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INITIAL-BOUNDARY VALUE PROBLEMS FOR A TIME-FRACTIONAL DIFFERENTIAL EQUATION WITH INVOLUTION PERTURBATION

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Abstract. Direct and inverse initial-boundary value problems of a time-fractional heat equation with involution perturbation are considered using both local and nonlocal boundary conditions. Results on existence of formal solutions to these problems are presented. Solutions are expressed in a form of series expansions using appropriate orthogonal basis obtained by separation of variables. Convergence of series solutions are obtained by imposing certain conditions on the given data. Uniqueness of the obtained solutions are also discussed. The obtained general solutions are illustrated by an example using an appropriate choice of the given data.

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1. Introduction

An involution is a mapping f, such that $f(x) \not\equiv x$, that maps a set of real numbers onto itself and satisfies the condition

$$f(f(x)) = x \quad \text{or} \quad f^{-1}(x) = f(x)$$

on this set. The simplest examples of involutions are reflection and inversion, *i.e.*,

$$f(x) = -x,\ x \in (-\infty, \infty) \quad \text{and} \quad f(x) = \frac{1}{x},\ x \in (-\infty, 0) \cup (0, \infty),$$

respectively. Other examples and basic properties of involutions can be found in [20] and [21]. A differential equation with involutions is defined as follows [16]:

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Definition 1.1. A relation of the form

$$F(x, y(f_1(x)), ..., y(f_m(x)), ..., y^{(n)}(f(x)), ..., y^{(n)}(f_m(x))) = 0,$$

where $f_i(f_i(x)) = x$ for every i, and $f_i(x) \not\equiv x$ for some i, is called a differential equation with involutions.

Differential equations with involutions received a lot of attention from both theoretical and practical applications points of view, see for example, [1, 3, 4, 6, 7, 9, 10, 17–19]. Przewoerska–Rolewicz studied different aspects of differential equations with involution and summarized her work in [15]. For a general overview about equations with involutions, we refer the reader to [5]. In this paper, we consider both direct and inverse initial-boundary value problems (IBVPs) of a time-fractional heat equation with involution, where the fractional derivative is defined in the Caputo sense. We first recall some basic definitions and notations regarding fractional derivatives [8]. The Riemann–Liouville and Caputo fractional derivatives of order $0 < \alpha < 1$ are defined, respectively, by:

$$D_{0t}^{\alpha} u\,(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{u(s)}{(t-s)^{\alpha}}\,\mathrm{d}s,$$

and

$${}_C D_{0t}^{\alpha} u\,(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{u'(s)}{(t-s)^{\alpha}}\,\mathrm{d}s,$$

where $\Gamma(\cdot)$ denotes the standard Gamma function. Next, we introduce the Mittag–Leffler function with two parameters, $E_{\alpha,\beta}(z)$, which plays an important role in the theory of fractional differential equations and which is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \mathrm{Re}(\alpha) > 0,\ \beta, z \in \mathbb{C}.$$

It generalizes the Mittag–Leffler function with one parameter, $E_{\alpha}(z) = E_{\alpha,1}(z)$. If $z = \lambda t^{\alpha}, \lambda \in \mathbb{R}$, we have

$${}_C D_{0t}^{\alpha} \left(E_{\alpha}(\lambda t^{\alpha})\right) = \lambda E_{\alpha}(\lambda t^{\alpha}),$$

and hence $E_{\alpha}(\lambda t^{\alpha})$ satisfies the following homogeneous fractional differential equation:

$${}_C D_{0t}^{\alpha} u(t) - \lambda u(t) = 0.$$

The general solution of the non-homogeneous fractional differential equation

$${}_C D_{0t}^{\alpha} u(t) - \lambda u(t) = f(t),$$

is given by [13]

$$u(t) = C E_{\alpha}(\lambda t^{\alpha}) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\,(t-s)^{\alpha}\right) f(s)\,\mathrm{d}s,$$

where C is a constant. If $f(t) = f_0$ is constant, then the solution reduces to

$$u(t) = C^* E_\alpha(\lambda t^\alpha) - \frac{f_0}{\lambda}, \quad \lambda \neq 0,$$

where $C^* = C + \dfrac{f_0}{\lambda}$ is constant and for $\lambda = 0$, it reduces to

$$u(t) = C + \frac{f_0}{\Gamma(\alpha+1)} t^\alpha.$$

Furthermore, we have [14]

$$E_{\alpha,\beta}(\lambda t^\alpha) \leq M, \qquad 0 < \alpha \leq \beta \leq 1,\ 0 \leq t \leq T < \infty,$$

for some positive constant M, and hence if $f(t)$ is continuous on $[0,T]$, we then have

$$\int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\beta}\left(\lambda (t-s)^\alpha\right) f(s) \mathrm{d}s < \infty.$$

In the remaining two sections of the paper, we present existence and uniqueness results for direct and inverse IBVPs with both local and nonlocal boundary conditions.

2. A direct initial-boundary value problem

Consider the following IBVP in a rectangular domain
$\Omega = \{(x,t) :\ -\pi < x < \pi,\ 0 < t < T\}$:

$$
\begin{aligned}
&{}_C D_{0t}^{\alpha} u\left(x,t\right) - u_{xx}\left(x,t\right) + \varepsilon u_{xx}\left(-x,t\right) = g\left(x,t\right), && (x,t) \in \Omega, && (2.1)\\
&u_x(-\pi,t) = 0, \quad u_x(\pi,t) = 0, && 0 \leq t \leq T, && (2.2)\\
&u(x,0) = 0, && -\pi \leq x \leq \pi. && (2.3)
\end{aligned}
$$

where ε is a nonzero real number such that $|\varepsilon| < 1$ and $g(x,t)$ is a given function of space and time.

Our aim is to prove the existence and uniqueness of a regular classical solution of the IBVP (2.1)–(2.3). By a regular solution we mean $u(x,t) \in C(\overline{\Omega}) \cap C_{t,x}^{1,2}(\Omega)$. In particular, we look for a solution in the form of a series expansion using orthogonal basis which can be obtained by considering the corresponding homogeneous equation and using the method of separation of variables. This will lead to the following spectral problem:

$$X''(x) - \epsilon X''(-x) + \mu X(x) = 0, \quad X'(-\pi) = X'(\pi) = 0,$$

which is self-adjoint and hence it has real eigenvalues and the corresponding eigenfunctions form a complete orthogonal basis in $L_2(-\pi,\pi)$ [11]. By expressing the function X as a sum of even and odd functions [10], it can be shown that this problem has the following eigenvalues:

$$\lambda_{1k} = (1-\varepsilon)\, k^2, \quad \lambda_{2k} = (1+\varepsilon)\left(k + \frac{1}{2}\right)^2, \quad k \in \mathbb{N} \cup \{0\},$$

and the corresponding eigenfunctions are given by

$$X_0 = 1, \quad X_{1k} = \cos kx,\ k \in \mathbb{N}, \quad X_{2k} = \sin\left(k + \frac{1}{2}\right) x,\ k \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

2.1. Existence of a formal solution

Since the system of eigenfunctions $\left\{1, \cos kx, \sin\left(k+\frac{1}{2}\right)x\right\}$ is complete and forms a basis in $L_2(-\pi,\pi)$, we seek solution to the IBVP (2.1)–(2.3) in the form:

$$u(x,t) = u_0(t) + \sum_{k=1}^{\infty} u_{1k}(t)\cos kx + \sum_{k=0}^{\infty} u_{2k}(t)\sin\left(k+\frac{1}{2}\right)x, \tag{2.5}$$

where the coefficients u_0, u_{1k} and u_{2k} are unknown functions of time. To find these functions, we expand the function $g(x,t)$ using the eigenfunction system (2.4) and substitute it together with the solution representation (2.5) into equations (2.1) and (2.3). Hence, we get the following fractional differential equations for u_0, u_{1k} and u_{2k}:

$$\begin{aligned}
&{}_C D_{0t}^{\alpha} u_0(t) = g_0(t), \quad u_0(0) = 0,\\
&{}_C D_{0t}^{\alpha} u_{ik}(t) + \lambda_{ik} u_{ik}(t) = g_{ik}(t), \quad u_{ik}(0) = 0, \qquad i = 1, 2.
\end{aligned}$$

The terms $g_0(t)$, $g_{1k}(t)$ and $g_{2k}(t)$ represent the coefficients of the series expansion of $g(x,t)$, which are given by

$$g_0(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,t)\mathrm{d}x, \quad g_{ik}(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x,t)X_{ik}\mathrm{d}x, \quad i = 1, 2,$$

where X_{1k} and X_{2k} are given in (2.4). Solving the above fractional differential equations, we get

$$\begin{aligned}
u_0(t) &= \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1} g_0(s)\mathrm{d}s\\
u_{ik}(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{ik}(t-s)^{\alpha}\right) g_{ik}(s)\mathrm{d}s, \quad i = 1, 2.
\end{aligned}$$

Now, substituting the obtained expressions of u_0, u_{1k} and u_{2k} into the solution representation (2.5), we obtain the following expression for $u(x,t)$:

$$\begin{aligned}
u(x,t) = {} & \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1} g_0(s)\,\mathrm{d}s\\
&+ \sum_{k=1}^{\infty}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{1k}(t-s)^{\alpha}\right) g_{1k}(s)\mathrm{d}s\,\cos kx\\
&+ \sum_{k=0}^{\infty}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2k}(t-s)^{\alpha}\right) g_{2k}(s)\mathrm{d}s\,\sin\left(k+\frac{1}{2}\right)x.
\end{aligned}$$

In order to justify that the above formal solution is indeed a true solution, we need to prove that the series representations of $u(x,t), u_{xx}(x,t)$ and ${}_C D_{0t}^{\alpha} u\,(x,t)$ uniformly converge to continuous functions in Ω. To do so, we first assume that

$$g(x,t) \in C^3(\bar{\Omega}), \quad \frac{\partial g}{\partial x}(\pm\pi, t) = \frac{\partial^3 g}{\partial x^3}(\pm\pi, t) = 0,$$

and then use integration by parts to rewrite g_{1k} and g_{2k} to be

$$g_{1k}(t) = \frac{1}{k^4}\left(g^{(4)}\right)_{1k}(t), \quad g_{2k}(t) = \frac{1}{\left(k+\frac{1}{2}\right)^4}\left(g^{(4)}\right)_{2k}(t)$$

where,

$$\left(g^{(4)}\right)_{ik}(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \frac{\partial^4 g(x,t)}{\partial x^4} X_{ik}\,\mathrm{d}x, \quad i = 1, 2.$$

Hence, the solution $u(x,t)$ can now be rewritten as

$$\begin{aligned} u\,(x,t) = {} & \frac{1}{\Gamma(\alpha+1)}\int\limits_0^t (t-s)^{\alpha-1} g_0(s)\,\mathrm{d}s \\ & + \sum_{k=1}^{\infty}\int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{1k}\,(t-s)^{\alpha}\right)\left(g^{(4)}\right)_{1k}(s)\,\mathrm{d}s\,\frac{\cos kx}{k^4} \\ & + \sum_{k=0}^{\infty}\int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2k}\,(t-s)^{\alpha}\right)\left(g^{(4)}\right)_{2k}(s)\,\mathrm{d}s\,\frac{\sin\left(k+\frac{1}{2}\right)x}{\left(k+\frac{1}{2}\right)^4}. \end{aligned}$$

The uniform convergence of each series in the representations of $u(x,t), u_{xx}(x,t)$ and ${}_C D_{0t}^{\alpha} u\,(x,t)$ to a continuous function in Ω can then be ensured by assuming that $\dfrac{\partial^4 g(\cdot,t)}{\partial x^4} \in L_1[-\pi,\pi]$ and $\dfrac{\partial^4 g(x,\cdot)}{\partial x^4} \in C[0,T]$. For example, we present here the convergence of the series representation of u_{xx}, which is given by

$$u_{xx}\,(x,t) = -\sum_{k=1}^{\infty} I_{1k}\frac{\cos kx}{k^2} - \sum_{k=0}^{\infty} I_{2k}\frac{\sin\left(k+\frac{1}{2}\right)x}{\left(k+\frac{1}{2}\right)^2},$$

where

$$I_{ik} = \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{ik}\,(t-s)^{\alpha}\right)\left(g^{(4)}\right)_{ik}(s)\,\mathrm{d}s < \infty.$$

Hence, we have the following estimate for u_{xx}:

$$|u_{xx}\,(x,t)\,| \le c\sum_{k=1}^{\infty}\frac{1}{k^2} + c\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^2},$$

for some positive constant c. Since the above two series converge, then by the Weierstrass M-test, the series representation of u_{xx} converges uniformly in Ω.

2.2. Uniqueness of solution

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the IBVP (2.1)–(2.3). Then, $u(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the following homogeneous IBVP:

$$\begin{aligned}
&{}_C D_{0t}^{\alpha} u\,(x,t) - u_{xx}\,(x,t) + \varepsilon u_{xx}\,(-x,t) = 0, \quad (x,t)\in\Omega,\\
&u_x(-\pi,t) = 0, \quad u_x(\pi,t) = 0, \qquad 0 \le t \le T,\\
&u(x,0) = 0, \qquad -\pi \le x \le \pi.
\end{aligned}$$

Now, by expressing u as a sum of even and odd functions, say $u = u_e + u_o$, it can be shown that both u_e and u_o satisfy the following IBVP:

$$ {}_C D_{0t}^{\alpha} w\,(x,t) = c\, w_{xx}\,(x,t), \quad (x,t)\in\Omega, \tag{2.6}$$

$$ w_x(-\pi,t) = 0, \quad w_x(\pi,t) = 0, \qquad 0 \le t \le T, \tag{2.7}$$

$$ w(x,0) = 0, \qquad -\pi \le x \le \pi, \tag{2.8}$$

for some positive constant c. More precisely, $c = 1-\varepsilon$ when $w = u_e$ and $c = 1+\varepsilon$ when $w = u_o$. Note that for sufficiently smooth function w with $w(0) = 0$, the Caputo derivative ${}_C D_{0t}^{\alpha} w$ coincides with the Riemann–Liouville derivative $D_{0t}^{\alpha} w$ [13], *i.e.*, ${}_C D_{0t}^{\alpha} w = D_{0t}^{\alpha} w$. Now, multiplying equation (2.6) by w and integrating over Ω, we get

$$\int_0^T (w, D_{0t}^{\alpha} w)\,\mathrm{d}t = \int_0^T (w, {}_C D_{0t}^{\alpha} w)\,\mathrm{d}t = -c\int_0^T (w_x, w_x)\,\mathrm{d}t \le 0,$$

where $(f,g) = \int_{-\pi}^{\pi} fg\,\mathrm{d}x$. However, since $(w, D_{0t}^{\alpha} w) \ge 0$, (see [12], Thm. 1.7.1), we must have

$$\int_0^T (w, D_{0t}^{\alpha} w)\,\mathrm{d}t = 0,$$

which in turn implies that $(w, D_{0t}^{\alpha} w) = 0$ and hence we have $w = 0$ (see [12], Thm. 1.7.1), or equivalently, $u_e = u_o = 0$. Therefore, the IBVP (2.1)–(2.3) has a unique solution.

2.3. Main result for the direct problem

The main result for the direct problem can be formulated in the following theorem:

Theorem 2.1. *If the nonhomogeneous term $g(x,t)$ in equation (2.1) satisfies the following conditions:*

$$\begin{aligned}
&g(x,t)\in C^3(\bar\Omega), \quad \frac{\partial^4 g(\cdot,t)}{\partial x^4}\in L_1[-\pi,\pi], \quad \frac{\partial^4 g(x,\cdot)}{\partial x^4}\in C[0,T]\\
&\frac{\partial g}{\partial x}(\pm\pi,t) = \frac{\partial^3 g}{\partial x^3}(\pm\pi,t) = 0, \quad 0\le t\le T,
\end{aligned}$$

then, for a nonzero real number ε such that $|\varepsilon| < 1$, the IBVP (2.1)–(2.3) has a unique solution which can be written in the form

$$u(x,t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} g_0(s)\mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{1k}(t-s)^{\alpha}\right) \left(g^{(4)}\right)_{1k}(s)\,\mathrm{d}s\, \frac{\cos kx}{k^4} + \sum_{k=0}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{2k}(t-s)^{\alpha}\right) \left(g^{(4)}\right)_{2k}(s)\,\mathrm{d}s\, \frac{\sin\left(k+\frac{1}{2}\right)x}{\left(k+\frac{1}{2}\right)^4},$$

where

$$g_0(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x,t)\mathrm{d}x, \quad \left(g^{(4)}\right)_{1k}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial^4 g(x,t)}{\partial x^4} \cos kx\,\mathrm{d}x$$

$$\left(g^{(4)}\right)_{2k}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial^4 g(x,t)}{\partial x^4} \sin\left(k+\frac{1}{2}\right) x\,\mathrm{d}x.$$

3. Inverse initial-boundary value problems

In this section, we consider the following equation:

$$ {}_C D_{0t}^{\alpha} u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \quad (x,t) \in \Omega, \tag{3.1}$$

where Ω is the rectangular domain given in the previous section and f is an unknown function of x. Our aim is to find a pair of functions $\{u(x,t), f(x)\}$ such that $u(x,t) \in C(\overline{\Omega}) \cap C_{t,x}^{1,2}(\Omega)$ and $f(x) \in C[-\pi,\pi]$ satisfying the following two problems with local and nonlocal boundary conditions (BCs), respectively.

IP1: Inverse problem with local BCs. Find a pair of functions $\{u(x,t),\ f(x)\}$ satisfying equation (3.1) and the following conditions:

$$u(x,0) = \phi(x), \quad u(x,T) = \psi(x), \qquad -\pi \le x \le \pi, \tag{3.2}$$

$$u(-\pi,t) = 0, \quad u(\pi,t) = 0, \qquad 0 \le t \le T, \tag{3.3}$$

where $\phi(x)$ and $\psi(x)$ are given functions.

IP2: Inverse problem with nonlocal BCs. Find a pair of functions $\{u(x,t),\ f(x)\}$ satisfying equation (3.1), conditions (3.2) and the nonlocal boundary condition:

$$u(-\pi,t) = u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t) \qquad 0 \le t \le T. \tag{3.4}$$

The above two inverse problems are a generalization of two problems considered in [2] for the classical heat equation, *i.e.*, for the case $\alpha = 1$.

3.1. Spectral problems

Using the separation of variables method for solving the homogeneous partial differential equation corresponding to equation (3.1) leads to the following spectral problems for IP1 and IP2, respectively:

$$X''(x) - \epsilon X''(-x) + \lambda X(x) = 0, \quad X(-\pi) = X(\pi) = 0, \tag{3.5}$$

$$X''(x) - \epsilon X''(-x) + \lambda X(x) = 0, \quad X(-\pi) = X(\pi), \, X'(-\pi) = X'(\pi), \tag{3.6}$$

which are self-adjoint. Problem (3.5) has the following eigenvalues:

$$\lambda_{1k} = (1-\varepsilon)\left(k + \frac{1}{2}\right)^2, \, k \in \mathbb{N} \cup \{0\}, \quad \lambda_{2k} = (1+\varepsilon)\, k^2, \, k \in \mathbb{N},$$

corresponding to the following eigenfunctions:

$$X_{1k} = \cos\left(k + \frac{1}{2}\right)x, \; k \in \mathbb{N} \cup \{0\}, \quad X_{2k} = \sin kx, \; k \in \mathbb{N}. \tag{3.7}$$

The eigenvalues of problem (3.6) are

$$\lambda_{1k} = (1-\varepsilon)\, k^2, \, k \in \mathbb{N} \cup \{0\}, \quad \lambda_{2k} = (1+\varepsilon)\, k^2, \, k \in \mathbb{N},$$

and the corresponding eigenfunctions are given by

$$X_0 = 1, \quad X_{1k} = \cos kx, \quad X_{2k} = \sin kx, \quad k \in \mathbb{N}. \tag{3.8}$$

Each one of the systems of eigenfunctions $\left\{\cos\left(k + \frac{1}{2}\right)x, \sin kx\right\}$ and $\{1, \cos kx, \sin kx\}$ is complete and forms a basis in $L_2(-\pi, \pi)$.

3.2. Existence and uniqueness of solution

Solution methods of proving existence and uniqueness of solutions to the inverse problems IP1 and IP2 are very similar which are based on series expansion of solutions using an appropriate orthogonal system. Hence, we present here the existence and uniqueness of solution to one of them, namely problem IP2. Using the orthogonal system (3.8), the set of solutions $\{u(x,t), \, f(x)\}$ can be represented as

$$u(x,t) = u_0(t) + \sum_{k=1}^{\infty} (u_{1k}(t) \cos kx + u_{2k}(t) \sin kx), \tag{3.9}$$

$$f(x) = f_0 + \sum_{k=1}^{\infty} (f_{1k} \cos kx + f_{2k} \sin kx), \tag{3.10}$$

where the coefficients $u_0(t)$, $u_{1k}(t)$ and $u_{2k}(t)$ are unknown functions of time and the coefficients f_0, f_{1k} and f_{2k} are unknown constants. Substituting the solution representation (3.9) and (3.10) into equation (3.1), we obtain the following fractional differential equations:

$$\begin{aligned} &{}_C D_{0t}^{\alpha} u_0(t) = f_0 \\ &{}_C D_{0t}^{\alpha} u_{ik}(t) + \lambda_{ik} u_{ik}(t) = f_{ik}, \qquad i = 1, 2, \end{aligned}$$

where $\lambda_{1k} = (1-\varepsilon)\, k^2$ and $\lambda_{2k} = (1+\varepsilon)\, k^2$ are the eigenvalues of problem (3.6). Solutions to these equations are found to be

$$u_0(t) = c_0 + \frac{f_0}{\Gamma(\alpha+1)} t^{\alpha}$$
$$u_{ik}(t) = c_{ik} E_{\alpha}\left(-\lambda_{ik} t^{\alpha}\right) + \frac{f_{ik}}{\lambda_{ik}}, \quad i = 1, 2,$$

where c_0, c_{1k} and c_{2k} are unknown constants. Now, expanding the functions $\phi(x)$ and $\psi(x)$ using the orthogonal system (3.8) and using the conditions in (3.2), we obtain a set of algebraic equations for c_0, c_{1k}, c_{2k} and f_0, f_{1k}, f_{2k}. Solving these equations, we obtain

$$c_0 = \phi_0, \qquad f_0 = \frac{\Gamma(\alpha+1)}{T^{\alpha}}(\psi_0 - \phi_0)$$
$$c_{ik} = \frac{\phi_{ik} - \psi_{ik}}{1 - E_{\alpha}\left(-\lambda_{ik} T^{\alpha}\right)}, \quad f_{ik} = \lambda_{ik}\left(\phi_{ik} - c_{ik}\right), \quad i = 1, 2,$$

where the constants ϕ_0, ϕ_{1k}, ϕ_{2k} and ψ_0, ψ_{1k}, ψ_{2k} are the coefficients of the series expansion of $\phi(x)$ and $\psi(x)$, respectively, which are given by

$$\phi_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\mathrm{d}x, \quad \phi_{ik} = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x) X_{ik}\mathrm{d}x, \quad i = 1, 2,$$
$$\psi_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(x)\mathrm{d}x, \quad \psi_{ik} = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x) X_{ik}\mathrm{d}x, \quad i = 1, 2,$$

and $X_{1k} = \cos kx$, $X_{2k} = \sin kx$ are the eigenfunctions of problem (3.6). Now, substituting for $u_0(t)$, $u_{1k}(t)$ and $u_{2k}(t)$ into (3.9) and for f_0, f_{1k} and f_{2k} into (3.10), we obtain the following expressions for $u(x,t)$ and $f(x)$:

$$\begin{aligned} u(x,t) = {} & \phi(x) + \frac{t^{\alpha}}{T^{\alpha}}(\psi_0 - \phi_0) \\ & + \sum_{k=1}^{\infty} \frac{1 - E_{\alpha}\left(-\lambda_{1k} t^{\alpha}\right)}{1 - E_{\alpha}\left(-\lambda_{1k} T^{\alpha}\right)}(\psi_{1k} - \phi_{1k}) \cos kx \\ & + \sum_{k=1}^{\infty} \frac{1 - E_{\alpha}\left(-\lambda_{2k} t^{\alpha}\right)}{1 - E_{\alpha}\left(-\lambda_{2k} T^{\alpha}\right)}(\psi_{2k} - \phi_{2k}) \sin kx, \end{aligned}$$

and

$$\begin{aligned} f(x) = {} & -\phi''(x) + \varepsilon\phi''(-x) + \frac{\Gamma(\alpha+1)}{T^{\alpha}}(\psi_0 - \phi_0) \\ & + \sum_{k=1}^{\infty}\left(\lambda_{1k}\frac{\psi_{1k} - \phi_{1k}}{1 - E_{\alpha}\left(-\lambda_{1k} T^{\alpha}\right)}\cos kx + \lambda_{2k}\frac{\psi_{2k} - \phi_{2k}}{1 - E_{\alpha}\left(-\lambda_{2k} T^{\alpha}\right)}\sin kx\right), \end{aligned}$$

In order to complete the proof of existence of a formal solution to problem IP2, we need to prove the uniform convergence of the series appearing in the above expressions for $u(x,t)$, $f(x)$ as well as the corresponding series in $u_{xx}(x,t)$ and ${}_{C}D_{0t}^{\alpha} u(x,t)$. For this purpose, we are going to rewrite the terms $\psi_{ik} - \phi_{ik}, i = 1, 2$ using integration by parts. Before doing so, we first note that conditions (3.2) and (3.4) imply that

$$\phi^{(j)}(-\pi)=\phi^{(j)}(\pi) \quad \text{and} \quad \psi^{(j)}(-\pi)=\psi^{(j)}(\pi) \quad j=0,1.$$

Further, assuming that $\phi''(-\pi)=\phi''(\pi)$ and $\psi''(-\pi)=\psi''(\pi)$, we then have

$$\psi_{1k}-\phi_{1k}=\frac{\left(\psi^{(3)}\right)_{2k}-\left(\phi^{(3)}\right)_{2k}}{k^3} \quad \text{and} \quad \psi_{2k}-\phi_{2k}=-\frac{\left(\psi^{(3)}\right)_{1k}-\left(\phi^{(3)}\right)_{1k}}{k^3},$$

where

$$\left(\phi^{(3)}\right)_{ik}=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\phi'''(x)X_{ik}\mathrm{d}x, \quad \text{and} \quad \left(\psi^{(3)}\right)_{ik}=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\psi'''(x)X_{ik}\mathrm{d}x, \quad i=1,2,$$

and $X_{1k}=\cos kx$, $X_{2k}=\sin kx$. The expressions for $u(x,t)$ and $f(x)$ can then be rewritten as:

$$\begin{aligned} u(x,t) &= \phi(x)+\frac{t^\alpha}{T^\alpha}(\psi_0-\phi_0) \\ &+\sum_{k=1}^{\infty}\frac{1-E_\alpha(-\lambda_{1k}t^\alpha)}{1-E_\alpha(-\lambda_{1k}T^\alpha)}\left(\frac{\left(\psi^{(3)}\right)_{2k}-\left(\phi^{(3)}\right)_{2k}}{k^3}\right)\cos kx \\ &-\sum_{k=1}^{\infty}\frac{1-E_\alpha(-\lambda_{2k}t^\alpha)}{1-E_\alpha(-\lambda_{2k}T^\alpha)}\left(\frac{\left(\psi^{(3)}\right)_{1k}-\left(\phi^{(3)}\right)_{1k}}{k^3}\right)\sin kx, \end{aligned}$$

and

$$\begin{aligned} f(x) &= -\phi''(x)+\varepsilon\phi''(-x)+\frac{\Gamma(\alpha+1)}{T^\alpha}(\psi_0-\phi_0) \\ &+\sum_{k=1}^{\infty}\left(\frac{1-\varepsilon}{k}\frac{\left(\psi^{(3)}\right)_{2k}-\left(\phi^{(3)}\right)_{2k}}{1-E_\alpha(-\lambda_{1k}T^\alpha)}\cos kx-\frac{1+\varepsilon}{k}\frac{\left(\psi^{(3)}\right)_{1k}-\left(\phi^{(3)}\right)_{1k}}{1-E_\alpha(-\lambda_{2k}T^\alpha)}\sin kx\right), \end{aligned}$$

The convergence of the above series and the ones corresponding to $u_{xx}(x,t)$ and ${}_CD^\alpha_{0t}u(x,t)$ can be shown in a very similar way. Hence, here we consider only one of them, namely, the one for ${}_CD^\alpha_{0t}u(x,t)$, which has the following representation:

$$\begin{aligned} {}_CD^\alpha_{0t}u(x,t) &= \frac{\Gamma(\alpha+1)}{T^\alpha}(\psi_0-\phi_0) \\ &+\sum_{k=1}^{\infty}\frac{(1-\varepsilon)E_\alpha(-\lambda_{1k}t^\alpha)}{1-E_\alpha(-\lambda_{1k}T^\alpha)}\left(\frac{\left(\psi^{(3)}\right)_{2k}-\left(\phi^{(3)}\right)_{2k}}{k}\right)\cos kx \\ &-\sum_{k=1}^{\infty}\frac{(1+\varepsilon)E_\alpha(-\lambda_{2k}t^\alpha)}{1-E_\alpha(-\lambda_{2k}T^\alpha)}\left(\frac{\left(\psi^{(3)}\right)_{1k}-\left(\phi^{(3)}\right)_{1k}}{k}\right)\sin kx. \end{aligned} \tag{3.11}$$

The convergence of the two series in the above representation is based on the following estimate:

$$\begin{aligned}|{}_C D_{0t}^{\alpha} u\,(x,t)| \leq{}& \frac{\Gamma(\alpha+1)}{T^{\alpha}}(|\psi_0|+|\phi_0|)\\ &+c\sum_{k=1}^{\infty}\frac{|\left(\psi^{(3)}\right)_{2k}|+|\left(\phi^{(3)}\right)_{2k}|}{k}+c\sum_{k=1}^{\infty}\frac{|\left(\psi^{(3)}\right)_{1k}|+|\left(\phi^{(3)}\right)_{1k}|}{k},\end{aligned}$$

for some positive constant c. Using the inequality $2ab \leq a^2+b^2$, the term $\left|\frac{(g^{(3)})_{ik}}{k}\right|$ can be estimated as follows:

$$\left|\frac{(g^{(3)})_{ik}}{k}\right| \leq \frac{1}{2}\left(\left|\left(g^{(3)}\right)_{ik}\right|^2+\frac{1}{k^2}\right) \quad \text{for} \quad g=\phi,\psi \quad \text{and} \quad i=1,2.$$

Now, assuming that $\phi'''(x)$ and $\psi'''(x) \in L_2(-\pi,\pi)$, then by Bessel inequality, the following series converge:

$$\sum_{k=1}^{\infty}\left|\left(g^{(3)}\right)_{ik}\right|^2 \leq c\,\|g'''(x)\|_{L_2(-\pi,\pi)} \quad \text{for} \quad g=\phi,\psi \quad \text{and} \quad i=1,2.$$

Therefore, by the Weierstrass M-test, the two series in the representation (3.11) of ${}_C D_{0t}^{\alpha} u\,(x,t)$ converge uniformly in Ω.

For uniqueness, we assume that problem IP2 has two pairs of solutions, say, $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$. Then, the pair

$$\{u(x,t)=u_1(x,t)-u_2(x,t), f(x)=f_1(x)-f_2(x)\}$$

satisfies equation (3.1) and the boundary conditions (3.4) along with the following homogeneous conditions:

$$u(x,0)=0, \quad u(x,T)=0, \qquad -\pi \leq x \leq \pi.$$

Hence, due to the completeness of system (3.8) in $L_2\,(-\pi,\pi)$, we then have only trivial solutions

$$u(x,t)=0, \quad f(x)=0, \qquad -\pi \leq x \leq \pi, \quad 0 \leq t \leq T.$$

Therefore, the inverse problem IP2 has a unique solution.

3.3. Main results for the inverse problems

The main results for the two inverse problems IP1 and IP2 can be formulated in the following theorems:

Theorem 3.1. *Let* $\phi\,(x), \psi\,(x) \in C^2\,[-\pi,\pi]\,, \phi'''(x), \psi'''(x) \in L_2(-\pi,\pi)$ *and* $\phi^{(i)}\,(\pm\pi)=0, \psi^{(i)}\,(\pm\pi)=0, i=0,2$. *Then, for a nonzero real number* ε *such that* $|\varepsilon|<1$, *the inverse problem IP1 has a unique solution pair*

$\{u(x,t), f(x)\}$ *which can be written in the form*

$$u(x,t) = \phi(x) - \sum_{k=1}^{\infty} \frac{1 - E_\alpha(-\lambda_{2k} t^\alpha)}{1 - E_\alpha(-\lambda_{2k} T^\alpha)} \left(\frac{\left(\psi^{(3)}\right)_{1k} - \left(\phi^{(3)}\right)_{1k}}{k^3} \right) \sin kx$$
$$+ \sum_{k=0}^{\infty} \frac{1 - E_\alpha(-\lambda_{1k} t^\alpha)}{1 - E_\alpha(-\lambda_{1k} T^\alpha)} \left(\frac{\left(\psi^{(3)}\right)_{2k} - \left(\phi^{(3)}\right)_{2k}}{\left(k + \frac{1}{2}\right)^3} \right) \cos\left(k + \frac{1}{2}\right) x,$$

$$f(x) = -\phi''(x) + \varepsilon \phi''(-x) - \sum_{k=1}^{\infty} \frac{1+\varepsilon}{k} \frac{\left(\psi^{(3)}\right)_{1k} - \left(\phi^{(3)}\right)_{1k}}{1 - E_\alpha(-\lambda_{2k} T^\alpha)} \sin kx$$
$$+ \sum_{k=0}^{\infty} \frac{1-\varepsilon}{\left(k + \frac{1}{2}\right)} \frac{\left(\psi^{(3)}\right)_{2k} - \left(\phi^{(3)}\right)_{2k}}{1 - E_\alpha(-\lambda_{1k} T^\alpha)} \cos\left(k + \frac{1}{2}\right) x,$$

where

$$\left(\phi^{(3)}\right)_{ik} = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi'''(x) X_{ik} \mathrm{d}x, \quad \left(\psi^{(3)}\right)_{ik} = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi'''(x) X_{ik} \mathrm{d}x, \quad i = 1, 2,$$

and $X_{1k} = \cos kx,\ X_{2k} = \sin\left(k + \frac{1}{2}\right) x.$

Theorem 3.2. *Let* $\phi(x), \psi(x) \in C^2[-\pi,\pi], \phi'''(x), \psi'''(x) \in L_2(-\pi,\pi)$ *and* $\phi^{(i)}(-\pi) = \phi^{(i)}(\pi), \psi^{(i)}(-\pi) = \psi^{(i)}(\pi), i = 0,1,2$. *Then, for a nonzero real number* ε *such that* $|\varepsilon| < 1$, *the inverse problem IP2 has a unique solution pair* $\{u(x,t), f(x)\}$ *which can be written in the form*

$$u(x,t) = \phi(x) + \frac{t^\alpha}{T^\alpha}(\psi_0 - \phi_0)$$
$$+ \sum_{k=1}^{\infty} \frac{1 - E_\alpha(-\lambda_{1k} t^\alpha)}{1 - E_\alpha(-\lambda_{1k} T^\alpha)} \left(\frac{\left(\psi^{(3)}\right)_{2k} - \left(\phi^{(3)}\right)_{2k}}{k^3} \right) \cos kx$$
$$- \sum_{k=1}^{\infty} \frac{1 - E_\alpha(-\lambda_{2k} t^\alpha)}{1 - E_\alpha(-\lambda_{2k} T^\alpha)} \left(\frac{\left(\psi^{(3)}\right)_{1k} - \left(\phi^{(3)}\right)_{1k}}{k^3} \right) \sin kx,$$

$$f(x) = -\phi''(x) + \varepsilon \phi''(-x) + \frac{\Gamma(\alpha+1)}{T^\alpha}(\psi_0 - \phi_0)$$
$$+ \sum_{k=1}^{\infty} \left(\frac{1-\varepsilon}{k} \frac{\left(\psi^{(3)}\right)_{2k} - \left(\phi^{(3)}\right)_{2k}}{1 - E_\alpha(-\lambda_{1k} T^\alpha)} \cos kx - \frac{1+\varepsilon}{k} \frac{\left(\psi^{(3)}\right)_{1k} - \left(\phi^{(3)}\right)_{1k}}{1 - E_\alpha(-\lambda_{2k} T^\alpha)} \sin kx \right),$$

where

$$\left(\phi^{(3)}\right)_{ik} = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi'''(x) X_{ik} \mathrm{d}x, \quad \left(\psi^{(3)}\right)_{ik} = \frac{1}{\pi} \int_{-\pi}^{\pi} \psi'''(x) X_{ik} \mathrm{d}x, \quad i = 1, 2,$$

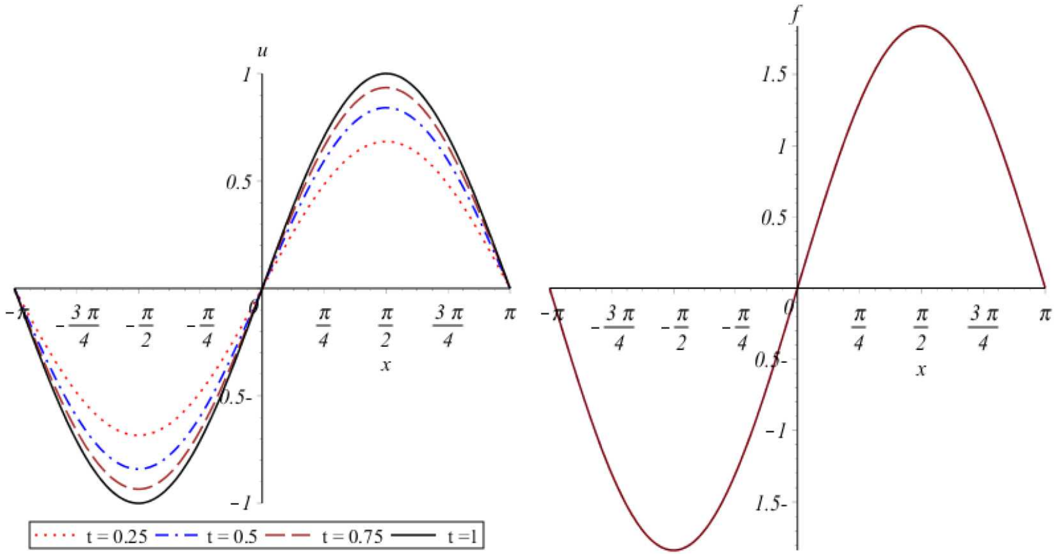

FIGURE 1. Graphs of $u(x,t)$ at different times (*left*) and $f(x)$ (*right*) for $\alpha = 0.5$ and $\varepsilon = 0.1$.

$$and \quad X_{1k} = \cos kx,\ X_{2k} = \sin kx.$$

It is worth mentioning here that for the special case of $\alpha = 1$, the above obtained results are in a full agreement with the ones obtained in [2].

3.4. Example

For the sake of illustration and for comparison with the case $\alpha = 1$, we present here a simple example solution for the inverse problem IP1 considering the following choices for $\phi(x)$ and $\psi(x)$:

$$\phi(x) = 0 \quad \text{and} \quad \psi(x) = \sin x.$$

Solutions corresponding to this choice are given by

$$u(x,t) = \frac{1 - E_\alpha\left(-\left(1+\varepsilon\right)t^\alpha\right)}{1 - E_\alpha\left(-\left(1+\varepsilon\right)T^\alpha\right)}\sin x, \quad \text{and} \quad f(x) = \frac{1+\epsilon}{1 - E_\alpha\left(-\left(1+\varepsilon\right)T^\alpha\right)}\sin x.$$

Again, this solution reduce to the one obtained in [2] for the case $\alpha = 1$. The obtained solutions are illustrated in Figures 1–3. Here, we took $T = 1$. Figure 1 shows the temperature profile at different times and the heat source for fixed values of the fractional order and the involution coefficient. The effect of the over-determination condition is clearly seen in the heat profile and the shape of the heat source. Moreover, the temperature is increasing with time and reaching its maximum when $t = T$. The effect of the involution term represented by the magnitude of the involution coefficient ε is illustrated in Figure 2. It shows very small effect on the heat profile, while its effect is more apparent in the strength of the heat source. Figure 3 illustrates the heat profile and the heat source for different values of the order α of the fractional derivative. It shows that smaller values of the fractional order result in more heat and stronger heat source.

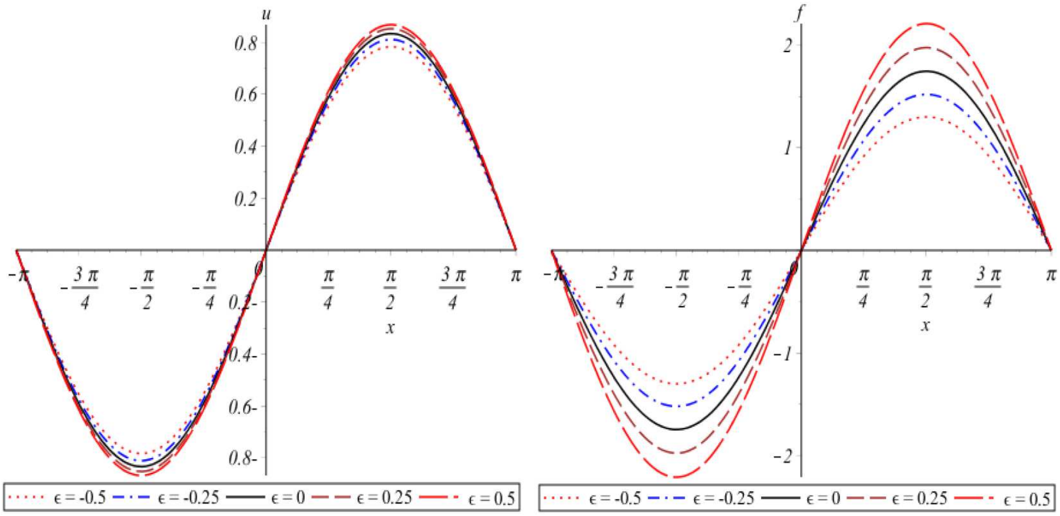

FIGURE 2. Graphs of $u(x,t)$ at $t = 0.5$ (*left*) and $f(x)$ (*right*) for different values of ε and for $\alpha = 0.5$.

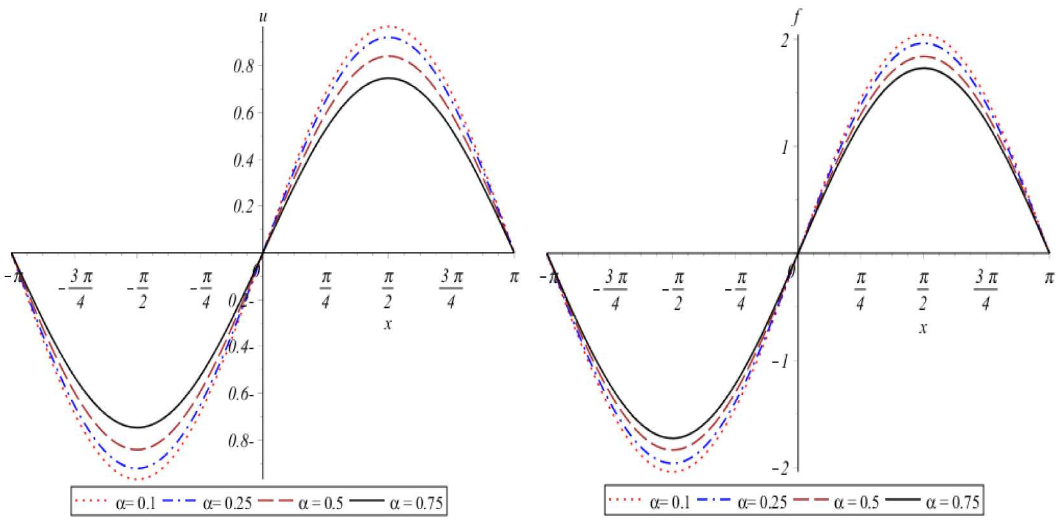

FIGURE 3. Graphs of $u(x,t)$ at $t = 0.5$ (*left*) and $f(x)$ (*right*) for different values of α and for $\varepsilon = 0.1$.

4. Conclusion

In this work, we have considered direct and inverse IBVPs of a time fractional heat equation with involution using three different types of boundary conditions, namely, Dirichlet, Neumann and periodic boundary conditions in a rectangular domain. We start by considering a direct problem with a given space and time dependent source term. A regular classical solution of the direct problem was obtained in a form of series expansion using orthogonal basis which are the eigenfunctions of a self-adjoint spectral problem obtained by considering the corresponding homogeneous equation and using the method of separation of variables. The uniform convergence of the series solution was obtained by imposing certain conditions on the given source term. The uniqueness of solution was obtained using certain properties of fractional operators. Moreover, two inverse source problems for a time-fractional heat equation with involution and a time-independent source term were considered using Dirichlet and periodic boundary conditions, respectively. The pair of solutions for each problem was obtained using the same approach as in the direct problem using appropriate system of eigenfunctions. Under certain conditions on the given data, convergence of solutions were obtained and using the completeness properties of the used system of eigenfunctions, one can prove the uniqueness of solutions. The obtained solutions generalize the solutions to the corresponding classical heat equation with involution. Finally, an illustrative example was presented showing the effect of the involution term and the order of the fractional derivative on the heat profile.

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References

[1] A.R. Aftabizadeh, Y.K. Huang and J. Wiener, Bounded solutions for differential equations with reflection of the argument. *J. Math. Anal. Appl.* **135** (1988) 31–37.
[2] N. Al-Salti, M. Kirane and B.T. Torebek, On a class of inverse problems for a heat equation with involution perturbation. To appear in: *Hacettepe J. Math. Stat.* **48** (2019). Doi: 10.15672/HJMS.2017.538.
[3] A.A. Andreev, On analogues of classical boundary value problems for a second-order differential equation with deviating argument. *Differ. Equ.* **40** (2004) 1192–1194.
[4] M.Sh. Burlutskaya and A.P. Khromov, The Fourier method in a mixed problem for a first-order partial differential equation with involution. *Comput. Math. Math. Phys.* **51** (2011) 2102–2114.
[5] A. Cabada and A.F. Tojo, Equations with Involutions. Atlantis Press, Paris (2015).
[6] C.P. Gupta, Existence and uniqueness theorems for boundary value problems involving reflection of the argument. *Nonlinear Anal.* **11** (1987) 1075–1083.
[7] C.P. Gupta, Two-point boundary value problems involving reflection of the argument. *Int. J. Math. Math. Sci.* **10** (1987) 361–371.
[8] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Vol. 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam (2006).
[9] M. Kirane and N. Al-Salti, Inverse problems for a nonlocal wave equation with an involution perturbation. *J. Nonlinear Sci. Appl.* **9** (2016) 1243–1251.
[10] A. Kopzhassarova and A. Sarsenbi, Basis properties of eigenfunctions of second-order differential operators with involution. *Abstr. Appl. Anal.* **2012** (2012) 6.
[11] M.A. Naimark, Linear Differential Operators, Part II: Linear Differential Operators in Hilbert Space. Ungar Publishing Co., New York (1968).
[12] A. Nakhushev, Fractional Calculus and Its Applications. Fizmatlit, Moscow (2003) (in Russian).
[13] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Vol 198 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA (1999).
[14] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19** (1971) 7–15.
[15] D. Przeworska-Rolewicz, Equations with Transformed Arguments: An Algebraic Approach, Modern Analytic and Computational Methods in Science and Mathematics. Elsevier Scientific Publishing and PWN-Polish Scientific Publishers, Amsterdam and Warsaw (1973).
[16] W. Watkins, Modified Wiener equations. *Int. J. Math. Math. Sci.* **27** (2001) 347–356.
[17] J. Wiener, Differential equations with involutions. *Differ. Equ.* **5** (1969) 1131–1137.
[18] J. Wiener, Differential equations in partial derivatives with involutions. *Differ. Equ.* **6** (1970) 1320–1322.
[19] J. Wiener and A.R. Aftabizadeh, Boundary value problems for differential equations with reflection of the argument. *Int. J. Math. Math. Sci.* **8** (1985) 151–163.
[20] J. Wiener and W. Watkins, A classroom approach to involutions. *College Math. J.* **19** (1988) 247–250.
[21] J. Wiener and W. Watkins, A glimpse into the wonderland of involutions. *Missouri J. Math. Sci.* **1** (2002) 175–185.