\Gamma(4\alpha+1)} + \cdots\right) \\ &\quad + x^2\left(-4\tfrac{2\alpha t^\alpha}{\Gamma(\alpha+1)} + 6\tfrac{(2\alpha t^\alpha)^2}{\Gamma(2\alpha+1)} - \tfrac{8(2\alpha t^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots\right)(\alpha-1) \\ &\quad + x^2\left(4 - 12\tfrac{2\alpha t^\alpha}{\Gamma(\alpha+1)} + 24\tfrac{(2\alpha t^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots\right)(\alpha-1)^2 + x^2\left(8 - 32\tfrac{2\alpha t^\alpha}{\Gamma(\alpha+1)} + \cdots\right)(\alpha-1)^3 \\ &\quad + x^2(16+\cdots)(\alpha-1)^4 + \cdots. \end{aligned} \tag{4.8}$$

For the special case $\alpha = 1$, we have the exact solution of the problem as $u(x,t) = x^2 e^{-2t}$ which is the same result as that obtained by using the NSFD operator. The numerical evaluation of equation (4.8) is shown in Figure 2 for the ABO sense. In addition, the numerical simulation of solution function (4.8) for different distance values is represented in Figure 3.

### 4.2. A new recursive approximation for fractional Burgers' equation

In this subsection, we consider the nonlinear modified KdV (mKdV) equation [36]

$$\frac{\partial^\alpha \xi}{\partial t^\alpha} + \frac{1}{2}\frac{\partial \xi^2}{\partial x} - \frac{\partial^2 \xi}{\partial x^2} = 0, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \tag{4.9}$$

subject to the initial condition

$$\xi(x,0) = x. \tag{4.10}$$

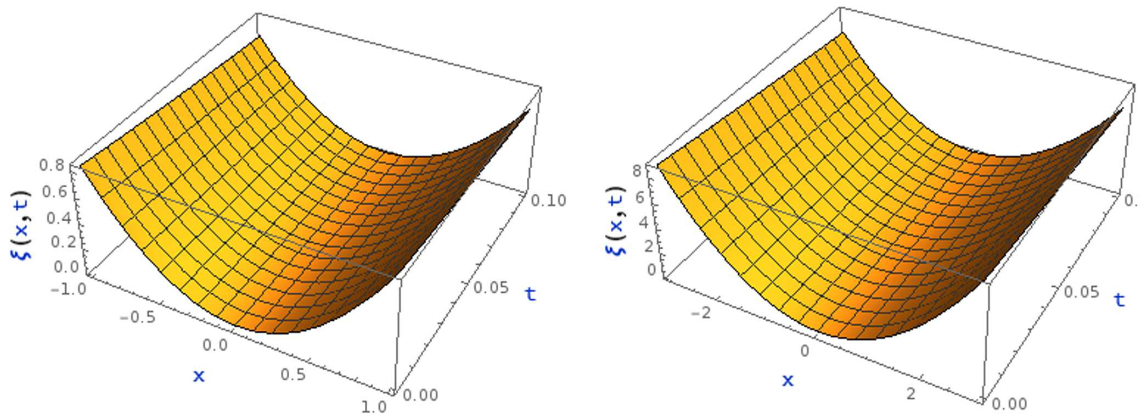


FIGURE 2. The solution function of (4.1) in the ABO sense for $\alpha = 0.9$ (*left*) and $\alpha = 0.99$ (*right*).

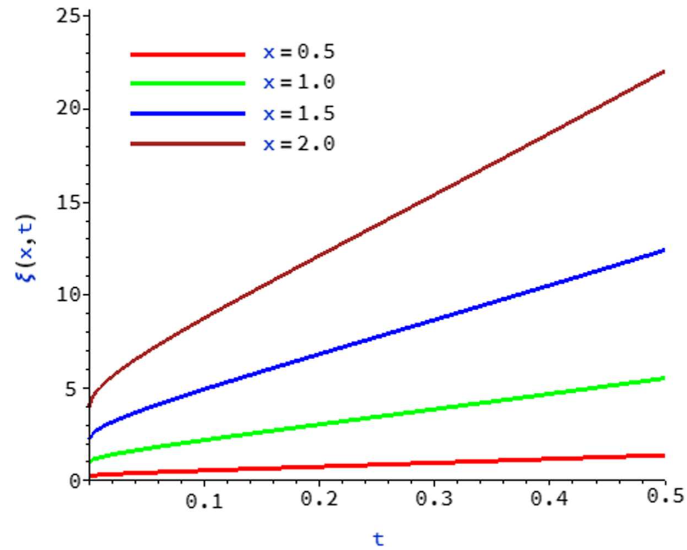


FIGURE 3. Numerical simulation of (4.8) in the ABO sense for $\alpha = 0.5$.

In order to obtain a recursive approximation which gives the solution of equation (4.9) by using the NSFDO, we apply the Laplace transform to equations (4.9) and (4.10). Then we get

$$\tilde{\psi}(x,s) = -\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\mathcal{L}\left\{\frac{1}{2}\frac{\partial\xi^2}{\partial x} - \frac{\partial^2\xi}{\partial x^2}\right\} + \frac{1}{s}\left[\xi(x,0)\right]. \tag{4.11}$$

Using the inverse LT of equation (4.11) we obtain

$$\xi(x,t) = \xi(x,0) - \mathcal{L}^{-1}\left\{\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\mathcal{L}\left\{\frac{1}{2}\frac{\partial\xi^2}{\partial x} - \frac{\partial^2\xi}{\partial x^2}\right\}\right\}. \tag{4.12}$$

Now, if we apply the LHTM, we have

$$\sum_{u=0}^{\infty}\lambda^{u}\tilde{\psi}_{u}(x,s)=x-\lambda\mathcal{L}^{-1}\left\{\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\mathcal{L}\left\{\sigma\left[\sum_{u=0}^{\infty}\lambda^{u}\tilde{\psi}_{u}(x,t)\right]+\sum_{u=0}^{\infty}\lambda^{u}\mathcal{N}_{u}(x,t)\right\}\right\}. \quad (4.13)$$

In the last equation, $\mathcal{N}_u(\xi)$ are the polynomials that show the nonlinear terms defined in (3.9). These polynomials are evaluated in the following way:

$$\begin{aligned}
\mathcal{N}_0(\xi) &= \left(\xi_0^2\right)_x,\\
\mathcal{N}_1(\xi) &= \left(\left[(\xi_0+\lambda\xi_1)^2\right]_{\lambda}\Big|_{\lambda=0}\right)_x = \left(2\xi_0\xi_1\right)_x,\\
\mathcal{N}_2(\xi) &= \tfrac{1}{2}\left(\left[\left(\xi_0+\lambda\xi_1+\lambda^2\xi_2\right)^2\right]_{\lambda\lambda}\Big|_{\lambda=0}\right)_x = \left(\xi_1^2+2\xi_0\xi_2\right)_x,\\
&\vdots
\end{aligned} \quad (4.14)$$

The other parts of $\mathcal{N}_u(\xi)$ can be calculated in this manner. According to the equation (4.14), we can obtain $\mathcal{N}_0(\xi)=2x$. Then, we have

$$\lambda^0:\xi_0(x,t)=\xi(x,0)+\mathcal{L}^{-1}\left\{\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\tilde{\theta}(x,s)\right\}=x, \quad (4.15)$$

$$\begin{aligned}
\lambda^1:\xi_1(x,t)&=\mathcal{L}^{-1}\left\{\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\left(\mathcal{L}\left\{\frac{\partial^2\xi_0}{\partial x^2}\right\}-\frac{1}{2}\mathcal{L}\left\{\mathcal{N}_0(\xi)\right\}\right)\right\}\\
&=\frac{4x(1-\alpha)\left(e^{\frac{\pi^2\alpha t}{2(1-\alpha)}}-1\right)}{\pi^2\alpha\varpi(\alpha)}-\frac{2tx}{\varpi(\alpha)},
\end{aligned}$$

$$\begin{aligned}
\lambda^2:\xi_2(x,t)&=\mathcal{L}^{-1}\left\{\left(\frac{\pi}{s\varpi(\alpha)\arctan\left(\frac{\alpha\pi}{s(1-\alpha)}\right)}\right)\left(\mathcal{L}\left\{\frac{\partial^2\xi_1}{\partial x^2}\right\}-\frac{1}{2}\mathcal{L}\left\{\mathcal{N}_1(\xi)\right\}\right)\right\}\\
&=(-4x)\left(\frac{24(\alpha-1)^2}{\pi^4\alpha^2(\varpi(\alpha))^2}\left(1-e^{\frac{\pi^2\alpha t}{2(1-\alpha)}}\right)+\frac{t^2}{(\varpi(\alpha))^2}-\frac{4(\alpha-1)te^{\frac{\pi^2\alpha t}{2(1-\alpha)}}}{\pi^2\alpha(\varpi(\alpha))^2}-\frac{8(\alpha-1)t}{\pi^2\alpha(\varpi(\alpha))^2}\right),
\end{aligned}$$

$$\vdots$$

Then we have the following series sum which gives the approximation solution:

$$\xi(x,t)=\sum_{u=0}^{\infty}\xi_u(x,t). \quad (4.16)$$

The analytical solution of the problem when $\alpha\to 1$ is obtained as

$$\xi(x,t)=\frac{x}{1+t}. \quad (4.17)$$

Now we apply the mentioned method in the ABO sense to the problems (4.9) and (4.10). Similarly, we set the recursive approximation as follows:

$$\tilde{\psi}(x,s)=\frac{\xi(x,0)}{s}-\left(\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{\frac{1}{2}\frac{\partial\xi^2}{\partial x}-\frac{\partial^2\xi}{\partial x^2}\right\}. \tag{4.18}$$

Then, taking the inverse LT of the last equation, we get

$$\xi(x,t)=\xi(x,0)-\mathcal{L}^{-1}\left\{\left(\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{\frac{1}{2}\frac{\partial\xi^2}{\partial x}-\frac{\partial^2\xi}{\partial x^2}\right\}\right\}. \tag{4.19}$$

Furthermore, if we apply the perturbation method, we have

$$\sum_{u=0}^{\infty}\lambda^{u}\tilde{\psi}_u(x,s)=x-\lambda\mathcal{L}^{-1}\left\{\left(\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\mathcal{L}\left\{\sigma\left[\sum_{u=0}^{\infty}\lambda^{u}\tilde{\psi}_u(x,t)\right]+\sum_{u=0}^{\infty}\lambda^{u}\mathcal{N}_u(x,t)\right\}\right\}. \tag{4.20}$$

The polynomials for the nonlinear terms $\mathcal{N}_u(\xi)$ in equation (4.20) are calculated as that in equation (4.14). Therefore, the first component of $\mathcal{N}_u(\xi)$ is given as $\mathcal{N}_0(\xi)=2x$. Then, we evaluate the homotopies as follows:

$$\lambda^0:\xi_0(x,t)=\xi(x,0)+\mathcal{L}^{-1}\left\{\left(\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\tilde{\theta}(x,s)\right\}=x, \tag{4.21}$$

$$\begin{aligned}\lambda^1:\xi_1(x,t)&=\mathcal{L}^{-1}\left\{\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\left(\mathcal{L}\left\{\tfrac{\partial^2\xi_0}{\partial x^2}\right\}-\tfrac{1}{2}\mathcal{L}\left\{\mathcal{N}_0(\xi)\right\}\right)\right\}\\&=-x\left(1-\alpha+\tfrac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right),\end{aligned}$$

$$\begin{aligned}\lambda^2:\xi_2(x,t)&=\mathcal{L}^{-1}\left\{\left(\tfrac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}}\right)\left(\mathcal{L}\left\{\tfrac{\partial^2\xi_1}{\partial x^2}\right\}-\tfrac{1}{2}\mathcal{L}\left\{\mathcal{N}_1(\xi)\right\}\right)\right\}\\&=2x\left(\tfrac{\alpha^2t^{\alpha+1}}{\Gamma(\alpha+2)}-(\alpha-1)\tfrac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}-(\alpha-1)(\alpha t-\alpha+1)\right),\end{aligned}$$

$$\vdots$$

Finally, we obtain the followings as the approximate and the analytical solution when $\alpha\to 1$ for equation (4.9), respectively:

$$\xi(x,t)=\sum_{u=0}^{\infty}\xi_u(x,t),\qquad \xi(x,t)=\frac{x}{1+t}. \tag{4.22}$$

Figure 4 shows the numerical evaluations obtained for equation (4.22).

## 5. Error measure of the method

In this subsection of the study, we analyse the convergence and the stability of the recommended method. If the series (3.14) converges where $\tilde{\psi}(x,s)$ is obtained by equation (3.7), it has to be the solution of equation (3.1). Moreover, the solution results declare that the mentioned method is stable. Our suggested method provides a good converge area of the solution by generative functions (3.10) and (3.11). Furthermore, the approximate

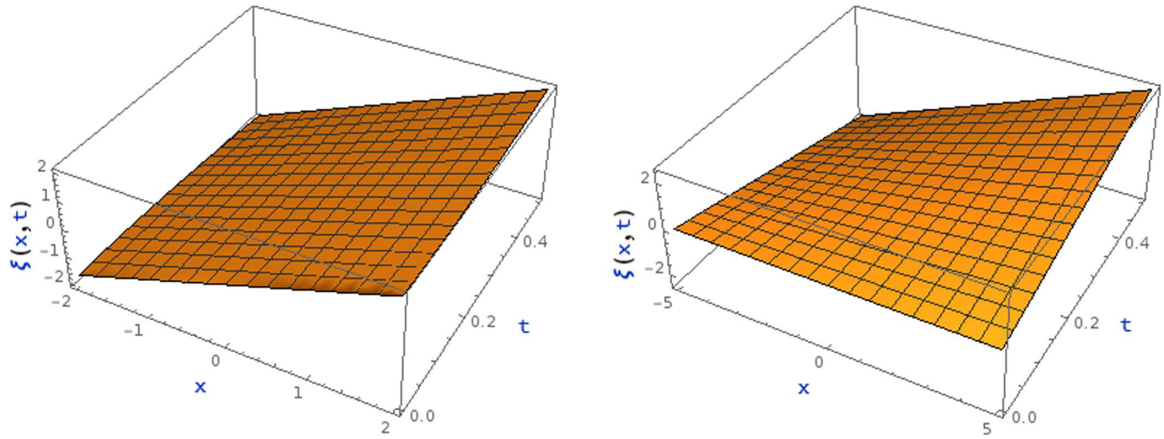


FIGURE 4. Numerical computation of equation (4.22) in the ABO sense for $\alpha = 0.75$ (*left*) and $\alpha = 1$ (*right*).

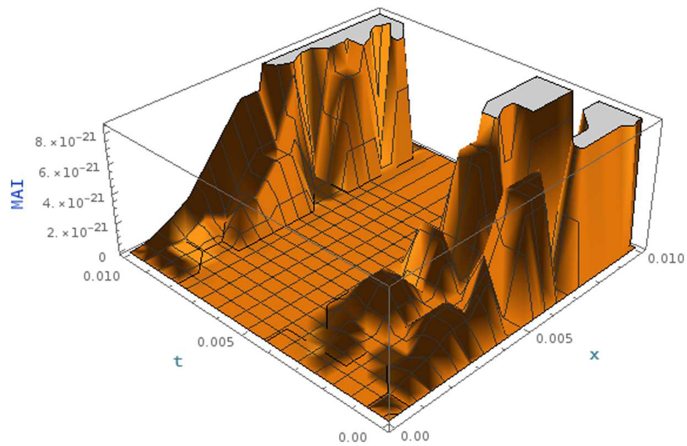


FIGURE 5. Measure of absolute inaccuracy ($MAI$) for some values of $x$ and $t$.

results with the homotopy are good settlement with the accurate solutions. In order to verify the convergent and stability of the proposed method defined in Section 3, the measure of absolute inaccuracy ($MAI$) is obtained for some values of space variable $x$ and time variable $t$. In Figure 5, we compare the numerical solution with the exact solution in equation (4.22), for $u = 8$.

## 6. Concluding remarks

In the present paper, we have investigated the character of two different fractional derivative operators defined with the NSF and MLF. The linear homogeneous Cauchy problem and time-fractional nonlinear Burgers' problem have been solved via the perturbation method with coupled LTs of these operators. The results obtained have been presented with figures for different values of the fractional operator $\alpha$, distance term $x$ and the time variable $t$. According to the results, it is conclude that the computations in the NSFDO is more complicated than the ABO, since it is easier than the LT of the MLF. Therefore, the results obtained with the ABO are closer to the exact solutions. Furthermore, it can be said that the ABO is preferred to the operator defined with the NSF in solving linear/nonlinear problems.

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