Math. Model. Nat. Phenom. 14 (2019) 313
https://doi.org/10.1051/mmnp/2018073

**Mathematical Modelling of Natural Phenomena**
www.mmnp-journal.org

# A NEW APPROACH TO CAPTURE HETEROGENEITY IN GROUNDWATER PROBLEM: AN ILLUSTRATION WITH AN EARTH EQUATION

Abdon Atangana[1,*] and Rubayyi T. Alqahtani[2]

**Abstract.** One of the major problem faced in modeling groundwater flow problems is perhaps how to capture heterogeneity of the geological formation within which the flow takes place. In this paper, we suggested applied a newly established approach to model real world problems that combines the concept of stochastic modeling in which parameters inputs are converted into distributions and the time differential operator is replaced by non-local differential operators. We illustrated this method with the Earth equation of groundwater recharge. For each case, we provided numerical and exact solution using the newly established numerical scheme and Laplace transform. We presented some numerical simulations. The numerical graphical representations let no doubt to think that this approach is the future way of modeling complex problems.

**Mathematics Subject Classification.** 34A08, 26A33, 34K28.

Received September 20, 2018. Accepted December 3, 2018.

## 1. Introduction

Modeling the flow of groundwater within a geological formation has been of great interest and has also attracted attention of many researchers from different fields of science, technology and engineering in the last decades [6, 12, 17, 18, 27, 28, 31]. This is mainly due to the fact that, geological formation within which such flow takes place cannot be seeing with naked eyes. More precisely, the heterogeneities of these media cannot be fully understood and model using existing available knowledge. Due to the complexity of this dynamical process, some researchers have suggested some important approaches that can be used to capture some heterogeneity but not all [6, 12, 17, 18, 27, 28, 31]. The stochastic approach was suggested where all parameters inputs can be converted to distributions. This technique is a mathematical or statistical tool for estimating probability distributions of potential output, which allows for random variation in all inputs parameters over a given time [19, 21, 22, 26, 29, 30]. The reason is that the random variation is commonly based on oscillations observed fact data for a particular period employing the standard time-series techniques. This for instance is the problem faced when collecting the storativity, hydraulic conductivity and the transmissivity of a given geological formation.

*Keywords and phrases:* Stochastic approach, non-local differential operators, Earth equation.

[1] Institute for Groundwater Studies, Faculty of Natural and Agricultural Science, University of Free State, 9300 Bloemfontein, South Africa.
[2] Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11566, Saudi Arabia.

[*] Corresponding author: AtanganaA@ufs.ac.za

These aquifers parameters cannot be determined for fixed value, their values change in time and space, thus to have a good prediction when using mathematical model with these aquifer parameters it is advised to consider them as distribution not as constant input parameters. This approach has been used in many scenarios with great success; however, it is important to point out that this process lead to Markovian process where we do not have time memory. With this concept, we cannot really capture long-range memory, in addition, we cannot capture heterogeneity linked to fracture network, thus to solve this problem, another concept was suggested [1–5, 8, 20]. This concept involved the replacement of time derivative by time fractional differential operator. In the literature, we can find three different types of fractional differential operators, including differential operator based on the power law kernel known as Riemann–Liouville–Caputo derivative, we have fractional differential operator based on the exponential decay law function with delta direct properties when the fractional order tends to 1, the derivative is called the Caputo–Fabrizio derivative and finally we have fractional differential operators with the generalized Mittag-Leffler function also with delta Dirac properties when alpha tends to 1, the derivative are called the Atangana–Baleanu derivatives [7, 13–16, 23–25]. Although the application of nonlocal differential operators have gained interest in modeling real-world problems, it is worth noting that they cannot still capture all the heterogeneities encountered in nature, in particular in groundwater flow problems [1–5, 8–11, 20]. Atangana and Bonyah suggested a new approach to model real world problems, in their approach they are argued that although some people believe that there are no new things under the sun, but pieces of existing results can be put together to obtain relatively new results. They suggested a new way of modeling real world problems, their approach is called fractional stochastic modeling, where the time derivative based on the concept of rate of change is replaced by a given non-local operator and the parameters inputs are replaced by distributions. In this paper, we will apply their approach on the Earth equation. In the next section, illustrate the approach using different fractional differential operators.

## 2. Application with Caputo–Fabrizio and normal distribution

While we are trying to apply the approach using the newly established method with Caputo–Fabrizio fractional derivative, we must recall that this operator has an interesting property that it provides a waiting distribution-based exponential decay law able to produce fading memory, and in addition the associate probability is a crossover from Gauss to non-Gaussian with steady state. In this study, we consider the following groundwater model.

$$S\frac{\mathrm{d}h(t)}{\mathrm{d}t} = R - \frac{h(t)}{DR}. \tag{2.1}$$

The above equation has for input parameter $S$ which is the specific yield, $R$ is the recharge, DR is the drainage resistance which is a site specific parameters and $h$ is the groundwater level. Now to apply the approach suggested by Atangana and Bonyah, we replace parameters by distribution as follow: we assume a range of DR in [a, b], $S$ in [c, d] and DR in [e, f]. Then for each sample we have the following distribution:

$$\overline{DR} = \overline{DR} + \rho N(0,1)\,, \quad \check{R} = \overline{R} + \rho N(0,1)\,, \quad \check{S} = \overline{S} + \rho N(0,1)$$

Such that equation (2.1) is converted to

$$\left(\overline{S} + \rho N(0,1)\right)\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \left(\overline{R} + \rho N(0,1)\right) - \frac{h(t)}{\left(\overline{DR} + \rho N(0,1)\right)} \tag{2.2}$$

Now to include the waiting distribution with fading memory, we replace the time derivative with Caputo–Fabrizio time derivative to obtain:

$$(\bar{S}+\rho N(0.1))\,{}_{0}^{CF}D_t^{\alpha}h(t)=(\bar{R}+\rho N(0,1))-\frac{h(t)}{(\overline{DR}+\rho N(0.1))}. \tag{2.3}$$

Or

$$\left(\overline{S}+\rho N\left(0,1\right)\right)\frac{M\left(\alpha\right)}{1-\alpha}\int\limits_0^t\exp\left[-\frac{\alpha}{1-\alpha}\left(t-y\right)\right]h^{'}(y)\mathrm{d}y=\left(\overline{R}+\rho N\left(0,1\right)\right)-\frac{h(t)}{\left(\overline{DR}+\rho N\left(0,1\right)\right)}$$

The above equation is more complex and can capture more heterogeneities the model with Caputo–Fabrizio only or with stochastic approach only. We make us of the Laplace transform to derive the exact solution of the equation (2.3). Thus, applying the Laplace transform, we obtain

$$\frac{\widehat{S}\left(p\;H-h\left(0\right)\right)}{\left(1-\alpha\right)\left(p+\frac{\alpha}{1-\alpha}\right)}=\frac{\hat{R}}{p}-\frac{H}{\widehat{DR}} \tag{2.4}$$

The above can be converted as follow

$$H=\frac{\frac{\hat{S}}{1-\alpha}h(0)}{\left(\frac{\hat{S}}{1-\alpha}+\frac{1}{\widehat{DR}}\right)p+\frac{\alpha}{\widehat{DR}(1-\alpha)}}+\frac{\hat{R}\alpha}{\left(1-\alpha\right)p}\frac{1}{\left(\frac{\hat{S}}{1-\alpha}+\frac{1}{\widehat{DR}}\right)p+\frac{\alpha}{\widehat{DR}(1-\alpha)}} \tag{2.5}$$

By simplification, we obtain

$$H=\frac{a_1}{a_2}h(0)\frac{1}{p+\frac{a_3}{a_2}}+\frac{a_4}{a_2p}\frac{1}{p+\frac{a_3}{a_2}}.$$

Then by taking the inverse Laplace transform, we obtain the following

$$h\left(t\right)=c\frac{a_1}{a_2}h(0)\exp\left[-\frac{a_3}{a_2}t\right]+\frac{a_4}{a_2}\int\limits_0^t\exp\left[-\frac{a_3}{a_2}(t-y)\right]\mathrm{d}y \tag{2.6}$$

Applying the initial condition, we obtain the constant c and we obtain the following exact

$$h\left(t\right)=h(0)\exp\left[-\frac{a_3}{a_2}t\right]+\frac{a_4}{a_2}\int\limits_0^t\exp\left[-\frac{a_3}{a_2}(t-y)\right]\mathrm{d}y$$

where

$$a_1=\frac{\overline{S}+\rho N\left(0,1\right)}{1-\alpha},\quad a_2=\left(\frac{\overline{S}+\rho N\left(0,1\right)}{1-\alpha}+\frac{1}{\overline{DR}+\rho N\left(0,1\right)}\right),$$
$$a_3=\frac{\alpha}{\left(\overline{DR}+\rho N\left(0,1\right)\right)\left(1-\alpha\right)},\quad a_4=\frac{\left(\overline{R}+\rho N\left(0,1\right)\right)\alpha}{1-\alpha}$$

Equation (2.6) is therefore the exact solution of the fractional stochastic Earth equation with the Caputo–Fabrizio derivative. If the distribution depends on the same time scale with the dynamical system, we consider the distribution as function of time, in this case the new equation cannot be solved analytically rather numerically. In the following, we present the numerical solution of the new equation

$$S(t)\,{}_0^{CF}D_t^{\alpha} h\left(t\right) = R(t) - \frac{h(t)}{DR(t)}. \tag{2.7}$$

Since we assume that the function $S(t)$ is nonzero thus applying the Caputo–Fabrizio integral on both sides we obtain

$$h\left(t\right) - h\left(0\right) = \frac{1-\alpha}{M(\alpha)}\left(\frac{R(t)}{S(t)} - \frac{h(t)}{S(t)DR(t)}\right) - \frac{\alpha}{M(\alpha)}\int_0^t \left(\frac{R(y)}{S(y)} - \frac{h(y)}{S(y)DR(y)}\right)\mathrm{d}y. \tag{2.8}$$

Now at a particular $t_{n+1}$

$$h\left(t_{n+1}\right) - h\left(0\right) = \left(\left(\frac{R(t_n)}{S(t_n)} - \frac{h(t_n)}{S(t_n)DR(t_n y)}\right)\right) + \frac{\alpha}{M(\alpha)}\int_0^{t_{n+1}} \left(\frac{R(y)}{S(y)} - \frac{h(y)}{S(y)DR(y)}\right)\mathrm{d}y. \tag{2.9}$$

Then at $t_n$

$$h\left(t_n\right) - h\left(0\right) = \frac{1-\alpha}{M(\alpha)}\left(\left(\frac{R(t_n)}{S(t_n)} - \frac{h(t_n)}{S(t_n)DR(t_n y)}\right)\right) + \frac{\alpha}{M(\alpha)}\int_0^{t_n} \left(\frac{R(y)}{S(y)} - \frac{h(y)}{S(y)DR(y)}\right)\mathrm{d}y. \tag{2.10}$$

Then equation (2.9) minus equation (2.10) we obtain the follows

$$\begin{aligned} h\left(t_{n+1}\right) - h\left(t_n\right) = {}& \frac{1-\alpha}{M(\alpha)}\left\{\left(\frac{R\left(t_n\right)}{S\left(t_n\right)} - \frac{h\left(t_n\right)}{S\left(t_n\right)DR\left(t_n y\right)}\right) - \left(\frac{R(t_{n-1})}{S(t_{n-1})} - \frac{h(t_{n-1})}{S(t_{n-1})DR(t_{n-1})}\right)\right\} \\ & + \frac{\alpha}{M(\alpha)}\int_{t_n}^{t_{n+1}} \left(\frac{R(y)}{S(y)} - \frac{h(y)}{S(y)DR(y)}\right)\mathrm{d}y. \end{aligned} \tag{2.11}$$

The above can be further discretized as

$$\begin{aligned} h\left(t_{n+1}\right) - h\left(t_n\right) = {}& \frac{1-\alpha}{M(\alpha)}\left\{\left(\frac{R\left(t_n\right)}{S\left(t_n\right)} - \frac{h\left(t_n\right)}{S\left(t_n\right)DR\left(t_n y\right)}\right) - \left(\frac{R(t_{n-1})}{S(t_{n-1})} - \frac{h(t_{n-1})}{S(t_{n-1})DR(t_{n-1})}\right)\right\} \\ & + \frac{\alpha}{M(\alpha)}\left\{\frac{3\triangle t}{2}\left(\frac{R\left(t_n\right)}{S\left(t_n\right)} - \frac{h\left(t_n\right)}{S\left(t_n\right)DR\left(t_n y\right)}\right) - \frac{\triangle t}{2}\left(\frac{R(t_{n-1})}{S(t_{n-1})} - \frac{h(t_{n-1})}{S(t_{n-1})DR(t_{n-1})}\right)\right\}. \end{aligned} \tag{2.12}$$

We finally obtain the following numerical solution that can be used for simulation

$$\begin{aligned} h^{n+1} - h^n = {}& \left(\frac{1-\alpha}{M\left(\alpha\right)} + \frac{3\triangle t}{2M\left(\alpha\right)}\right)\left(\frac{R\left(t_n\right)}{S\left(t_n\right)} - \frac{h^n}{S\left(t_n\right)DR\left(t_n y\right)}\right) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\triangle t}{2M\left(\alpha\right)}\right) \\ & \times \left(\frac{R(t_{n-1})}{S(t_{n-1})} - \frac{h^{n-1}}{S(t_{n-1})DR(t_{n-1})}\right) \end{aligned} \tag{2.13}$$

## 3. Application with Caputo and normal distribution

In this section, we apply the approach with Caputo classical differential operator, such that (2.1) becomes:

$$\left(\overline{S}+\rho N\left(0,1\right)\right){}_{0}^{C}D_{t}^{\alpha}h(t)=\left(\overline{R}+\rho N\left(0,1\right)\right)-\frac{h(t)}{\left(\overline{DR}+\rho N\left(0,1\right)\right)}. \tag{3.1}$$

Or

$$\left(\overline{S}+\rho N\left(0,1\right)\right)\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-y)^{-\alpha}h^{'}(y)\mathrm{d}y=\left(\overline{R}+\rho N\left(0,1\right)\right)-\frac{h(t)}{\left(\overline{DR}+\rho N\left(0,1\right)\right)}.$$

The difference between this model and the previous one is based on the waiting distribution and probability distribution, as this version waiting time is power law and probability distribution is non-Gaussian only. In case the distribution does not have same time scale with the dynamical system, we can use the Laplace transform to obtain:

$$\hat{S}[P^{\alpha}H-h(0)]=\frac{\hat{R}}{P}-\frac{H}{\widehat{\mathrm{DR}}}. \tag{3.2}$$

Rearranging and taking the inverse Laplace transform we obtain the following exact solution

$$h\left(t\right)=h\left(0\right)E_{\alpha}\left[\frac{t^{\alpha}}{\widehat{DR}\hat{S}}\right]+\frac{\hat{R}}{\widehat{DR}}\int_{0}^{t}E_{\alpha}\left[\frac{(t-y)^{\alpha}}{\widehat{DR}\hat{S}}\right]\mathrm{d}y.$$

Again we assume that, the distribution depends on the same time scale of the dynamical system, the mathematical model cannot solve analytically, and therefore we rely on the numerical method. Since we assume that the function $S(t)$ is nonzero thus applying the Riemann–Liouville integral on both sides we obtain

$$h\left(t\right)-h\left(0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)(t-y)^{\alpha-1}\mathrm{d}y \tag{3.3}$$

Now at a particular $t_{n+1}$

$$h\left(t_{n+1}\right)-h\left(0\right)=+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t_{n+1}}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)(t_{n+1}-y)^{\alpha-1}\mathrm{d}y. \tag{3.4}$$

Then at $t_n$

$$h\left(t_{n}\right)-h\left(0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t_{n}}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)(t_{n}-y)^{\alpha-1}\mathrm{d}y. \tag{3.5}$$

Then equation (3.4) minus equation (3.5) we obtain the follows

$$\begin{aligned}h\left(t_{n+1}\right)-h\left(t_{n}\right)=&\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t_{n+1}}\left(\frac{R\left(y\right)}{S\left(y\right)}-\frac{h\left(y\right)}{S\left(y\right)DR\left(y\right)}\right)(t_{n+1}-y)^{\alpha-1}\mathrm{d}y\\&-\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t_{n}}\left(\frac{R\left(y\right)}{S\left(y\right)}-\frac{h\left(y\right)}{S\left(y\right)DR\left(y\right)}\right)(t_{n}-y)^{\alpha-1}\mathrm{d}y.\end{aligned} \tag{3.6}$$

For simplicity, we put

$$\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)=F(h\left(y\right),y)$$

Then without loss of generality, we evaluate the following integral

$$\int\limits_{0}^{t_{n+1}}F(h\left(y\right),y)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y=\sum_{k=0}^{n}\left\{\frac{F\left(h_k,t_k\right)}{\triangle t}\int\limits_{t_k}^{t_{k+1}}\left(y-t_{k-1}\right)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\right.$$
$$\left.-\frac{F\left(h_{k-1},t_{k-1}\right)}{\triangle t}\int\limits_{t_k}^{t_{k+1}}\left(y-t_{k-1}\right)\left(t_n-y\right)^{\alpha-1}\mathrm{d}y\right\}. \tag{3.7}$$

The above can be further extended as:

$$\int\limits_{0}^{t_{n+1}}F(h\left(y\right),y)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y$$
$$=\sum_{k=0}^{n}\left\{\frac{\left(\triangle t\right)^{\alpha}F\left(h_k,t_k\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n+1-k\right)^{\alpha}\left(n-k+2+\alpha\right)-\left(n-k\right)^{\alpha}\left(n-k+2+2\alpha\right)\right)\right.$$
$$\left.-\frac{\left(\triangle t\right)^{\alpha}F\left(h_{k-1},t_{k-1}\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n-k\right)^{\alpha}\left(n-k+1+\alpha\right)-\left(n-1-k\right)^{\alpha}\left(n-k+1+2\alpha\right)\right)\right\}. \tag{3.8}$$

Then the second integral is evaluated as

$$\int\limits_{0}^{t_{n}}F(h\left(y\right),y)\left(t_{n}-y\right)^{\alpha-1}\mathrm{d}y$$
$$=\sum_{k=1}^{n-1}\left\{\frac{\left(\triangle t\right)^{\alpha}F\left(h_k,t_k\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n-k\right)^{\alpha}\left(n-k+1+\alpha\right)-\left(n-k-1\right)^{\alpha}\left(n-k+1+2\alpha\right)\right)\right.$$
$$\left.-\frac{\left(\triangle t\right)^{\alpha}F\left(h_{k-1},t_{k-1}\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n-k-1\right)^{\alpha}\left(n-k+\alpha\right)-\left(n-2-k\right)^{\alpha}\left(n-k+2\alpha\right)\right)\right\} \tag{3.9}$$

Putting together (3.9) and (3.8) into (3.6) we obtain the following:

$$h(t_{n+1})-h(t_n)$$
$$=\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=0}^{n}\left\{\frac{(\triangle t)^{\alpha}F\left(h_k,t_k\right)}{\Gamma(\alpha+2)}((n+1-k)^{\alpha}(n-k+2+\alpha)-(n-k)^{\alpha}(n-k+2+2\alpha))\right.\right.$$
$$\left.-\frac{(\triangle t)^{\alpha}F(h_{k-1},t_{k-1})}{\Gamma(\alpha+2)}((n-k)^{\alpha}(n-k+1+\alpha)-(n-1-k)^{\alpha}(n-k+1+2\alpha))\right\}$$
$$-\sum_{k=1}^{n-1}\left\{\frac{(\triangle t)^{\alpha}F(h_k,t_k)}{\Gamma(\alpha+2)}((n-k)^{\alpha}(n-k+1+\alpha)-(n-k-1)^{\alpha}(n-k+1+2\alpha))\right.$$
$$\left.\left.-\frac{(\triangle t)^{\alpha}F(h_{k-1},t_{k-1})}{\Gamma(\alpha+2)}((n-k-1)^{\alpha}(n-k+\alpha)-(n-2-k)^{\alpha}(n-k+2\alpha))\right\}\right\} \tag{3.10}$$

The above is the numerical solution of the equation under investigation, the proof of stability and error analysis can be found in [32]. We now finally present our investigation using the Atangana–Baleanu fractional differential and integral operators

## 4. Application with Atangana–Baleanu derivative and normal distribution

In this section, we analyze the model with the Atangana–Baleanu fractional differential operator in Caputo sense. We can recall and motivate this version of the model with the fact that the Atangana–Baleanu differential operator provides a waiting time distribution that is a crossover from stretched exponential to power law. For small time, this derivative is able to capture the Brownian motion and for a later time it can capture power law like Riemann–Liouville derivative. On the other hand, the operator kernel has a density distribution with a crossover behavior too from Gaussian to non-Gaussian without any steady state, these properties are found in several instances in nature. We therefore apply such operator to this problem to see the possible effect. We start with assuming that the distribution does not depend on the same time with the dynamical system, thus, one can use the Laplace transform to derive the exact solution as follow:

$$\left(\overline{S}+\rho N\left(0,1\right)\right){}_{0}^{ABC}D_{t}^{\alpha}h(t)=\left(\overline{R}+\rho N\left(0,1\right)\right)-\frac{h(t)}{\left(\overline{DR}+\rho N\left(0,1\right)\right)}.$$

Or

$$\left(\overline{S}+\rho N\left(0,1\right)\right)\frac{AB(\alpha)}{1-\alpha}\int\limits_{0}^{t}(t-y)^{-\alpha}h^{'}(y)\mathrm{d}y=\left(\overline{R}+\rho N\left(0,1\right)\right)-\frac{h(t)}{\left(\overline{DR}+\rho N\left(0,1\right)\right)}$$

$$\frac{AB(\alpha)}{1-\alpha}\frac{\hat{S}\left[p^{\alpha}H-h(0)\right]}{p^{\alpha}+\frac{\alpha}{1-\alpha}}=\frac{\hat{R}}{p}-\frac{H}{\widehat{DR}}$$

Rearranging, taking the inverse Laplace transform we obtain the following exact solution

$$h\left(t\right)=h\left(0\right)E_{\alpha}\left[\frac{\alpha t^{\alpha}}{AB\left(\alpha\right)-\left(1-\alpha\right)\widehat{DR}\hat{S}}\right]+\frac{\hat{R}}{\widehat{DR}}\int\limits_{0}^{t}E_{\alpha}\left[\frac{\left(t-y\right)^{\alpha}}{AB\left(\alpha\right)-\left(1-\alpha\right)\widehat{DR}\hat{S}}\right]\mathrm{d}y \tag{4.1}$$

$$h\left(t\right)-h\left(0\right)=\frac{1-\alpha}{B(\alpha)}\left(\frac{R(t)}{S(t)}-\frac{h(t)}{S(t)DR(t)}\right)-\frac{\alpha}{\Gamma\left(\alpha\right)B(\alpha)}\int\limits_{0}^{t}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)\left(t-y\right)^{\alpha-1}\mathrm{d}y$$

Now at a particular $t_{n+1}$

$$\begin{aligned}h\left(t_{n+1}\right)-h\left(0\right)=&\frac{1-\alpha}{B(\alpha)}\left(\left(\frac{R(t_n)}{S(t_n)}-\frac{h(t_n)}{S(t_n)DR(t_ny)}\right)\right)\\&+\frac{\alpha}{\Gamma\left(\alpha\right)B(\alpha)}\int\limits_{0}^{t_{n+1}}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\end{aligned} \tag{4.2}$$

Then at $t_n$

$$h\left(t_n\right)-h\left(0\right)=\frac{1-\alpha}{M(\alpha)}\left(\left(\frac{R(t_n)}{S(t_n)}-\frac{h(t_n)}{S(t_n)DR(t_n y)}\right)\right)$$
$$+\frac{\alpha}{\Gamma\left(\alpha\right)B(\alpha)}\int\limits_0^{t_n}\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)(t_n-y)^{\alpha-1}\mathrm{d}y \tag{4.3}$$

Then equation (4.1) minus equation (4.2) we obtain the follows

$$\begin{aligned}h\left(t_{n+1}\right)-h\left(t_n\right)\\ &=\frac{1-\alpha}{M\left(\alpha\right)}\left\{\left(\frac{R\left(t_n\right)}{S\left(t_n\right)}-\frac{h\left(t_n\right)}{S\left(t_n\right)DR\left(t_n y\right)}\right)-\left(\frac{R\left(t_{n-1}\right)}{S\left(t_{n-1}\right)}-\frac{h\left(t_{n-1}\right)}{S\left(t_{n-1}\right)DR\left(t_{n-1}\right)}\right)\right\}\\ &+\frac{\alpha}{\Gamma\left(\alpha\right)B\left(\alpha\right)}\int\limits_0^{t_{n+1}}\left(\frac{R\left(y\right)}{S\left(y\right)}-\frac{h\left(y\right)}{S\left(y\right)DR\left(y\right)}\right)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\\ &-\frac{\alpha}{\Gamma\left(\alpha\right)B\left(\alpha\right)}\int\limits_0^{t_n}\left(\frac{R\left(y\right)}{S\left(y\right)}-\frac{h\left(y\right)}{S\left(y\right)DR\left(y\right)}\right)\left(t_n-y\right)^{\alpha-1}\mathrm{d}y\end{aligned} \tag{4.4}$$

For simplicity, we put

$$\left(\frac{R(y)}{S(y)}-\frac{h(y)}{S(y)DR(y)}\right)=F(h\left(y\right),y) \tag{4.5}$$

Then without loss of generality, we evaluate the following integral

$$\begin{aligned}\int\limits_0^{t_{n+1}}F(h\left(y\right),y)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\\ &=\sum_{k=0}^{n}\left\{\frac{F\left(h_k,t_k\right)}{\triangle t}\int\limits_{t_k}^{t_{k+1}}\left(y-t_{k-1}\right)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\right.\\ &\left.-\frac{F\left(h_{k-1},t_{k-1}\right)}{\triangle t}\int\limits_{t_k}^{t_{k+1}}\left(y-t_{k-1}\right)\left(t_n-y\right)^{\alpha-1}\mathrm{d}y\right\}\end{aligned} \tag{4.6}$$

The above can be further extended as:

$$\begin{aligned}\int\limits_0^{t_{n+1}}F(h\left(y\right),y)\left(t_{n+1}-y\right)^{\alpha-1}\mathrm{d}y\\ &=\sum_{k=0}^{n}\left\{\frac{\left(\triangle t\right)^{\alpha}F\left(h_k,t_k\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n+1-k\right)^{\alpha}\left(n-k+2+\alpha\right)-\left(n-k\right)^{\alpha}\left(n-k+2+2\alpha\right)\right)\right.\\ &\left.-\frac{\left(\triangle t\right)^{\alpha}F\left(h_{k-1},t_{k-1}\right)}{\Gamma\left(\alpha+2\right)}\left(\left(n-k\right)^{\alpha}\left(n-k+1+\alpha\right)-\left(n-1-k\right)^{\alpha}\left(n-k+1+2\alpha\right)\right)\right\}\end{aligned} \tag{4.7}$$

Then the second integral is evaluated as

$$\begin{aligned}
&\int_0^{t_n} F(h(y),y)(t_n-y)^{\alpha-1}\mathrm{d}y \\
&\quad=\sum_{k=1}^{n-1}\left\{\frac{(\triangle t)^{\alpha}F(h_k,t_k)}{\Gamma(\alpha+2)}\left((n-k)^{\alpha}(n-k+1+\alpha)-(n-k-1)^{\alpha}(n-k+1+2\alpha)\right)\right.\\
&\qquad\left.-\frac{(\triangle t)^{\alpha}F(h_{k-1},t_{k-1})}{\Gamma(\alpha+2)}\left((n-k-1)^{\alpha}(n-k+\alpha)-(n-2-k)^{\alpha}(n-k+2\alpha)\right)\right\}
\end{aligned}$$

Putting together (4.7) and (4.6) into (4.3) we obtain the following:

$$\begin{aligned}
&h(t_{n+1})-h(t_n)\\
&\quad=\frac{1-\alpha}{M(\alpha)}\left\{\left(\frac{R(t_n)}{S(t_n)}-\frac{h(t_n)}{S(t_n)DR(t_n y)}\right)-\left(\frac{R(t_{n-1})}{S(t_{n-1})}-\frac{h(t_{n-1})}{S(t_{n-1})DR(t_{n-1})}\right)\right\}\\
&\qquad+\frac{\alpha}{\Gamma(\alpha)B(\alpha)}\left\{\sum_{k=0}^{n}\left\{\frac{(\triangle t)^{\alpha}F(h_k,t_k)}{\Gamma(\alpha+2)}\left((n+1-k)^{\alpha}(n-k+2+\alpha)-(n-k)^{\alpha}(n-k+2+2\alpha)\right)\right.\right.\\
&\qquad\left.-\frac{(\triangle t)^{\alpha}F(h_{k-1},t_{k-1})}{\Gamma(\alpha+2)}\left((n-k)^{\alpha}(n-k+1+\alpha)-(n-1-k)^{\alpha}(n-k+1+2\alpha)\right)\right\}\\
&\qquad-\sum_{k=1}^{n-1}\left\{\frac{(\triangle t)^{\alpha}F(h_k,t_k)}{\Gamma(\alpha+2)}\left((n-k)^{\alpha}(n-k+1+\alpha)-(n-k-1)^{\alpha}(n-k+1+2\alpha)\right)\right.\\
&\qquad\left.\left.-\frac{(\triangle t)^{\alpha}F(h_{k-1},t_{k-1})}{\Gamma(\alpha+2)}\left((n-k-1)^{\alpha}(n-k+\alpha)-(n-2-k)^{\alpha}(n-k+2\alpha)\right)\right\}\right\}
\end{aligned}$$

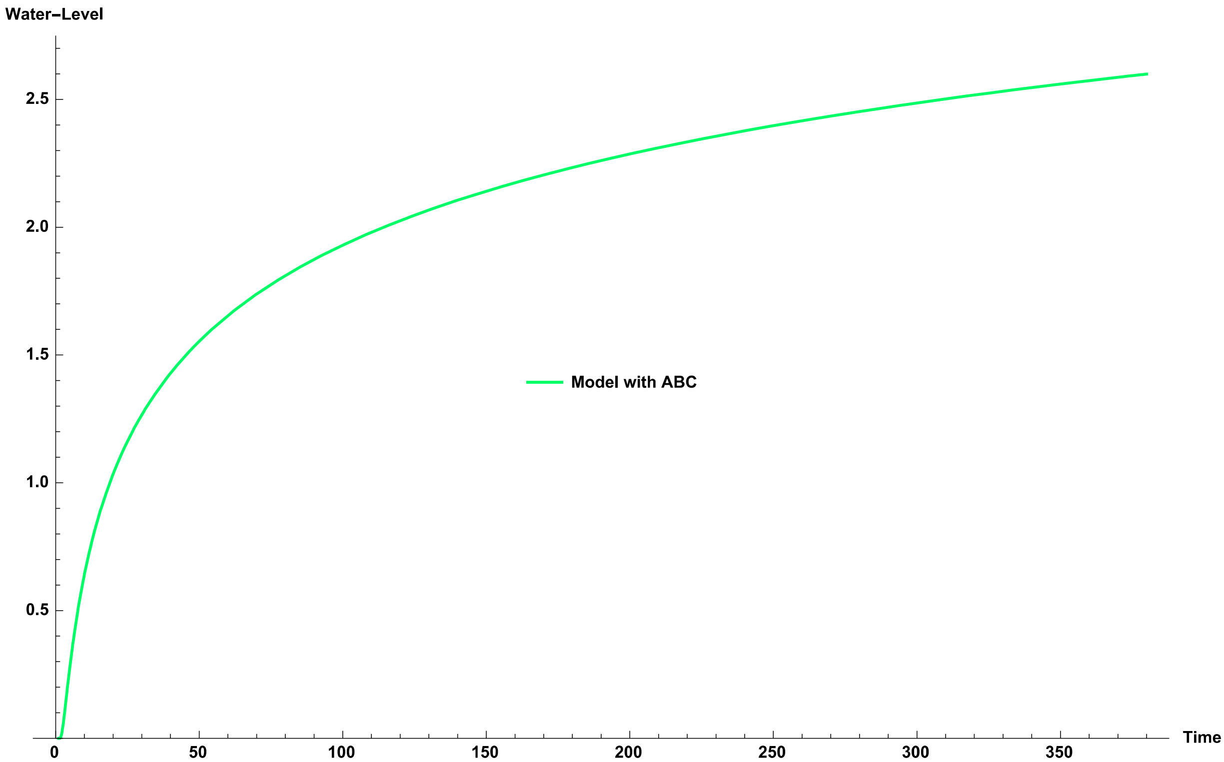
The above is the numerical solution of the equation under investigation, the proof of stability and error analysis can be found in [32]. We now finally present our investigation using the Atangana-Baleanu fractional differential and integral operators.

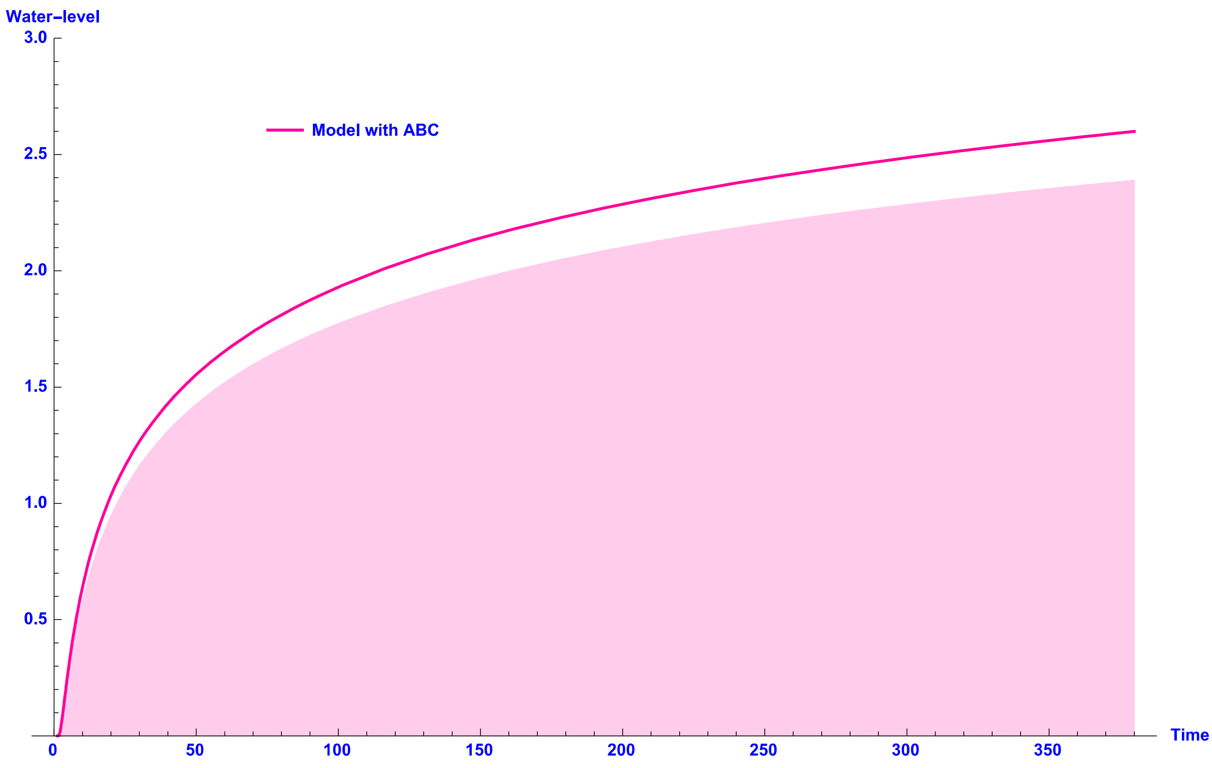
## 5. Numerical simulations and test with experimental

In this section, we present some numerical simulation for different values of fractional order alpha and normal distribution for each model. To achieve this, we assume the following ranges within which the aquifer parameters can be found, the storativity is assumed to be $S \in [0.0001,\ 0.1]$, the drainage coefficient can be assumed to be $\mathrm{DR} \in [0.1,\ 1]$, and $R \in [200, 2000]$.

## 6. Conclusion

The complexity of nature made mankind to invest more in producing knowledge that can be used to understand it. The knowledge suggested by modelers become more and more complex each day due to limitations of the previous proposed works. We can recall that prediction started by using the rate of change as differential operator, then mankind realized that such mathematical operator cannot really captured heterogeneities encounter in nature, then, researcher suggested the concept of stochastic to capture a type of heterogeneity linked statistical setting of nature. With the limitation of this last suggestion, non-local differential operators to capture memory, elasticity and also long range dependency. But those mathematical tools are still limited


Water–Level
2.5
2.0
1.5
1.0
0.5
0
50
100
150
200
250
300
350
Time
Model with ABC


Water–level
3.0
2.5
2.0
1.5
1.0
0.5
0
50
100
150
200
250
300
350
Time
Model with ABC

and thus this approach was suggested. The suggested approach seems to be the future of modeling especially when dealing with groundwater problems.

*Acknowledgements.* The authors thank the Deanship of Scientific research of Al Imam Mohammad Ibn Saud Islamic University (IMSIU) for their fund to support this research in 1438, Grant number: 371211.

## References


[1] A. Atangana, Non validity of index law in fractional calculus: a fractional differential operator with Markovian and non-Markovian properties. *Phys. A: Stat. Mech. Appl.* **505** (2018) 688–706.

[2] A. Atangana and B. Dumitru, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **20** (2016) 763–769.

[3] A. Atangana and J.F. Gomez Aguila, Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena. *Eur. Phys. J. Plus* **133** (2018) 166.

[4] D. Benson, S. Wheatcraft and M. Meerschaert, Application of a fractional advection-dispersion equation. *Water Resour. Res.* **36** (2000) 1403–1412.

[5] M. Caputo, Linear model of dissipation whose Q is almost frequency independent-II. *Geophys. J. Int.* **13** (1967) 529–539.

[6] W.-H. Chiang, W. Kinzelbach and R. Rausch, Aquifer Simulation Model for WINdows – Groundwater Flow and Transport Modeling, An Integrated Program. Gebrüder Borntraeger, Berlin, Stuttgart (1998).

[7] A. Coronel-Escamilla, J.F. Gómez-Aguilar, L. Torres and R.F. Escobar-Jiménez, A numerical solution for a variable-order reaction-diffusion model by using fractional derivatives with non-local and non-singular kernel. *Phys. A: Stat. Mech. Appl.* **491** (2018) 406–424.

[8] L. Debnath, A brief historical introduction to fractional calculus. *Int. J. Math. Educ. Sci. Technol.* **35** (2004) 487–501.

[9] E.F. Doungmo Goufo, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg–de Vries–Burgers equation. *Math. Model. Anal.* **21** (2016) 188–198.

[10] E.F. Doungmo Goufo, Stability and convergence analysis of a variable order replicator-mutator process in a moving medium. *J. Theor. Biol.* **403** (2016) 178–187.

[11] E.F. Doungmo Goufo, Speeding up chaos and limit cycles in evolutionary language and learning processes. *Math. Methods Appl. Sci.* **40** (2017) 3055–3065.

[12] S.D. Foster, R. Hirata and K.W.F. Howard, Groundwater use in developing cities: Policy issues arising from current trends. *Hydrogeol. J.* **19** (2010) 271–4.

[13] J.F. Gómez-Aguilar, A. Atangana, New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications. *Eur. Phys. J. Plus* **132** (2017) 13.

[14] J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, M.G. López-López and V.M. Alvarado-Martínez, Universal character of the fractional space-time electromagnetic waves in dielectric media. *J. Electromagn. Waves Appl.* **29** (2015) 727–740.

[15] J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, M.G. López-López and V.M. Alvarado-Martínez, Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media. *J. Electromagn. Waves Appl.* **30** (2016) 1937–1952.

[16] J.F. Gómez-Aguilar, H. Yépez-Martínez, J. Torres-Jiménez, T. Córdova-Fraga, R.F. Escobar-Jiménez and V.H. Olivares-Peregrino, Homotopy perturbation transform method for nonlinear differential equations involving to fractional operator with exponential kernel. *Adv. Differ. Equ.* **2017** (2017) 68.

[17] M.S. Hantush, Drawdown around a partially penetrating well. In: *Proc. of the Am. Soc. of Civil Engineers. J. Hyd. Div.* **87** (1961a) 83–98.

[18] M.S. Hantush, Aquifer tests on partially penetrating wells. In: *Proc. of the Am. Soc. of Civil Engineers. J. Hyd. Div.* **87** (1961b) 171–194.

[19] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.* **43** (2001) 525–546.

[20] S. Holm and S.P. Näsholm, A causal and fractional all-frequency wave equation for lossy media. *J. Acoust. Soc. Am.* **130** (2011) 2195–2201.

[21] R. Jordan, D. Kinderlehrer and F. Otto, The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** (1998) 1–17.

[22] S. Kadry, A Solution of Linear Stochastic Differential Equation. *WSEAS Trans. Math.* **6** (2007) 618.

[23] I. Koca, Analysis of rubella disease model with non-local and non-singular fractional derivatives. *Int. J. Optim. Control Theories* (2017).

[24] I. Koca, Modelling the spread of Ebola virus with Atangana-Baleanu fractional operators. *Eur. Phys. J. Plus* **133** (2018) 100.

[25] I. Koca, Numerical analysis of coupled fractional differential equations with Atangana-Baleanu fractional derivative. *Discrete Continuous Dyn. Syst. Ser. S* **12** (2019).

[26] G. Parisi, N. Sourlas, Random magnetic fields, supersymmetry, and negative dimensions. *Phys. Rev. Lett.* **43** (1979) 744–745.

[27] R. Rausch, W. Schäfer, R. Therrien and C. Wagner, Solute Transport Modelling – An Introduction to Models and Solution Strategies. Borntraeger, Berlin, Stuttgart (2005).

[28] L.A. Richards, Capillary conduction of liquids through porous mediums. *J. Appl. Phys.* **1** (1931) 318–333.

[29] A. Slavík, Generalized differential equations: differentiability of solutions with respect to initial conditions and parameters. *J. Math. Anal. Appl.* **402** (2013) 261–274.

[30] J. Teugels and B. Sund, Encyclopedia of Actuarial Science. Wiley, Chichester (2004) 523–527.
[31] C.V. Theis, The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using groundwater storage. *Am. Geophys. Union Trans.* **16** (1935) 519–524.
[32] M. Toufik and A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models. *Eur. Phys. J. Plus* **132** (2017) 444.